

1. If $\vec{r}(t) = r \cos(\omega t)\vec{i} + r \sin(\omega t)\vec{j}$ is the position vector of a point at time t , $\vec{v}(t)$ is the velocity vector of $\vec{r}(t)$ and $\vec{a}(t)$ is the acceleration vector of $\vec{r}(t)$, show that
 - (a) $\vec{r} \cdot \vec{v} = 0$,
 - (b) $\vec{r} \times \vec{v} = \text{constant vector}$,
 - (c) $\vec{a} = \omega^2 \vec{r}$.
2. (a) Compute the divergence and curl of the vector functions:
 - (i) $\vec{v} = e^x \cos y \vec{i} + xy^2 \vec{j} + yz^3 \vec{k}$
 - (ii) $\vec{v} = yz \vec{i} + 3zx \vec{j} + z \vec{k}$
 - (b) (i) Find $\text{div}(\text{grad } f)$, for $f(x, y, z) = 1 - x^2 - 4y^2 + 2z^2$
 - (ii) Find $\nabla \times \nabla(\nabla \cdot \vec{v})$, for $\vec{v}(x, y, z) = e^x \vec{i} + e^y \vec{j} + e^z \vec{k}$
 - (c) Verify the formula $\text{div}(\vec{v}f) = f \text{div} \vec{v} + \vec{v} \cdot \text{grad } f$ for $f = e^{xyz}$ and $\vec{v} = x \vec{i} + y \vec{j} + z \vec{k}$.
 - (d) Prove that for any vector fields $\vec{v}$ and $\vec{w}$ on $\mathbf{R}^3$,
 - (i) $\text{curl}(\vec{v} + \vec{w}) = \text{curl} \vec{v} + \text{curl} \vec{w}$
 - (ii) $\text{div}(\text{curl} \vec{v}) = 0$
3. It is given that $\vec{r} = x \vec{i} + y \vec{j} + z \vec{k}$ and $\vec{p} = a \vec{i} + b \vec{j} + c \vec{k}$ is a constant vector and $\vec{u} = (\vec{p} \cdot \vec{r}) \vec{r}$.
 - (a) Evaluate $\vec{u} = (\vec{p} \cdot \vec{r}) \vec{r}$.
 - (b) Show that
 - (i) $\nabla \cdot \vec{u} = 4 \vec{p} \cdot \vec{r}$,
 - (ii) $\nabla \times \vec{u} = \vec{p} \times \vec{r}$,
 - (iii) $\nabla \times (\vec{p} \times \vec{r}) = 2 \vec{p}$.
4. Let $\vec{F}(x, y, z) = (x + 2y + az) \vec{i} + (bx - 3y - z) \vec{j} + (4x + cy + 2z) \vec{k}$ be a vector field on $\mathbf{R}^3$, where a, b and c are real constants.
 - (a) Find the values of a, b and c such that $\vec{F}$ is irrotational.
 - (b) With the values of a, b and c obtained in (a), determine a potential function ϕ on $\mathbf{R}^3$ for which $\nabla \phi = \vec{F}$.
5. Let $\vec{G}(x, y, z) = 3yz \vec{i} + x^2 \vec{j} + x \cos y \vec{k}$ be a vector field on $\mathbf{R}^3$.
 - (a) Show that $\vec{G}$ is solenoidal.
 - (b) Find a vector field $\vec{F}(x, y, z) = f_1(x, y, z) \vec{i} + f_2(x, y, z) \vec{j}$ on $\mathbf{R}^3$ such that $\nabla \times \vec{F} = \vec{G}$.
6. (a) A vector field $\vec{F}$ is said to be **solenoidal** if $\nabla \cdot \vec{F} = 0$. Let $\vec{F} = (y + z) \vec{i} + (x + z) \vec{j} + (x + y) \vec{k}$. Show that $\vec{F}$ is solenoidal.
 - (b) As a consequence of $\vec{F}$ being solenoidal, there exists a vector field $\vec{H}$ such that $\vec{F} = \nabla \times \vec{H}$. Find a vector field $\vec{H} = h_1(x, y, z) \vec{i} + h_2(x, y, z) \vec{j} + h_3(x, y, z) \vec{k}$ with $h_2(x, y, z) \equiv 0$ such that $\vec{F} = \nabla \times \vec{H}$.
 - (c) Observe that if ϕ is a scalar field and $\vec{H}, \vec{F}$ are vector fields such that $\vec{F} = \nabla \times \vec{H}$, then we have $\nabla \times (\vec{H} + \nabla \phi) = \nabla \times \vec{H} + \nabla \times \nabla \phi = \nabla \times \vec{H} = \vec{F} \dots \dots (I)$. Using (b) and observation (I), find a vector field $\vec{G} = g_1(x, y, z) \vec{i} + g_2(x, y, z) \vec{j} + g_3(x, y, z) \vec{k}$ such that $\vec{F} = \nabla \times \vec{G}$ and $g_2(x, y, z) = 2y$.