

MA3160: Probability and Stochastic Processes
Homework assignment #1 – solutions

1. (i) Consider two events A and B , with $\mathbb{P}(A) = 0.3$, $\mathbb{P}(B) = 1$. Compute $\mathbb{P}(A \cap B)$, $\mathbb{P}(A^c \cap B)$, and $\mathbb{P}(A \cap B^c)$ (where we denote by $A^c = \Omega \setminus A$ the complement of an event A).

First, it follows from $A \cap B^c \subseteq B^c$ that

$$0 \leq \mathbb{P}(A \cap B^c) \leq \mathbb{P}(B^c) = 1 - \mathbb{P}(B) = 0.$$

We thus have $\mathbb{P}(A \cap B^c) = 0$.

Second, we can write A as the disjoint union $A = (A \cap B) \cup (A \cap B^c)$, which implies

$$\mathbb{P}(A) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cap B^c).$$

We deduce

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) - \mathbb{P}(A \cap B^c) = \mathbb{P}(A) = 0.3.$$

Finally, for similar reasons we have

$$\mathbb{P}(A^c \cap B) = \mathbb{P}(A^c) = 1 - \mathbb{P}(A) = 0.7.$$

- (ii) We now consider three independent events A , B , and C . Using the definition of independence, show that the two events A^c and $B^c \cup C^c$ are independent.

In order to prove the independence of these two events, we need to check the relation

$$\mathbb{P}(A^c \cap (B^c \cup C^c)) = \mathbb{P}(A^c) \cdot \mathbb{P}(B^c \cup C^c),$$

and we have the following four relations at our disposal, coming from the independence of the three events A , B , and C :

1. $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$,
2. $\mathbb{P}(B \cap C) = \mathbb{P}(B)\mathbb{P}(C)$,
3. $\mathbb{P}(A \cap C) = \mathbb{P}(A)\mathbb{P}(C)$,
4. $\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$.

We can for example proceed as follows. First,

$$\mathbb{P}(A^c) = \mathbb{P}(A^c \cap (B^c \cup C^c)) + \mathbb{P}(A^c \cap (B^c \cup C^c)^c)$$

(since we have a disjoint union), so

$$\begin{aligned} \mathbb{P}(A^c \cap (B^c \cup C^c)) &= \mathbb{P}(A^c) - \mathbb{P}(A^c \cap (B^c \cup C^c)^c) \\ &= \mathbb{P}(A^c) - \mathbb{P}(A^c \cap B \cap C). \end{aligned}$$

Then (for similar reasons),

$$\mathbb{P}(A^c \cap B \cap C) + \mathbb{P}(A \cap B \cap C) = \mathbb{P}(B \cap C)$$

so

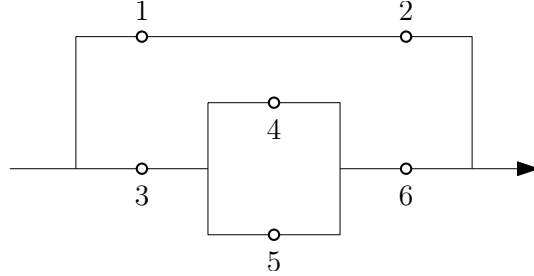
$$\begin{aligned}
\mathbb{P}(A^c \cap B \cap C) &= \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B \cap C) \\
&= \mathbb{P}(B)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) \\
&= (1 - \mathbb{P}(A))\mathbb{P}(B)\mathbb{P}(C) \\
&= \mathbb{P}(A^c)\mathbb{P}(B)\mathbb{P}(C)
\end{aligned}$$

(where we used relations 2. and 4. to go from the first line to the second one). Finally, we write

$$\begin{aligned}
\mathbb{P}(A^c \cap (B^c \cup C^c)) &= \mathbb{P}(A^c) - \mathbb{P}(A^c \cap B \cap C) \\
&= \mathbb{P}(A^c) - \mathbb{P}(A^c)\mathbb{P}(B)\mathbb{P}(C) \\
&= \mathbb{P}(A^c)(1 - \mathbb{P}(B)\mathbb{P}(C)) \\
&= \mathbb{P}(A^c)\mathbb{P}(B^c \cup C^c)
\end{aligned}$$

(since $\mathbb{P}(B)\mathbb{P}(C) = \mathbb{P}(B \cap C) = 1 - \mathbb{P}((B \cap C)^c) = 1 - \mathbb{P}(B^c \cup C^c)$).

2. We consider the electrical circuit set up as in the figure below, with 6 elements. We denote by E_i the event that the i^{th} element does not work ($1 \leq i \leq 6$), and we assume that the events $(E_i)_{1 \leq i \leq 6}$ are independent. We also know that $\mathbb{P}(E_i) = \frac{1}{3}$ for $i = 1, 2, 3$, $\mathbb{P}(E_i) = \frac{1}{2}$ for $i = 4, 5$, and $\mathbb{P}(E_6) = \frac{1}{4}$. Compute the probability of the event E that the circuit does not work.



We have

$$\mathbb{P}(F) = \mathbb{P}((F_1 \cup F_2) \cap (F_3 \cup (F_4 \cap F_5) \cup F_6)) = \mathbb{P}(F_1 \cup F_2)\mathbb{P}(F_3 \cup (F_4 \cap F_5) \cup F_6)$$

(using independence of the events for the second equality). We then compute separately the two factors in this product. On the one hand,

$$\mathbb{P}(F_1 \cup F_2) = \mathbb{P}(F_1) + \mathbb{P}(F_2) - \mathbb{P}(F_1 \cap F_2) = \frac{1}{3} + \frac{1}{3} - \frac{1}{9} = \frac{5}{9}.$$

On the other hand,

$$\begin{aligned}
\mathbb{P}(F_3 \cup (F_4 \cap F_5) \cup F_6) &= \mathbb{P}(F_3) + \mathbb{P}(F_4 \cap F_5) + \mathbb{P}(F_6) - \mathbb{P}(F_3 \cap F_4 \cap F_5) - \mathbb{P}(F_3 \cap F_6) \\
&\quad - \mathbb{P}(F_4 \cap F_5 \cap F_6) + \mathbb{P}(F_3 \cap F_4 \cap F_5 \cap F_6) \\
&= \frac{1}{3} + \frac{1}{4} + \frac{1}{4} - \frac{1}{12} - \frac{1}{12} - \frac{1}{16} + \frac{1}{48} = \frac{5}{8}.
\end{aligned}$$

Note that we could also compute the probability of the complement, which involves simpler calculations:

$$\begin{aligned}\mathbb{P}(F_3 \cup (F_4 \cap F_5) \cup F_6) &= 1 - \mathbb{P}((F_3 \cup (F_4 \cap F_5) \cup F_6)^c) \\ &= 1 - \mathbb{P}(F_3^c \cap (F_4 \cap F_5)^c \cap F_6^c) \\ &= 1 - \mathbb{P}(F_3^c) \mathbb{P}((F_4 \cap F_5)^c) \mathbb{P}(F_6^c),\end{aligned}$$

and we have $\mathbb{P}(F_3^c) = 1 - \mathbb{P}(F_3) = \frac{2}{3}$, $\mathbb{P}(F_6^c) = 1 - \mathbb{P}(F_6) = \frac{3}{4}$, and $\mathbb{P}((F_4 \cap F_5)^c) = 1 - \mathbb{P}(F_4 \cap F_5) = 1 - \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$, so we find again

$$\mathbb{P}(F_3 \cup (F_4 \cap F_5) \cup F_6) = 1 - \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{3}{4} = 1 - \frac{3}{8} = \frac{5}{8}.$$

Finally, we obtain

$$\mathbb{P}(F) = \mathbb{P}(F_1 \cup F_2) \mathbb{P}(F_3 \cup (F_4 \cap F_5) \cup F_6) = \frac{5}{9} \cdot \frac{5}{8} = \frac{25}{72}.$$

3. *We can perform a test to determine if a given electronic component has a defect, which occurs with a probability 0.001. If the component is defective, the test detects it correctly with probability 0.99, while if the component is not defective, the test wrongly detects a defect with probability 0.01.*

Let us denote by D the event that the component is defective, so that $\mathbb{P}(D) = 0.001$. We also denote by T the event that the test is positive, i.e. it says that the component has a defect. We know that $\mathbb{P}(T|D) = 0.99$, and $\mathbb{P}(T|D^c) = 0.01$.

- (i) *If the test says that a component is defective, find the probability that this component indeed has a defect.*

We want to determine $\mathbb{P}(D|T)$, which can be done by using Bayes' formula:

$$\begin{aligned}\mathbb{P}(D|T) &= \frac{\mathbb{P}(T|D)\mathbb{P}(D)}{\mathbb{P}(T|D)\mathbb{P}(D) + \mathbb{P}(T|D^c)\mathbb{P}(D^c)} \\ &= \frac{(0.99) \cdot (0.001)}{(0.99) \cdot (0.001) + (0.01) \cdot (0.999)} \\ &\simeq 0.09 \dots\end{aligned}$$

- (ii) *If the test says that a component is not defective, find the probability that this component indeed has no defect.*

We use again Bayes' formula:

$$\begin{aligned}\mathbb{P}(D^c|T^c) &= \frac{\mathbb{P}(T^c|D^c)\mathbb{P}(D^c)}{\mathbb{P}(T^c|D^c)\mathbb{P}(D^c) + \mathbb{P}(T^c|D)\mathbb{P}(D)} \\ &= \frac{(0.99) \cdot (0.999)}{(0.99) \cdot (0.999) + (0.01) \cdot (0.001)} \\ &\simeq 0.99 \dots\end{aligned}$$

4. Consider n independent random variables $X_1, \dots, X_n$, each having density function $f_{X_i}(x) = 3e^{-3x} \mathbb{1}_{[0, +\infty)}(x)$ ($1 \leq i \leq n$). Let $S_n = \sum_{i=1}^n X_i$.

(i) Compute $\mathbb{E}[S_n]$ and $\text{Var}(S_n)$.

For $\mathbb{E}[S_n]$, we use the linearity property of expectation:

$$\mathbb{E}[S_n] = \sum_{i=1}^n \mathbb{E}[X_i] = n\mathbb{E}[X_1] = \frac{n}{3}.$$

For $\text{Var}(S_n)$, we use the property that “the variance of the sum is the sum of the variances”, since the random variables $(X_i)_{1 \leq i \leq n}$ are assumed to be independent:

$$\text{Var}(S_n) = \sum_{i=1}^n \text{Var}(X_i) = n\text{Var}(X_1) = \frac{n}{9}.$$

Here, we used the formulas seen in class for the expectation and the variance of a random variable with exponential distribution (here, with parameter 3). Let us recall quickly how to obtain them.

- We can compute $\mathbb{E}[X_i]$ as follows, using the definition of the expectation for a continuous random variable, and then an integration by parts:

$$\begin{aligned} \mathbb{E}[X_i] &= \int_{-\infty}^{+\infty} x f_{X_i}(x) dx \\ &= \int_{-\infty}^{+\infty} x 3e^{-3x} \mathbb{1}_{[0, +\infty)}(x) dx \\ &= - \int_0^{+\infty} x d(e^{-3x}) \\ &= (-xe^{-3x}) \Big|_0^{+\infty} + \int_0^{+\infty} e^{-3x} dx \\ &= -\frac{1}{3} e^{-3x} \Big|_0^{+\infty} \\ &= \frac{1}{3}. \end{aligned}$$

- A similar computation (but now, with two integrations by parts) yields $\mathbb{E}[X_i^2] = \frac{2}{9}$, so $\text{Var}(X_i) = \mathbb{E}[X_i^2] - (\mathbb{E}[X_i])^2 = \frac{2}{9} - \left(\frac{1}{3}\right)^2 = \frac{1}{9}$.

(ii) Let $Y = \min(X_1, \dots, X_n)$: compute the cumulative distribution function of Y , and then its density. What is the distribution of Y ?

First, $F_Y(a) = 0$ and $f_Y(a) = 0$ for all $a < 0$. For $a \geq 0$, we have

$$\begin{aligned} F_Y(a) &= \mathbb{P}(Y \leq a) \\ &= 1 - \mathbb{P}(Y > a) \\ &= 1 - \mathbb{P}(\min(X_1, \dots, X_n) > a). \end{aligned}$$

We then notice that $\min(X_1, \dots, X_n) > a$ if and only if for all $i = 1, \dots, n$, $X_i > a$. Hence,

$$\begin{aligned}\mathbb{P}(\min(X_1, \dots, X_n) > a) &= \mathbb{P}\left(\bigcap_{i=1}^n \{X_i > a\}\right) \\ &= \prod_{i=1}^n \mathbb{P}(X_i > a) \\ &= (e^{-3a})^n = e^{-3na}.\end{aligned}$$

We thus have $F_Y(a) = 1 - e^{-3na}$, and we can recover the density by differentiating:

$$f_Y(a) = \frac{d}{da} F_Y(a) = \frac{d}{da} (1 - e^{-3na}) = 3ne^{-3na}.$$

Hence, $f_Y(a) = 3ne^{-3na} \mathbb{1}_{[0, +\infty)}(a)$, which is the density of the exponential distribution with parameter $3n$.

(iii) *Determine the cumulative distribution function of $Z = X_1 + X_2$, and then its density.*

Since Z is the sum of two independent r.v. with exponential distribution of parameter 3, its distribution is the $\gamma_{n=2, \lambda=3}$ distribution seen in class. We can compute its cumulative distribution function “by hand”: $\mathbb{P}(Z \leq a) = 0$ for $a \leq 0$, and for $a > 0$,

$$\begin{aligned}F_Z(a) &= \mathbb{P}(Z \leq a) = \mathbb{P}(X_1 + X_2 \leq a) \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{(X_1, X_2)}(x_1, x_2) \mathbb{1}_{x_1 + x_2 \leq a} dx_1 dx_2.\end{aligned}$$

The joint density function $f_{(X_1, X_2)}(x_1, x_2)$ is the product of the densities of X_1 and X_2 , since these two random variables are independent:

$$f_{(X_1, X_2)}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = (3e^{-3x_1} \mathbb{1}_{[0, +\infty)}(x_1)) \cdot (3e^{-3x_2} \mathbb{1}_{[0, +\infty)}(x_2)).$$

We deduce

$$\begin{aligned}F_Z(a) &= \int_{-\infty}^{+\infty} dx_1 3e^{-3x_1} \mathbb{1}_{[0, +\infty)}(x_1) \left(\int_{-\infty}^{+\infty} dx_2 3e^{-3x_2} \mathbb{1}_{[0, +\infty)}(x_2) \mathbb{1}_{x_1 + x_2 \leq a} \right) \\ &= \int_0^a dx_1 3e^{-3x_1} \left(\int_0^{a-x_1} dx_2 3e^{-3x_2} \right) \\ &= \int_0^a dx_1 3e^{-3x_1} (1 - e^{-3(a-x_1)}) \\ &= \int_0^a dx_1 3e^{-3x_1} - \int_0^a dx_1 3e^{-3a} \\ &= 1 - e^{-3a} - 3ae^{-3a}.\end{aligned}$$

We can then recover the density f_Z by differentiating: $f_Z(a) = 0$ for $a \leq 0$, and for $a > 0$,

$$f_Z(a) = \frac{d}{da} F_Z(a) = \frac{d}{da} (1 - e^{-3a} - 3ae^{-3a}) = 9ae^{-3a}.$$

Note that this is indeed the density (seen in class) of a r.v. with the $\gamma_{n=2,\lambda=3}$ distribution:

$$f_Z(a) = \lambda e^{-\lambda a} \frac{(\lambda a)^{n-1}}{(n-1)!} \mathbb{1}_{[0,+\infty)}(a) \quad \text{with } n = 2 \text{ and } \lambda = 3.$$

5. Let X be a continuous random variable, with density $f_X(x)$. We denote by F_X its cumulative distribution function.

- (i) Let $Y_1 = 3X + 2$. Express the cumulative distribution function of Y_1 in terms of F_X . What is the density of Y_1 , in terms of f_X (hint: the density of a r.v. can be recovered as the derivative of its cumulative distribution function)?

For the cumulative distribution function F_{Y_1} of Y_1 , we have: for all $y \in \mathbb{R}$,

$$\begin{aligned} F_{Y_1}(y) &= \mathbb{P}(Y_1 \leq y) = \mathbb{P}(3X + 2 \leq y) \\ &= \mathbb{P}\left(X \leq \frac{y}{3} - \frac{2}{3}\right) = F_X\left(\frac{y}{3} - \frac{2}{3}\right). \end{aligned}$$

As we explained in class, the density f_{Y_1} of Y_1 can then be recovered as the derivative of $F_{Y_1}(y)$ with respect to y :

$$f_{Y_1}(y) = \frac{d}{dy} F_{Y_1}(y) = \frac{d}{dy} F_X\left(\frac{y}{3} - \frac{2}{3}\right) = \frac{1}{3} f_X\left(\frac{y}{3} - \frac{2}{3}\right).$$

- (ii) Let $Y_2 = 1 - X$. Find its cumulative distribution function and its density, in terms of F_X and f_X .

This is very similar to the previous question:

$$\begin{aligned} F_{Y_2}(y) &= \mathbb{P}(Y_2 \leq y) = \mathbb{P}(1 - X \leq y) \\ &= \mathbb{P}(X \geq 1 - y) = 1 - \mathbb{P}(X < 1 - y) = 1 - F_X(1 - y) \end{aligned}$$

(we used the fact that X is continuous: where?). Hence,

$$f_{Y_2}(y) = \frac{d}{dy} F_{Y_2}(y) = \frac{d}{dy} (1 - F_X(1 - y)) = f_X(1 - y).$$

- (iii) Let $Y_3 = X^2$. Find its cumulative distribution function and its density, in terms of F_X and f_X .

We have $F_{Y_3}(y) = 0$ and $f_{Y_3}(y) = 0$ for $y \leq 0$. For $y > 0$,

$$\begin{aligned} F_{Y_3}(y) &= \mathbb{P}(Y_3 \leq y) = \mathbb{P}(X^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\ &= \mathbb{P}(X \leq \sqrt{y}) - \mathbb{P}(X \leq -\sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y}) \end{aligned}$$

(where did we use the fact that the random variable is continuous?). Hence,

$$f_{Y_3}(y) = \frac{d}{dy} F_{Y_3}(y) = \frac{d}{dy} (F_X(\sqrt{y}) - F_X(-\sqrt{y})) = \frac{1}{2\sqrt{y}} (f_X(\sqrt{y}) + f_X(-\sqrt{y})).$$

- (iv) Assume now that X has standard normal distribution $\mathcal{N}(0, 1)$: what is the density of X^2 ?

We use the result from the previous question with $f_X(y) = \frac{1}{\sqrt{2\pi}}e^{-y^2/2}$ (the density of an $\mathcal{N}(0, 1)$ -distributed r.v.): $f_{X^2}(y) = 0$ for $y \leq 0$, and for $y > 0$,

$$f_{X^2}(y) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\sqrt{y}}(f_X(\sqrt{y}) + f_X(-\sqrt{y})) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{2\sqrt{y}}(2e^{-y/2}) = \frac{1}{\sqrt{2\pi}}y^{-1/2}e^{-y/2}$$

(note that this is a classical distribution: it is known as the $\gamma_{\frac{1}{2}, \frac{1}{2}}$ distribution).

6. Let X and Y be two jointly continuous random variables, with joint density function

$$f_{(X,Y)}(x, y) = c\mathbb{1}_{[0,2]}(x)\mathbb{1}_{[0,1]}(y)$$

for some constant $c > 0$.

- (i) Determine the value of c , and then the marginal density functions $f_X(x)$ and $f_Y(y)$ of, respectively, X and Y (hint: $f_X(x)$ can be obtained by keeping x fixed, and integrating over all values of y , and similarly for $f_Y(y)$).

First, the joint density $f_{(X,Y)}$ should satisfy the normalizing condition

$$1 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{(X,Y)}(x, y) dx dy = \int_0^2 \left(\int_0^1 c dy \right) dx = 2c,$$

so we must have $c = \frac{1}{2}$.

We then obtain the marginal density functions, as explained in the hint:

$$f_X(x) = \int_{-\infty}^{+\infty} f_{X,Y}(x, y) dy = \frac{1}{2}\mathbb{1}_{[0,2]}(x),$$

and similarly, $f_Y(y) = \mathbb{1}_{[0,1]}(y)$.

- (ii) Compute $\mathbb{P}(X \geq Y)$.

We have

$$\begin{aligned} \mathbb{P}(X \geq Y) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \mathbb{1}_{x \geq y} dx dy \\ &= \frac{1}{2} \int_0^1 \left(\int_y^2 dx \right) dy \\ &= \frac{1}{2} \int_0^1 (2 - y) dy \\ &= \frac{1}{2} \left(2 - \frac{1}{2} \right) = \frac{3}{4}. \end{aligned}$$

(iii) Let $Z = X + Y$, and $a \in [0, 1]$. Compute $\mathbb{P}(Z \leq a)$.

We have

$$\begin{aligned}\mathbb{P}(Z \leq a) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(x, y) \mathbb{1}_{x+y \leq a} dx dy \\ &= \frac{1}{2} \int_0^a \left(\int_0^{a-y} dx \right) dy \\ &= \frac{1}{2} \int_0^a (a - y) dy \\ &= \frac{1}{2} \left(a^2 - \frac{1}{2} a^2 \right) = \frac{1}{4} a^2\end{aligned}$$

(this can be seen on a picture, since (X, Y) is uniform on the rectangle $[0, 2] \times [0, 1]$).

7. In a factory, the failure times of machines can be modeled by a Poisson process $(N_t)_{t \geq 0}$ with some rate $\lambda > 0$. On average, there are three failures per unit of time.

(i) Find the value of λ .

We know that the expected number of failures per unit of time is $\mathbb{E}[N_{t+1} - N_t] = \lambda$ ($N_{t+1} - N_t$ having Poisson distribution with parameter λ). Hence, $\lambda = 3$.

(ii) Evaluate the following probabilities:

(a) the probability that the first failure occurs after time 2,

This is

$$\mathbb{P}(N_2 = 0) = e^{-2\lambda} = e^{-6}$$

(using $\lambda = 3$).

(b) the probability that the third failure occurs before time 4,

This is

$$\mathbb{P}(N_4 \geq 3) = 1 - \mathbb{P}(N_4 \leq 2),$$

where

$$\begin{aligned}\mathbb{P}(N_4 \leq 2) &= \mathbb{P}(N_4 = 0) + \mathbb{P}(N_4 = 1) + \mathbb{P}(N_4 = 2) \\ &= e^{-12} + \frac{12}{1} e^{-12} + \frac{12^2}{2} e^{-12} = 85e^{-12}.\end{aligned}$$

Hence,

$$\mathbb{P}(N_4 \geq 3) = 1 - 85e^{-12}.$$

(c) the probability that two failures occur in $[2, 6]$, no failure in $[4, 10]$, and at least one failure in $[8, 16]$.

This is

$$\begin{aligned}\mathbb{P}(N_4 - N_2 = 2, N_{10} - N_4 = 0, N_{16} - N_{10} \geq 1) \\ &= \mathbb{P}(N_4 - N_2 = 2) \cdot \mathbb{P}(N_{10} - N_4 = 0) \cdot \mathbb{P}(N_{16} - N_{10} \geq 1) \\ &= \left(\frac{6^2}{2} e^{-6} \right) \cdot e^{-18} \cdot (1 - e^{-18}),\end{aligned}$$

using the fact that the increments are independent, and then that they are Poisson distributed.

(iii) For $t \geq 0$, compute $\text{Cov}(N_t, N_{t+4})$.

We make appear independent increments: since $N_{t+4} = (N_{t+4} - N_t) + N_t$,

$$\text{Cov}(N_t, N_{t+4}) = \text{Cov}(N_t, N_{t+4} - N_t) + \text{Cov}(N_t, N_t) = \text{Var}(N_t)$$

($\text{Cov}(N_t, N_{t+4} - N_t) = 0$, since these two random variables are independent). Since N_t has Poisson distribution with parameter $3t$,

$$\text{Cov}(N_t, N_{t+4}) = 3t.$$

(iv) Let $n \geq 1$: what is the expectation $\mathbb{E}[S_n]$ of the n^{th} failure time S_n ?

We can write

$$S_n = \sum_{i=1}^n (S_i - S_{i-1}),$$

and we know that each interarrival time $S_i - S_{i-1}$, $1 \leq i \leq n$, has exponential distribution with parameter $\lambda = 3$.

We have

$$\mathbb{E}[S_n] = \mathbb{E}\left[\sum_{i=1}^n (S_i - S_{i-1})\right] = \sum_{i=1}^n \mathbb{E}[S_i - S_{i-1}] = n \cdot \frac{1}{3}$$

(since each $S_i - S_{i-1}$ has expectation $\frac{1}{\lambda} = \frac{1}{3}$).