

EE3109 Applied Electromagnetics
2022/23 Semester B
Test 2

Name: _____ Solution _____

Student #: _____

Date: _____

1. **(21 Marks)** A plane wave impinges from air onto a medium with $\{\epsilon_r, \mu_r, \sigma\} = \{1, 0.25, 0\}$. The air-medium boundary is at $y = 0$ (i.e. on the xz -plane), with air on the side of $y > 0$.

- (a) Is there a critical angle? If so, what is this angle? (3)
 (b) Is there a Brewster angle (i.e. an angle where there is no reflection), for either the TE wave or the TM wave? If so, what is this angle and for which polarization (TE or TM) does this angle exist? (3)

Now consider the following incident electric field: $\mathbf{E}^i = E_0 \exp(j20\pi(x + 3y)) \hat{\mathbf{z}}$

- (c) What is the polarization of this wave? (2)
 (d) Find the wave vectors of the reflected and transmitted waves. (4)
 (e) Find the transmission coefficient. (3)
 (f) Write the transmitted electric field. (3)
 (g) If the power intensity of the incident plane wave is 1 W/m^2 , what is the amplitude of the transmitted electric field? (3)

Solution:

Note that $\{\epsilon_{r1}, \mu_{r1}, \sigma_1\} = \{1, 1, 0\}$, $\{\epsilon_{r2}, \mu_{r2}, \sigma_2\} = \{1, 0.25, 0\}$, so $\eta_1 = 120\pi$; $n_1 = 1$; $\eta_2 = 60\pi$; $n_2 = 0.5 \Rightarrow \eta_1 = 2\eta_2$, $n_1 = 2n_2$. These relations will be useful for the following question parts.

- (a) There is a critical angle since $n_1 > n_2$.

$$n_1 \sin \theta_c = n_2 \Rightarrow \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}(0.5) = 30^\circ$$

- (b) Note that $\eta_1 > \eta_2$, and $n_1 > n_2 \Rightarrow \sin \theta_1 < \sin \theta_2 \Rightarrow \cos \theta_1 > \cos \theta_2$. Therefore Γ_{TE} which has the numerator of $\eta_1 \cos \theta_2 = \eta_2 \cos \theta_1$ can be zero. Hence there is a Brewster angle for the TE polarization.

Let the input angle be the Brewster angle:

From Snell's Law:

$$n_2 \sin \theta_2 = n_1 \sin \theta_B \Rightarrow \sin^2 \theta_2 = \left(\frac{n_1}{n_2}\right)^2 \sin^2 \theta_B = 4 \sin^2 \theta_B.$$

From the reflection coefficient equation, $\Gamma_{TE} = 0$ when

$$\eta_1 \cos \theta_2 = \eta_2 \cos \theta_B \Rightarrow \cos^2 \theta_2 = \left(\frac{\eta_1}{\eta_2}\right)^2 \cos^2 \theta_B = \frac{1}{4} \cos^2 \theta_B.$$

Add the above equations to get

$$\begin{aligned} 1 &= \frac{1}{4} \cos^2 \theta_B + 4 \sin^2 \theta_B \\ 1 &= \frac{1}{4} + \left(4 - \frac{1}{4}\right) \sin^2 \theta_B \\ \sin^2 \theta_B &= \frac{\left(1 - \frac{1}{4}\right)}{\left(4 - \frac{1}{4}\right)} = \frac{1}{5} = 0.2 \\ \Rightarrow \theta_B &= \sin^{-1}(\sqrt{0.2}) = 26.57^\circ \end{aligned}$$

- (c) $\mathbf{E}^i$ does not have a component in the direction of the interface normal (y-direction).
 $\Rightarrow$ The wave is TE-polarized.

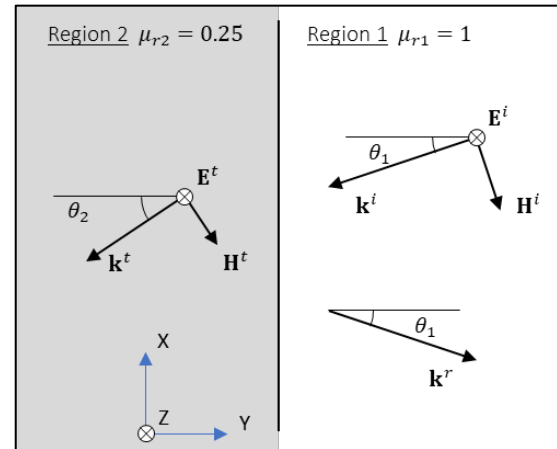
- (d) A diagram of the plane of incidence is as shown ($\mathbf{E}^r$ and $\mathbf{H}^r$ are not drawn b/c they are not needed in this question).

From the phase of the plane wave expression in the question, we see that

$$\mathbf{k}^i = 20\pi(-1, -3, 0).$$

The reflected wave will have the wavevector direction reversed along the direction of the surface normal, so

$$\mathbf{k}^r = 20\pi(-1, 3, 0).$$



The transmitted wave will stay on the plane of incidence, so $k_z^t = 0$. It has the same tangential component as the incident wave, so $k_x^t = k_x^i = -20\pi$. The longitudinal component should be such that the transmitted wave is at the correct angle:

$$\begin{aligned} n_2 \sin \theta_2 &= n_1 \sin \theta_1 \\ \Rightarrow \sin \theta_2 &= \frac{n_1}{n_2} \sin \theta_1 = \frac{n_1}{n_2} \left(\frac{k_x^i}{\sqrt{k_x^{i2} + k_y^{i2}}} \right) = (2) \left(\frac{1}{\sqrt{1^2 + 3^2}} \right) = \frac{2}{\sqrt{10}} \\ \theta_2 &= \sin^{-1} \frac{2}{\sqrt{10}} = 39.23^\circ \end{aligned}$$

Now from the trigonometry of the question

$$\frac{k_y^t}{k_x^t} = \frac{1}{\tan \theta_2} \Rightarrow k_y^t = \frac{k_x^t}{\tan \theta_2} = \frac{(20\pi)(-1)}{\tan 39.23^\circ} = (20\pi)(-1.225).$$

So putting all three components together, $\mathbf{k}^t = (k_x^t, k_y^t, k_z^t) = 20\pi(-1, -1.225, 0)$.

- (e) The transmission coefficient for the TE polarization is

$$\tau_{TE} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}.$$

Now we proceed to calculate the parts needed:

$$\theta_i = \tan^{-1} \left(\frac{k_x^i}{k_y^i} \right) = \tan^{-1} \left(\frac{-1}{-3} \right) = 18.43^\circ; \cos \theta_i = 0.9487;$$

$$\theta_t = 39.23^\circ \text{ (from part (d)); } \cos \theta_t = 0.7746;$$

$$\text{From before: } \eta_1 = \eta_0; \eta_2 = 0.5\eta_0$$

Substitution into τ_{TE} gives:

$$\tau_{TE} = \frac{2(0.5\eta_0)(0.9487)}{(0.5\eta_0)(0.9487) + \eta_0(0.7746)} = 1.4202$$

(f) The transmitted electric field is

$$\mathbf{E}^t = \tau_{TE} E_0 \exp(-j\mathbf{k}^t \cdot \mathbf{r}) \hat{\mathbf{z}} = 1.4202 E_0 \exp(-j(x + 1.225y)) \hat{\mathbf{z}}$$

(g) We find the incident field magnitude from the incident power density as follows:

$$S_{av} = \frac{|E_0|^2}{2\eta_0} \Rightarrow |E_0|^2 = \sqrt{2\eta_0 S_{av}} = \sqrt{(2)(120\pi)(1)} = 27.46 \text{ [V/m]}.$$

From this we find the transmitted electric field magnitude to be

$$|\mathbf{E}^t| = |\tau_{TE} E_0| = (1.4202)(27.46) = 39.00 \text{ [V/m]}.$$

2. **(16 Marks)** A transmission line with $R' = 0 \Omega/\text{m}$; $L' = 337.5 \text{ nH/m}$; $G' = 0 \text{ S/m}$; $C' = 60 \text{ pF/m}$ is used to measure the unknown load. After connection and measurement, it is found that the S11 de-embedded to the load (that is to say, the Γ_L) is $0.3 \angle 30^\circ$. For the frequency of 2.4 GHz ,
- Find the phase velocity, phase constant and attenuation constant of the wave along the line (3)
 - Find the characteristic impedance of the line. (2)
 - Find the voltage standing wave ratio. (2)
 - Find the input impedance at the beginning of the transmission line, if the line is 0.3λ long. (3)
 - Write an expression for the voltage along the line. Leave the amplitude V_0^+ as an unknown constant. (3)
 - If a power of 100 mW is transferred to the load, find $|V_0^+|$. (3)

Solution:

(a)

$$v_p = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{(337.5 \times 10^{-9})(60 \times 10^{-12})}} = 2.222 \times 10^8 \text{ [m/s]},$$

$$\beta = \omega\sqrt{L'C'} = (2\pi \times 2.4 \times 10^9)\sqrt{(337.5 \times 10^{-9})(60 \times 10^{-12})} = 67.86 \text{ [rad/m]},$$

$$\alpha = 0 \text{ [Np/m]} \quad (R' = 0 \text{ and } G' = 0 \Rightarrow \text{lossless TL}).$$

(b)

$$Z_0 = \sqrt{\frac{L'}{C'}} = \sqrt{\frac{(337.5 \times 10^{-9})}{(60 \times 10^{-12})}} = 75.00 \text{ } [\Omega].$$

(c)

$$S = \frac{1 + |\Gamma_L|}{1 - |\Gamma_L|} = \frac{1.3}{0.7} = 1.857.$$

(d) Note that $2\beta l = 2\left(\frac{2\pi}{\lambda}\right)(0.3\lambda) = 1.2\pi = 216^\circ$.

$$\Gamma_{in} = \Gamma_L e^{-j2\beta l} = 0.3 \angle (30^\circ - 216^\circ) = 0.3 \angle -186^\circ = -0.2984 + j0.0314.$$

$$Z_{in} = Z_0 \frac{1 + \Gamma_{in}}{1 - \Gamma_{in}} = (75) \frac{1 + (-0.2984 + j0.0314)}{1 - (-0.2984 + j0.0314)} = 40.46 + j2.79 \text{ } \Omega.$$

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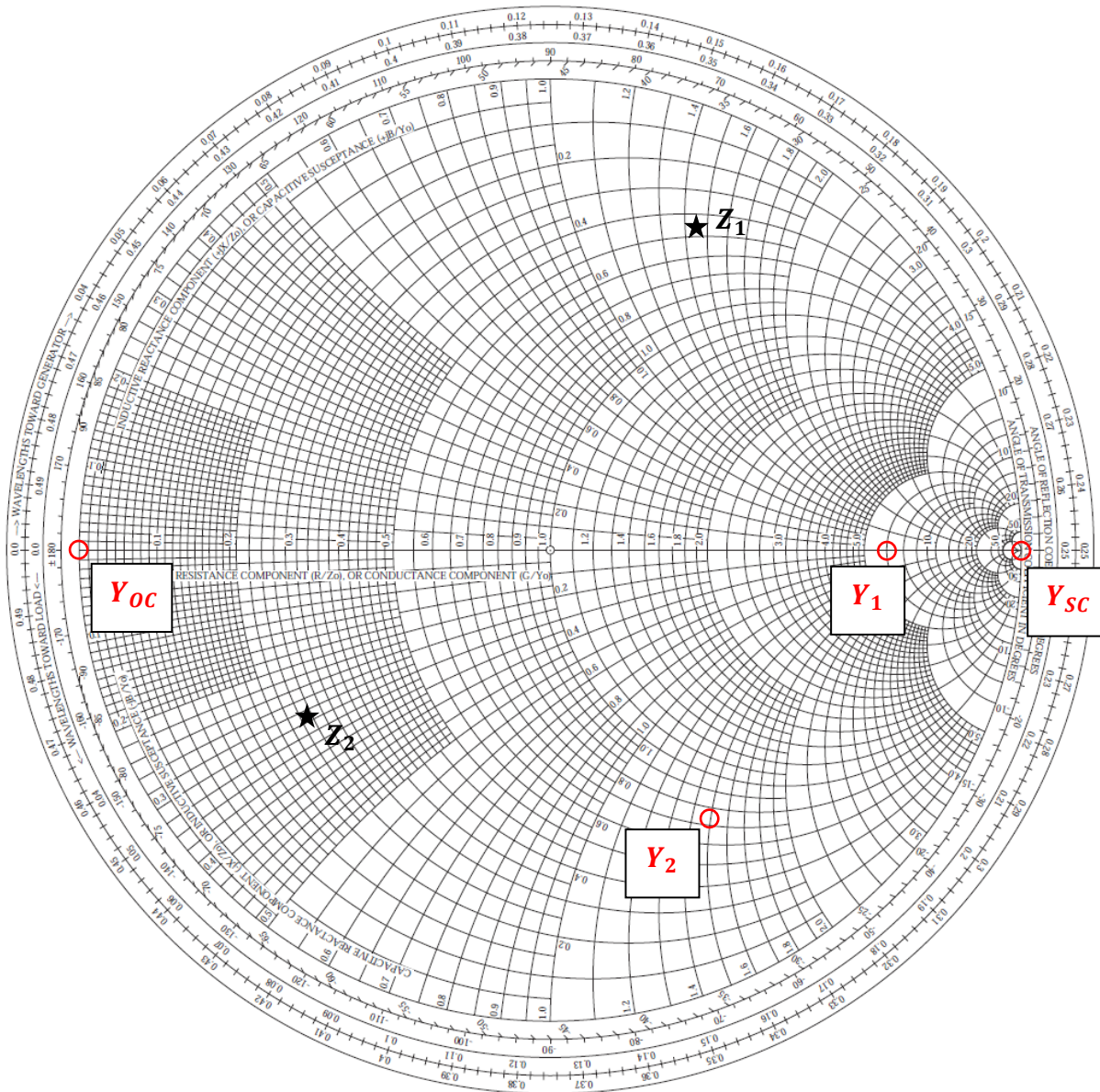
(e)

$$V(z) = V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{+j\beta z}) = V_0^+ e^{-j67.86z} \left(1 + 0.3e^{\frac{j\pi}{6}} e^{+j135.7z}\right).$$

(f)

$$P_{disp} = \frac{|V_0^+|^2}{2Z_0} (1 - |\Gamma_L|^2) \Rightarrow |V_0^+| = \sqrt{\frac{2Z_0 P_{disp}}{1 - |\Gamma_L|^2}} = \sqrt{\frac{2(75)(0.1)}{1 - (0.3)^2}} = 4.060 \text{ [V]}.$$

3. (7 Marks) In the Smith Chart below,
- Label the following points: Y_{OC} , Y_{SC} , $Y_1 = 0.12 \text{ S}$ and $Y_2 = 0.015 - j0.030 \text{ S}$. For this part of the question, consider the Smith Chart in admittance mode with $Y_0 = 20 \text{ mS}$. (3)
 - Read the impedances Z_1 and Z_2 which are labelled on the plot. $Z_0 = 50 \Omega$. (2)
 - Classify each of $\{Y_1, Y_2, Z_1, Z_2\}$ as capacitive, inductive or neither. (2)



Solution:

- The admittances are labelled in red in the plot above.
- $$z'_1 = 0.55 + j1.45 \Rightarrow Z_1 = z'_1 Z_0 = 27.5 + j72.5 [\Omega]$$

$$z'_2 = 0.24 - j0.32 \Rightarrow Z_2 = z'_2 Z_0 = 12 - j16 [\Omega]$$
- For impedances, the top half of the S.C. is inductive; for admittances, the bottom half of the S.C. is inductive. Therefore, Y_1 is neither, Y_2 is inductive, Z_1 is inductive, Z_2 is capacitive.