

EE 3114 **Systems and Control**

2. Mathematical Modeling

Semester B, 2017-18

Mathematical Representation of a System

To control a system, a mathematical representation of the system is needed, which can be:

- 1) a functional relationship between input and output of the system
 - a map
- 2) a state-space description of the system
 - an ordinary differential equation

1) Input-Output Representation

The mathematical representation of a system can be described by a functional relationship between **input** and **output**:



Relationship between **y** and **u** is described by an **operator** (a map) **G**:

$$\mathbf{y} = \mathbf{G}[\mathbf{u}]$$

Transfer Function

- Laplace Transforms:



Transfer Function :

$$G(s) = \frac{\text{Laplace Transform of Output}}{\text{Laplace Transform of Input}} = \frac{Y(s)}{U(s)}$$

(with zero initial conditions)

Review of Laplace Transform

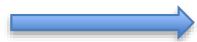
$$a\ddot{y} + b\dot{y} + cy = du$$

→ Laplace transforms: $Y = L\{y\}, \quad U = L\{u\}$

$$as^2Y + bsY + cY = dU$$

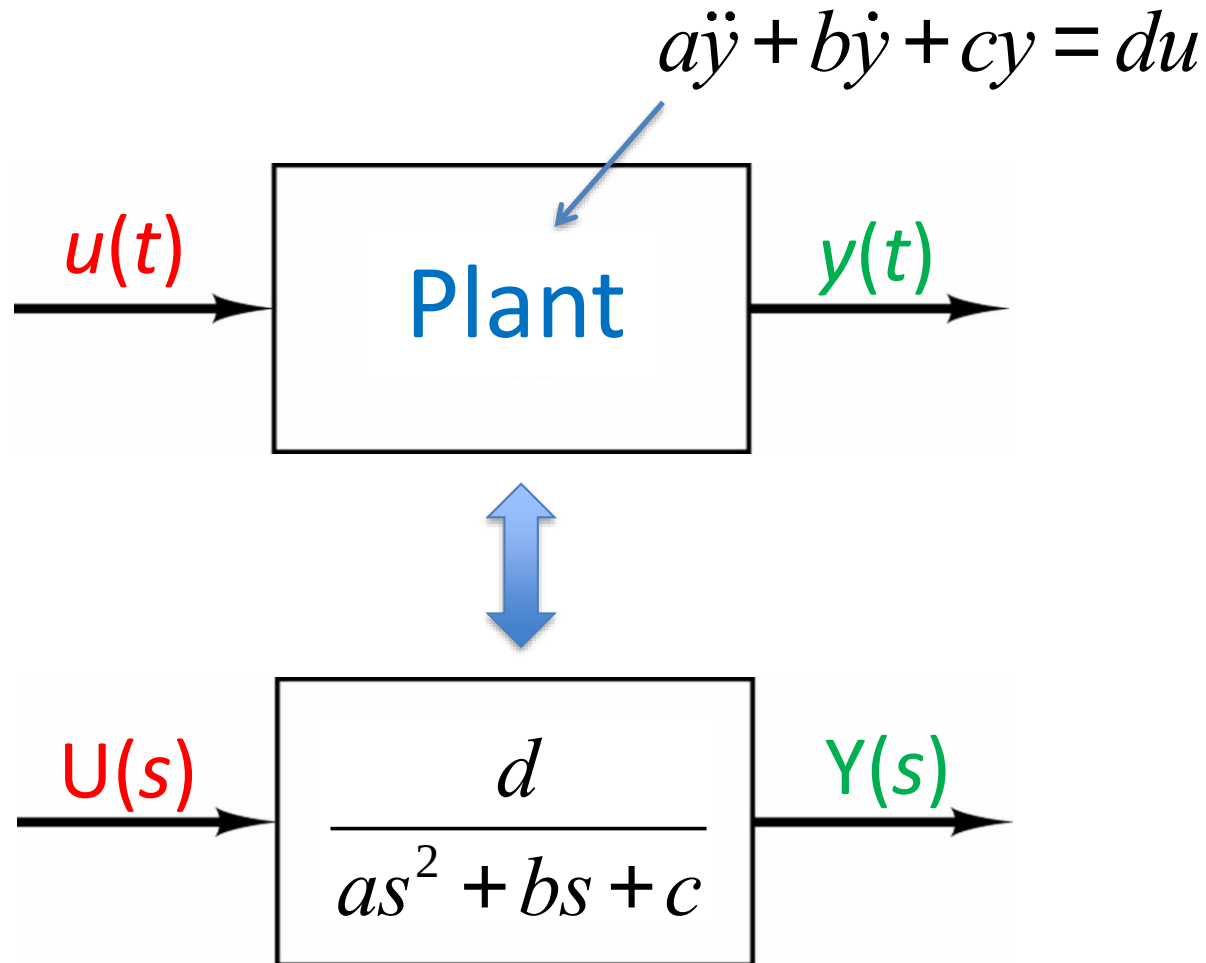


$$(as^2 + bs + c)Y = dU$$



$$Y = \frac{d}{as^2 + bs + c}U \quad \text{i.e., } Y = GU$$

Laplace Transform converts **differential equation** description to **transfer function** description:



In-class Exercise

Given: ODE

$$a\ddot{y} - by = c\dot{u} + du$$

Find: Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = ?$$

5 minutes

In-class Exercise

Given: ODE

$$a\ddot{y} - by = c\dot{u} + du$$

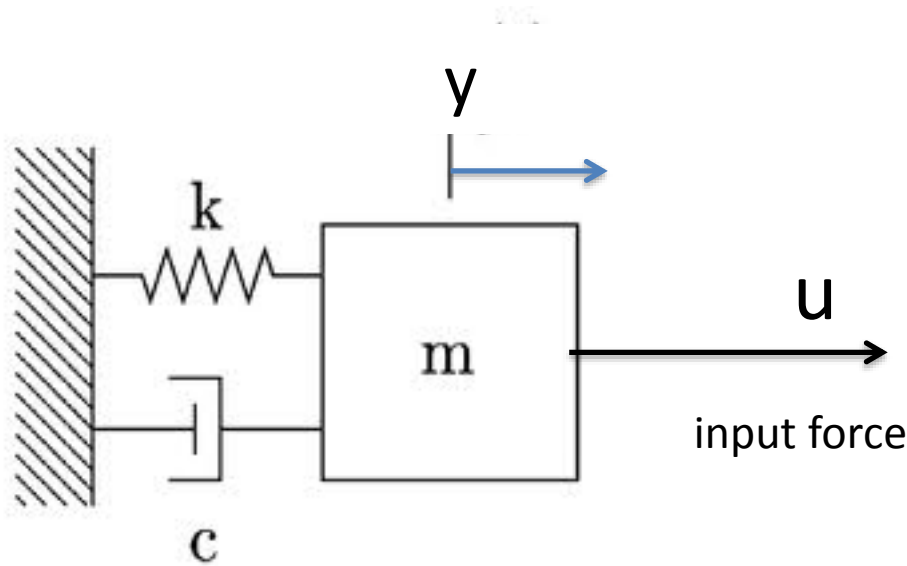
Find: Transfer Function

$$G(s) = \frac{Y(s)}{U(s)} = ?$$

Answer:

$$G(s) = \frac{cs + d}{as^3 - b}$$

Example: Mass-Spring-Damper System



$$m\ddot{y} + c\dot{y} + ky = u$$

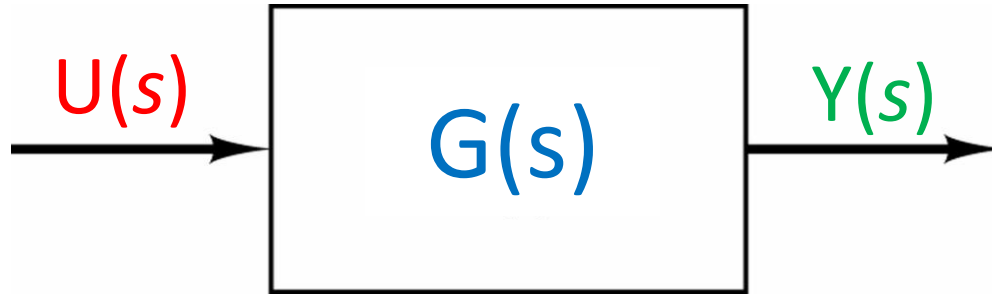
Taking Laplace Transform (with zero initial conditions)

$$(ms^2 + cs + k)Y(s) = U(s)$$

yields the transfer function

$$G(s) = \frac{1}{ms^2 + cs + k}$$

Input-output relationship of the system:



$$Y(s) = G(s)U(s)$$

Hence,

$$y(t) = L^{-1}\{G(s)U(s)\}$$

(Output obtained by **Inverse Laplace Transform**)

Reduction Rules For Complicated Systems

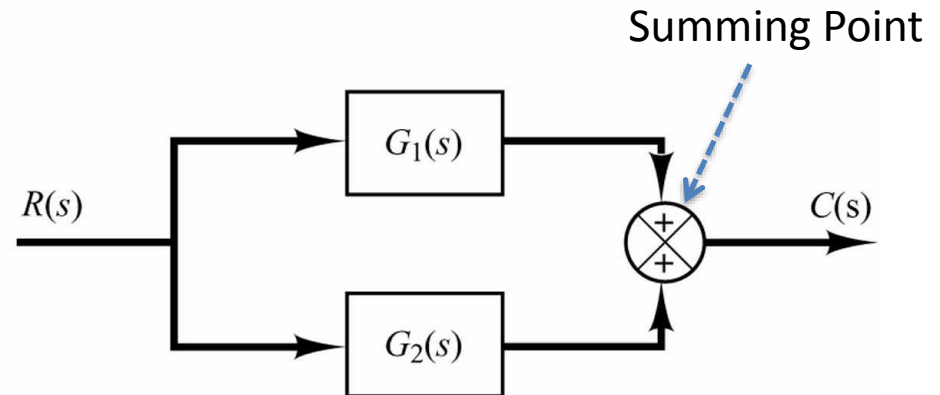
(1) Cascaded system

$$C/R = G_1 G_2$$



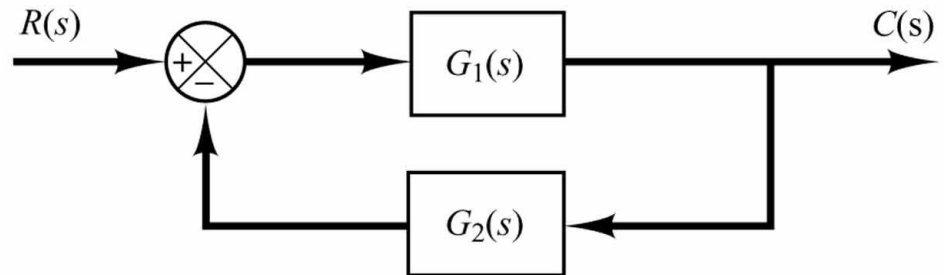
(2) Parallel system

$$C/R = G_1 + G_2$$

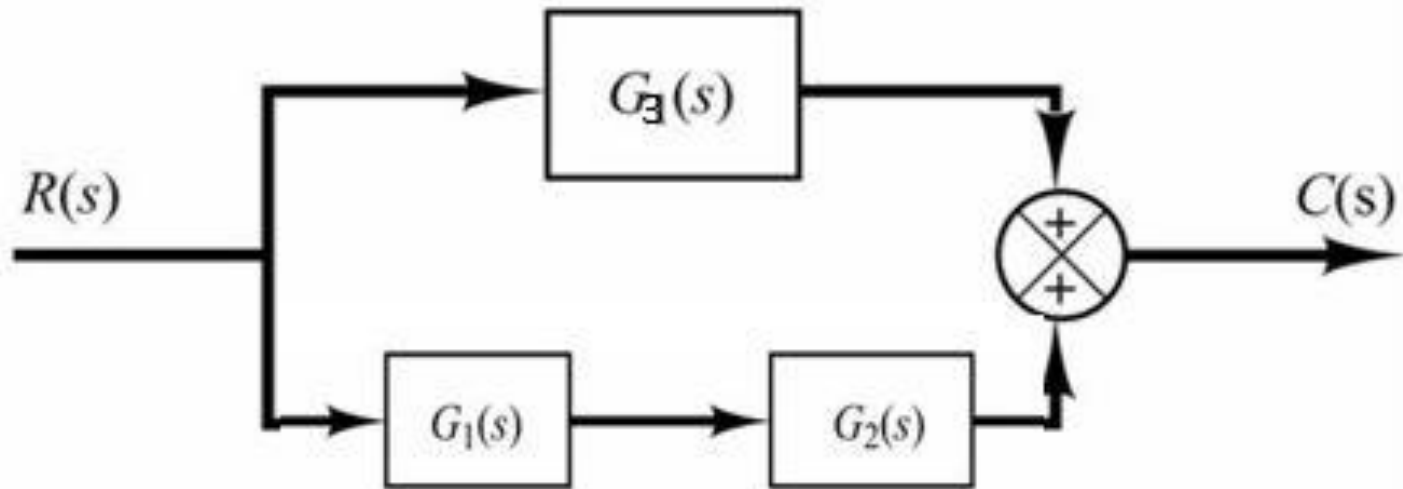


(3) Feedback (closed-loop) system

$$\frac{C(s)}{R(s)} = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$



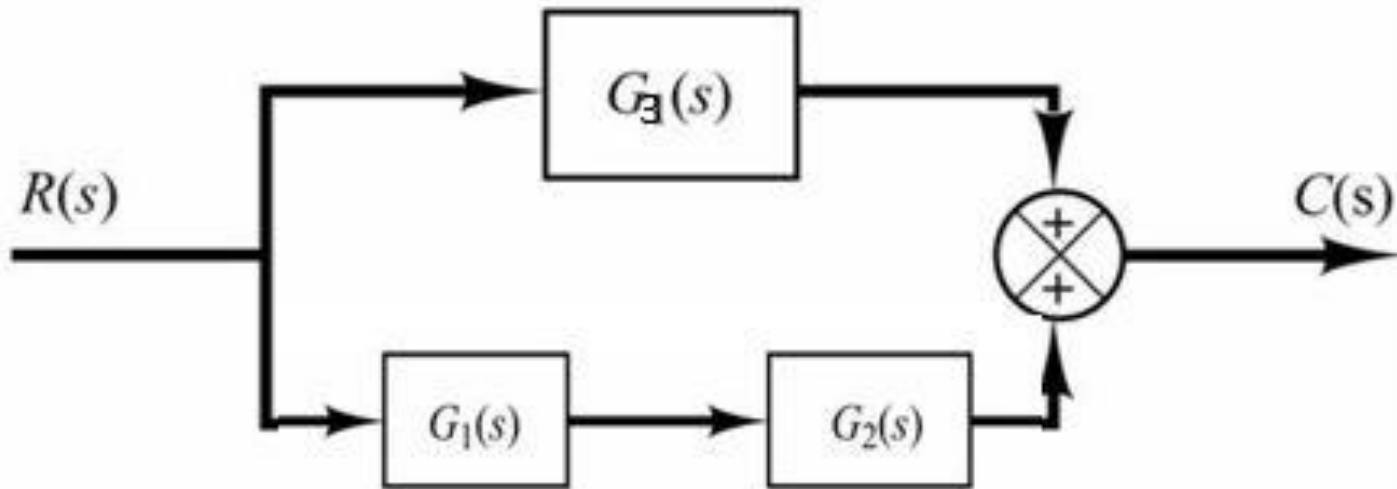
In-class Exercise



$$\frac{C(s)}{R(s)} = ?$$

5 minutes

In-class Exercise



$$\frac{C(s)}{R(s)} = ?$$

Answer:

$$\frac{C(s)}{R(s)} = G_1(s) \cdot G_2(s) + G_3(s)$$

2) State-Space Representation

Dynamical Relationship

The concept of **state** is to introduce a set of **internal variables** that define completely the dynamical behavior of the system

For example, the **states** of a moving object include:

Position

Velocity

Acceleration

In general, state equations can be written as

State equations

$$\begin{aligned}\dot{x}_1(t) &= f_1(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t)) \\ \dot{x}_2(t) &= f_2(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t)) \\ &\vdots \\ \dot{x}_m(t) &= f_m(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t))\end{aligned}$$

Output equations

$$\begin{aligned}y_1(t) &= g_1(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t)) \\ &\vdots \\ y_k(t) &= g_k(t, x_1(t), \dots, x_m(t), u_1(t), \dots, u_n(t))\end{aligned}$$

Define the **state**, **input** and **output** vectors:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_m(t) \end{bmatrix}$$

For example:

Position

Velocity

Acceleration

$$\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$$

For example:

Current

Force

Torque

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_k(t) \end{bmatrix}$$

For example:

Displacement

Angle

Voltage

And, the following vector functions:

$$f(t, \mathbf{x}(t), \mathbf{u}(t)) = \begin{bmatrix} f_1(t, \mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ f_m(t, \mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix}$$

$$g(t, \mathbf{x}(t), \mathbf{u}(t)) = \begin{bmatrix} g_1(t, \mathbf{x}(t), \mathbf{u}(t)) \\ \vdots \\ g_k(t, \mathbf{x}(t), \mathbf{u}(t)) \end{bmatrix}$$

Then, one can rewrite the above OEDs as

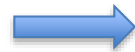
State equation: $\dot{\mathbf{x}}(t) = f(t, \mathbf{x}(t), \mathbf{u}(t))$

Output equation: $\mathbf{y}(t) = g(t, \mathbf{x}(t), \mathbf{u}(t))$

where $\mathbf{x} \in \mathbb{R}^m$ is the state vector
 $\mathbf{u} \in \mathbb{R}^n$ is the control vector
 $\mathbf{y} \in \mathbb{R}^k$ is the output vector

Example: Mass-Spring-Damper System

$$m\ddot{y} + c\dot{y} + ky = u$$


$$\ddot{y} = -\frac{c}{m}\dot{y} - \frac{k}{m}y + \frac{1}{m}u$$

Assume that the output (measurement) is the velocity

Define the state $x_1(t) = y(t)$ Displacement

$x_2(t) = \dot{y}(t)$ Velocity

System dynamics:
$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{c}{m}x_2 - \frac{k}{m}x_1 + \frac{1}{m}u \end{cases}$$

Output relationship: $y = x_1$

Formal Definition of “State”

Definition: A **state** of a dynamic system is a set of variables which, if specified at the time $t = t_0$, provide all the necessary information to find the behavior of the output at any time $t > t_0$, given the input to the system at interval $[t_0, t)$

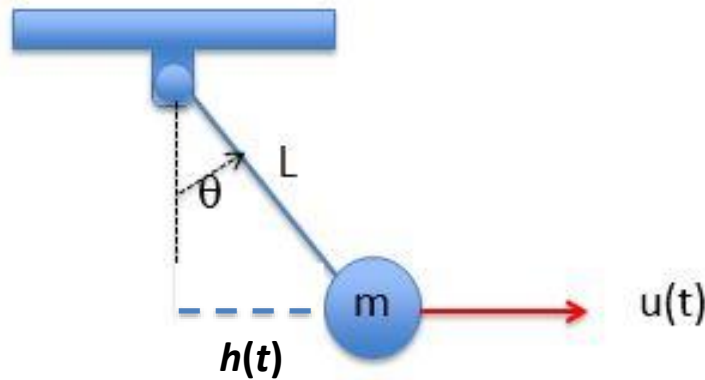
Note:

The input-output relationship is **unique**

But a definition of the state is **not unique**

Example: Pendulum

Let the force input be $u(t)$ and the horizontal displacement $h(t)$ of the mass m is the output (measurement)



Physical relationship between input and output:

$$mL^2\ddot{\theta} = -mgL \sin \theta + u(t)L \cos \theta$$

Secondary School Physics

$$mL^2\ddot{\theta} = -mgL \sin \theta + u(t)L \cos \theta$$

Representation 1

Define:

$$\begin{aligned} x_1 &= \theta \\ x_2 &= \dot{\theta} \end{aligned}$$

System dynamics

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{L} \sin x_1 + \frac{1}{mL} u \cos x_1 \end{cases}$$

Output relationship

$$y = L \sin x_1 = h$$

Representation 2

Define:

$$\begin{aligned} x_1 &= \theta \\ x_2 &= mL^2 \dot{\theta} \end{aligned}$$

System dynamics

$$\begin{cases} \dot{x}_1 = \frac{1}{mL^2} x_2 \\ \dot{x}_2 = -mgL \sin x_1 + Lu \cos x_1 \end{cases}$$

Output relationship

$$y = L \sin x_1 = h$$

Not unique

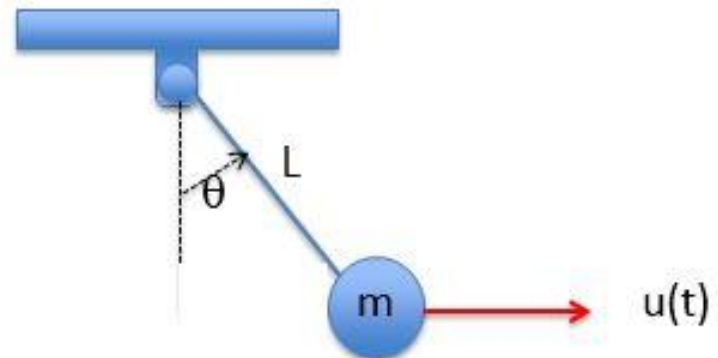
Pendulum: Representation 1

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}) \quad x_1 = \theta, x_2 = \dot{\theta}$$

$$\mathbf{y} = g(\mathbf{x})$$

$$f = \begin{bmatrix} x_2(t) \\ -\frac{g}{L} \sin x_1(t) + \frac{1}{mL} \cos x_1 u(t) \end{bmatrix}$$

$$g = L \sin x_1(t)$$



Note:

One shall consider systems that are described by a **finite number** of states

The **order** of the system is the **minimal number** of states required to describe its dynamics

This number can be found by determining the number of initial conditions required

Linear Time-Invariant (LTI) Systems

Let a system be described by an operator G as follows:

$$y(t) = G[u(t)]$$

Definition of a Linear System

The system described by the operator G is linear, if given

$$y_1(t) = G[u_1(t)]$$

$$y_2(t) = G[u_2(t)]$$

then for any constants α and β , one has

$$G[\alpha u_1(t) + \beta u_2(t)] = \alpha y_1(t) + \beta y_2(t)$$

Definition of a Time-Invariant System

System input-output relationship is time-independent

or

System can be represented by differential equations with constant coefficients

Case Study 1:



$$G: \frac{dy}{dt} = u(t) \Rightarrow y(t) = \int u(t) dt$$

Let $y(t) = f(u(t))$. Then

$$f(\alpha u_1(t) + \beta u_2(t)) = \int (\alpha u_1(t) + \beta u_2(t)) dt$$

$$= \int \alpha u_1(t) dt + \int \beta u_2(t) dt = \alpha \int u_1(t) dt + \beta \int u_2(t) dt$$

$$= \alpha f(u_1(t)) + \beta f(u_2(t))$$

➔ It is linear

Also, the coefficient of $u(t)$ is 1, which is a constant

➔ System is time-invariant

Case Study 2:



$$G: \frac{dy}{dt} = t(u(t))^2 \Rightarrow y(t) = \int t(u(t))^2 dt$$

Let $y(t) = f(u(t))$. Then

$$f(\alpha u_1(t) + \beta u_2(t)) = \int t(\alpha u_1(t) + \beta u_2(t))^2 dt$$

$$= \int t(\alpha^2 u_1^2 + 2\alpha u_1 u_2 + \beta^2 u_2^2) dt$$

$$= \alpha^2 f(u_1(t)) + 2\alpha\beta \int t u_1(t) u_2(t) dt + \beta^2 f(u_2(t))$$

$$\neq \alpha f(u_1(t)) + \beta f(u_2(t))$$

➔ It is nonlinear

Also, the term 't' in $t(u(t))^2$ is not constant ➔ System is time-varying

Finite-dimensional **linear time-varying** systems are described in state space as follows:

State equations

$$\begin{aligned}\dot{x}_1(t) &= a_{11}(t)x_1(t) + \cdots + a_{1m}(t)x_m(t) + b_{11}(t)u_1 + \cdots + b_{1n}(t)u_n \\ \dot{x}_2(t) &= a_{21}(t)x_1(t) + \cdots + a_{2m}(t)x_m(t) + b_{21}(t)u_1 + \cdots + b_{2n}(t)u_n \\ &\vdots \\ \dot{x}_m(t) &= a_{m1}(t)x_1(t) + \cdots + a_{mm}(t)x_m(t) + b_{m1}(t)u_1 + \cdots + b_{mn}(t)u_n\end{aligned}$$

Output equations

$$\begin{aligned}y_1(t) &= c_{11}(t)x_1(t) + \cdots + c_{1m}(t)x_m(t) + d_{11}(t)u_1 + \cdots + d_{1n}(t)u_n \\ y_2(t) &= c_{21}(t)x_1(t) + \cdots + c_{2m}(t)x_m(t) + d_{21}(t)u_1 + \cdots + d_{2n}(t)u_n \\ &\vdots \\ y_k(t) &= c_{k1}(t)x_1(t) + \cdots + c_{km}(t)x_m(t) + d_{k1}(t)u_1 + \cdots + d_{kn}(t)u_n\end{aligned}$$

Or, in matrix form:

State equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

Output equation

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & \cdots & a_{1m}(t) \\ \vdots & \ddots & \vdots \\ a_{m1}(t) & \cdots & a_{mm}(t) \end{bmatrix}$$

State Matrix

$$\mathbf{A}(t) \in \mathbb{R}^{m \times m}$$

$$\mathbf{B}(t) = \begin{bmatrix} b_{11}(t) & \cdots & b_{1n}(t) \\ \vdots & \ddots & \vdots \\ b_{m1}(t) & \cdots & b_{mn}(t) \end{bmatrix}$$

Input Matrix

$$\mathbf{B}(t) \in \mathbb{R}^{m \times n}$$

$$\mathbf{C}(t) = \begin{bmatrix} c_{11}(t) & \cdots & c_{1m}(t) \\ \vdots & \ddots & \vdots \\ c_{k1}(t) & \cdots & c_{km}(t) \end{bmatrix}$$

Output Matrix

$$\mathbf{C}(t) \in \mathbb{R}^{k \times m}$$

$$\mathbf{D}(t) = \begin{bmatrix} d_{11}(t) & \cdots & d_{1n}(t) \\ \vdots & \ddots & \vdots \\ d_{k1}(t) & \cdots & d_{kn}(t) \end{bmatrix}$$

Direct Transmission Matrix

$$\mathbf{D}(t) \in \mathbb{R}^{k \times n}$$

LTI system can be described by

State equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Output equation

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mm} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} c_{11} & \cdots & c_{1m} \\ \vdots & \ddots & \vdots \\ c_{k1} & \cdots & c_{km} \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{k1} & \cdots & d_{kn} \end{bmatrix}$$

State Matrix

$$\mathbf{A} \in \mathbb{R}^{m \times m}$$

Input Matrix

$$\mathbf{B} \in \mathbb{R}^{m \times n}$$

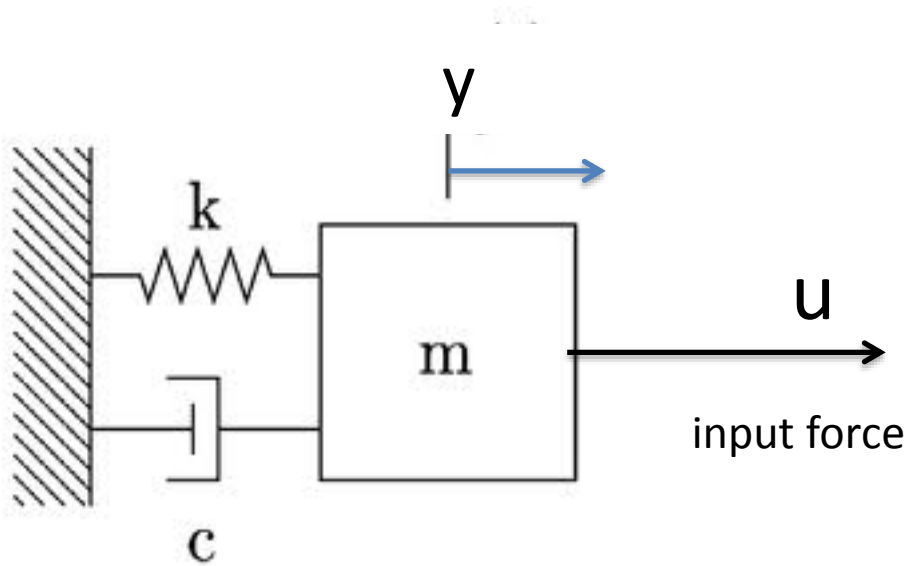
Output Matrix

$$\mathbf{C} \in \mathbb{R}^{k \times m}$$

Direct Transmission Matrix

$$\mathbf{D} \in \mathbb{R}^{k \times n}$$

Example: Mass-Spring-Damper System



$$m\ddot{y} + c\dot{y} + ky = u$$

Taking Laplace Transform (with zero initial conditions)

$$(ms^2 + cs + k)Y(s) = U(s)$$

yields the transfer function

$$G(s) = \frac{1}{ms^2 + cs + k}$$

Example: Mass-Spring-Damper System

$$m\ddot{y} + c\dot{y} + ky = u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u(t)$$

$$y_1 = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\rightarrow A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{bmatrix}, B = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}, C = \begin{bmatrix} 0 & 1 \end{bmatrix}, D = 0$$

→ The matrices A, B, C, D are time-invariant

→ This is a linear time-invariant (LTI) system

Transfer Function and State-Space Equation

$$G(s) = \frac{Y(s)}{U(s)} \quad \longleftrightarrow \quad \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Laplace Transform
with zero initial conditions

$$\begin{cases} sX(s) = AX(s) + BU(s) \\ Y(s) = CX(s) + DU(s) \end{cases}$$

$$(sI - A)X(s) = BU(s)$$

$$X(s) = (sI - A)^{-1} BU(s)$$

$$Y(s) = C(sI - A)^{-1} BU(s) + DU(s)$$

$$G(s) = C(sI - A)^{-1} B + D$$

Single-Input Single-Output (SISO) Systems

There are many possible state representations of a given transfer function.

Some has characteristics that will be later used for different purposes such as controller or estimator designs.

Given a system:

$$\frac{Y(s)}{U(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1} s + b_m}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n}$$

For physical systems, because of casualty and delay, $m < n$

Some models have same order of input and output, $m = n$

SISO refers to the dimension, not the order, of the input-output (i.e., both are scalars but not vectors)

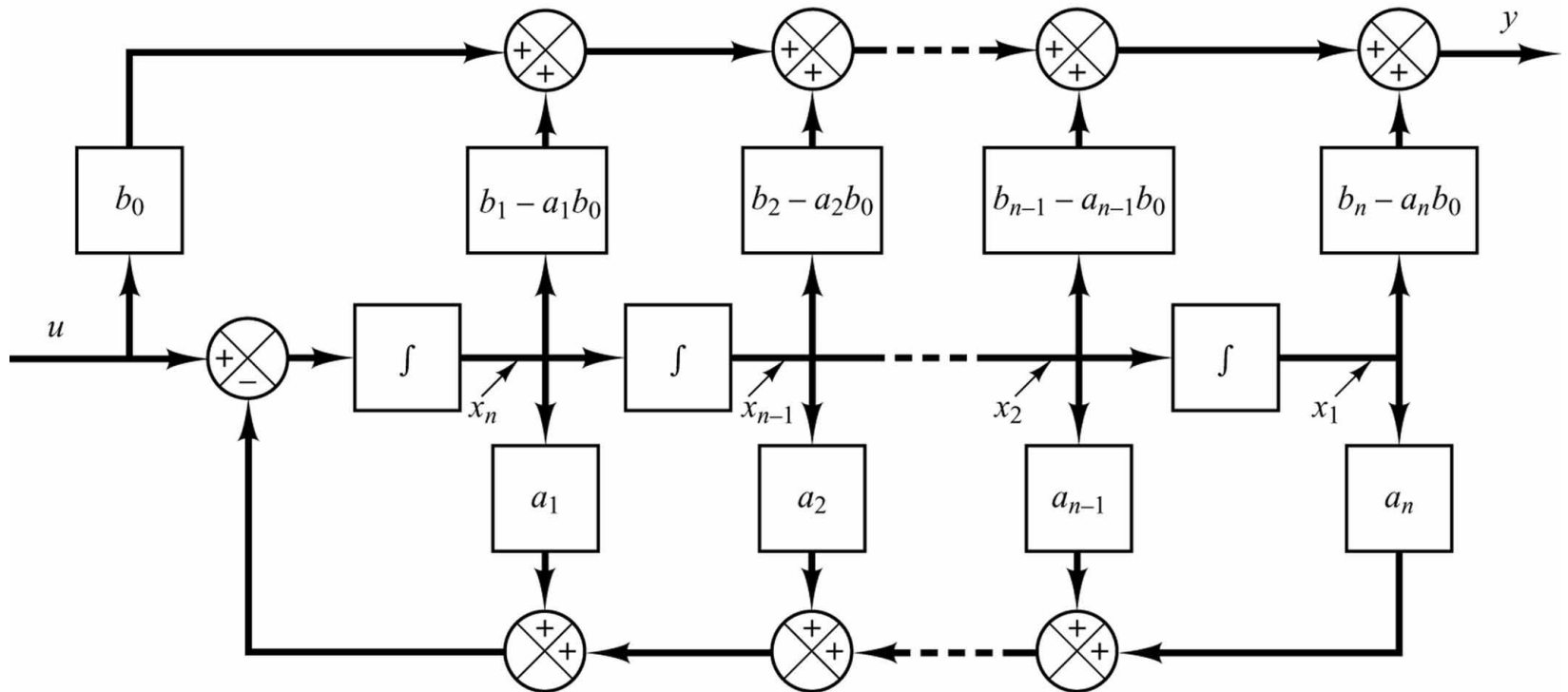
Controllable Canonical Form

For $m = n$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & \cdot & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdot & \cdot & \cdot & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} b_n - a_n b_0 & b_{n-1} - a_{n-1} b_0 & \cdot & \cdot & \cdot & b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

Block diagram representation of the system (Controllable canonical form)



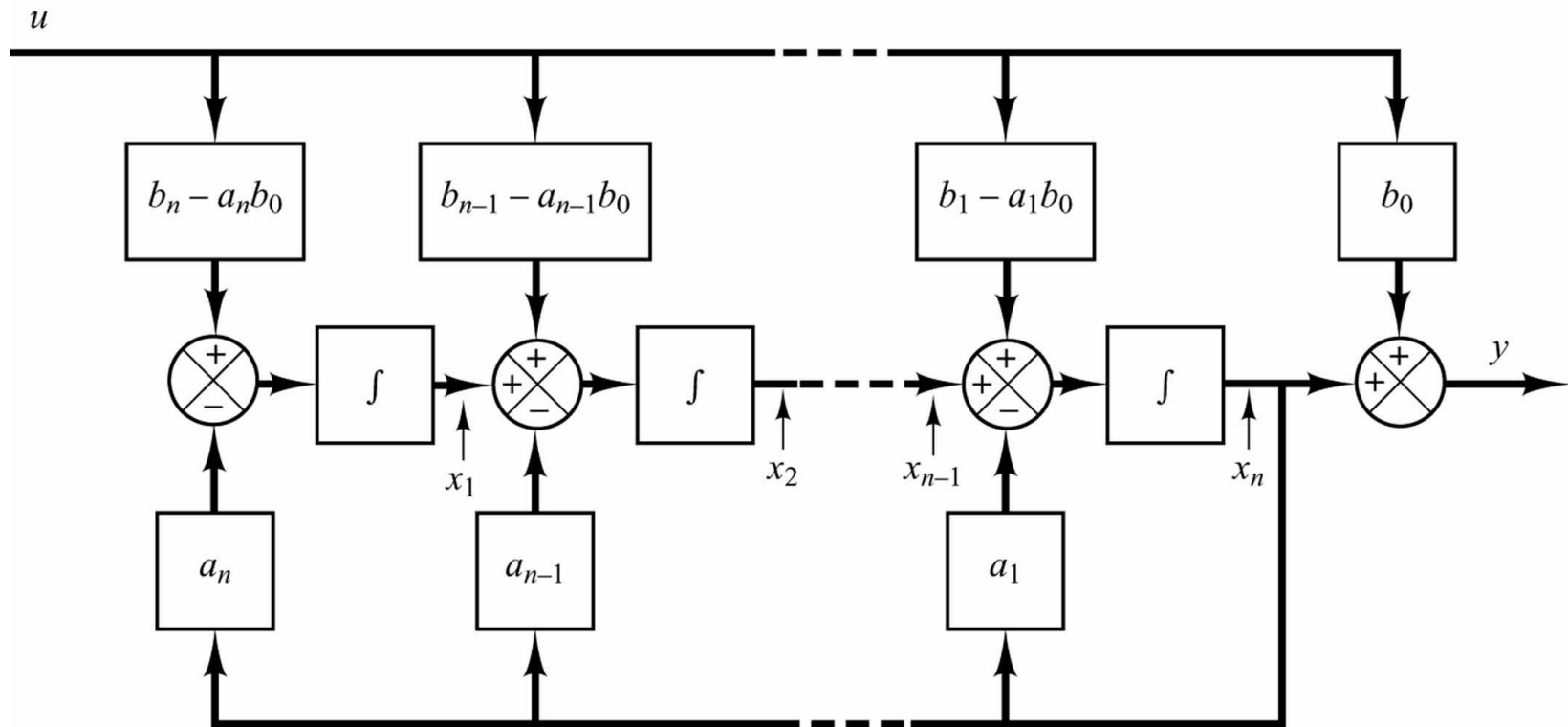
Observable Canonical Form

For $m = n$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \cdot & \cdot & 0 & -a_n \\ 1 & 0 & 0 & \cdot & \cdot & 0 & -a_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdot & \cdot & 0 & -a_2 \\ 0 & 0 & 0 & \cdot & \cdot & 1 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} b_n - a_n b_0 \\ b_{n-1} - a_{n-1} b_0 \\ \vdots \\ b_2 - a_2 b_0 \\ b_1 - a_1 b_0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + b_0 u$$

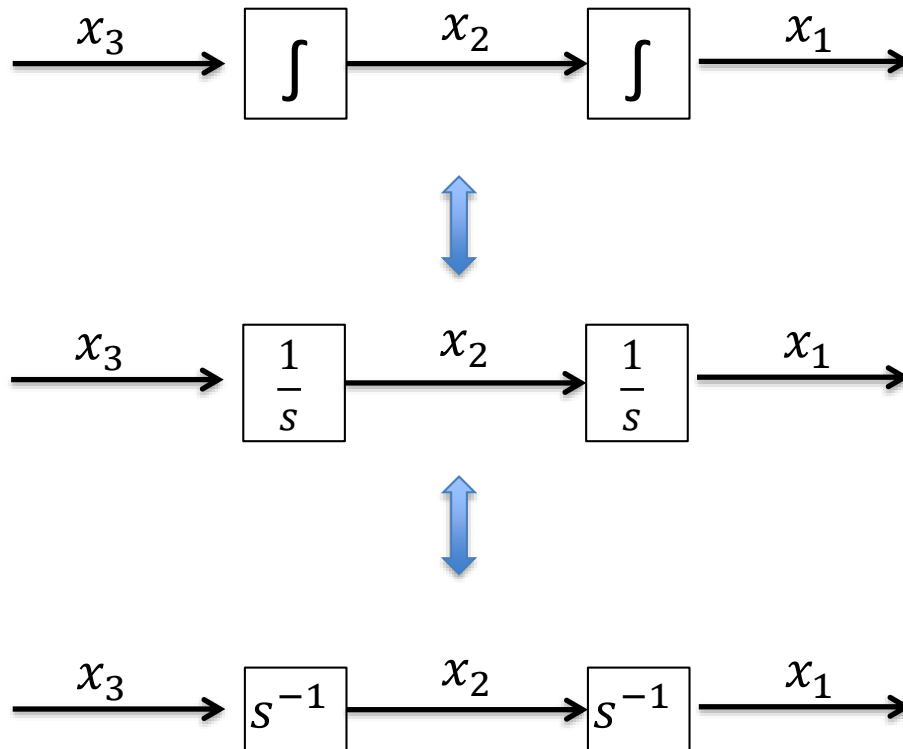
Block diagram representation of the system (Observable canonical form)



Note:

Integrator block can be represented in different but equivalent forms

Example: $\dot{x}_1 = x_2$ and $\dot{x}_2 = x_3$



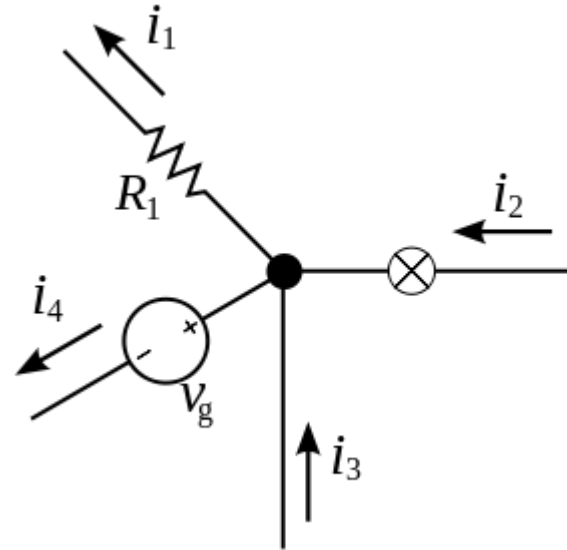
Models of Electric Circuits

Basic Equations of Electric Circuits:

- Kirchhoff's Current Law
 - The algebraic sum of currents leaving a junction (a node) is equal to the algebraic sum of currents entering that node
- Kirchhoff's Voltage Law
 - The algebraic sum of all voltages taken around a closed path in a circuit is 0

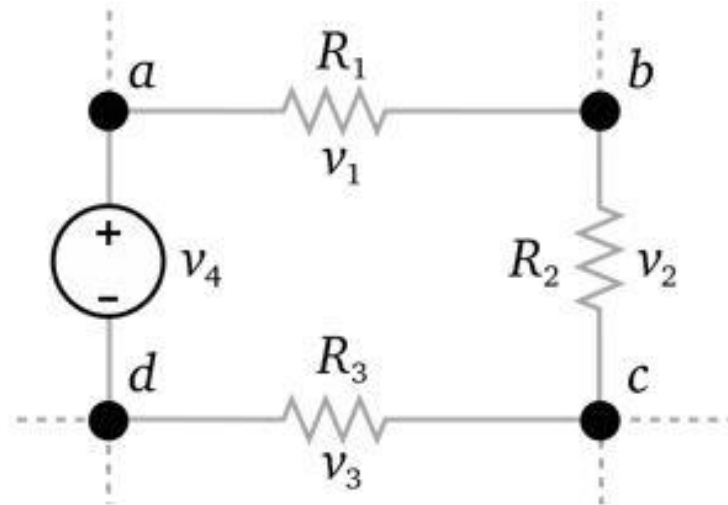
Kirchhoff's Current Law

$$\sum_{k=1}^n I_k = 0$$

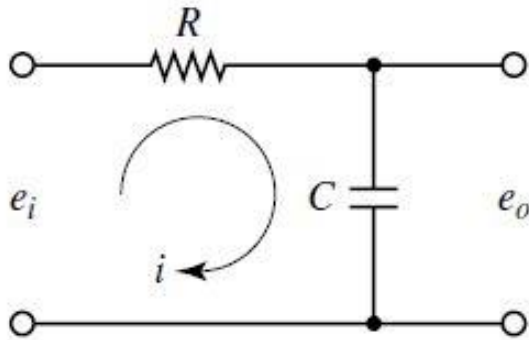


Kirchhoff's Voltage Law

$$\sum_{k=1}^n V_k = 0$$



RC Circuit



$$i = \frac{e_i - e_o}{R}$$

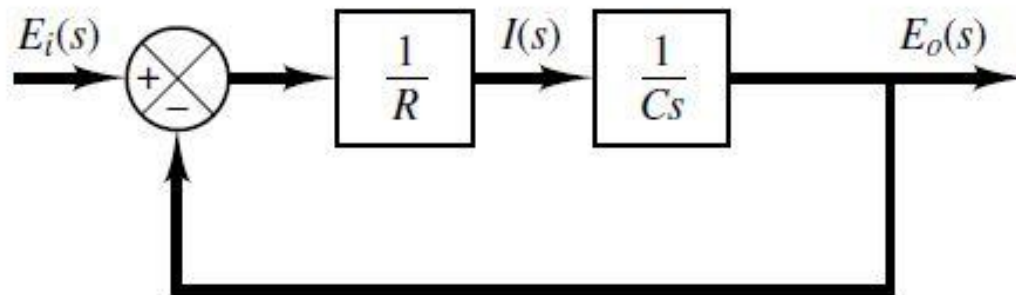
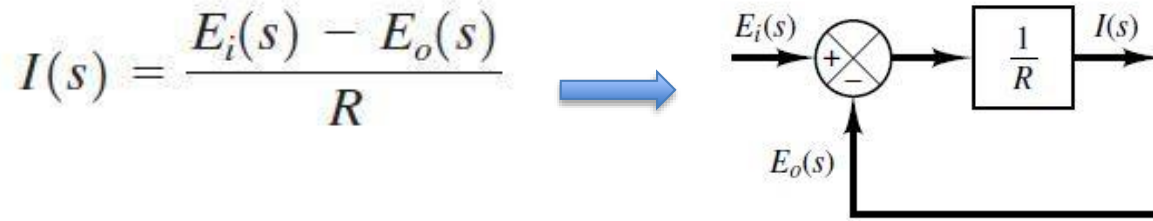
$$e_o = \frac{\int i dt}{C}$$

Parameters: R, C

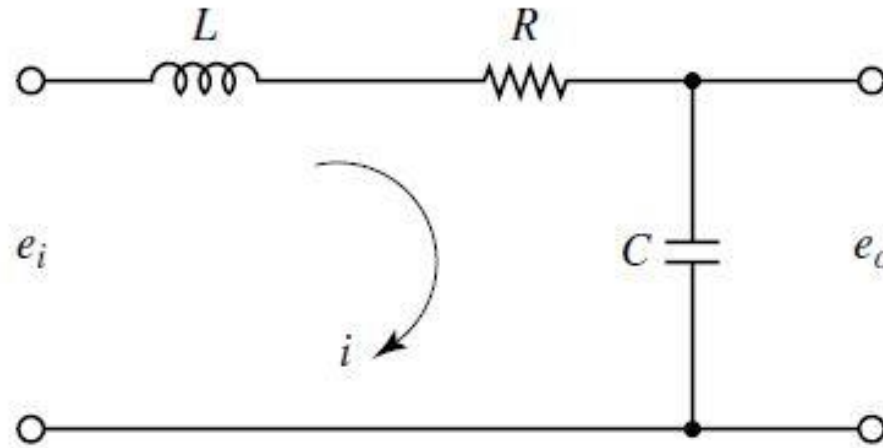
Independent variables (inputs): e_i

Dependent variables (outputs): e_o and i

Laplace Transforms:

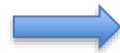


LRC Circuit



$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i dt = e_i$$

$$\frac{1}{C} \int i dt = e_o$$



$$LsI(s) + RI(s) + \frac{1}{C} \frac{1}{s} I(s) = E_i(s)$$

$$\frac{1}{C} \frac{1}{s} I(s) = E_o(s)$$

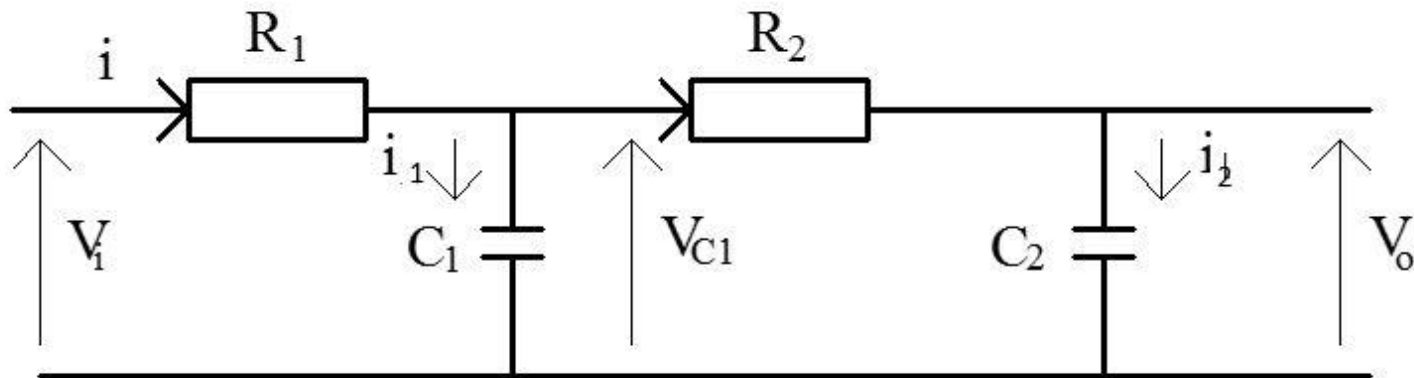
Transfer Function:

$$\frac{E_o(s)}{E_i(s)} = \frac{1}{LCs^2 + RCs + 1}$$

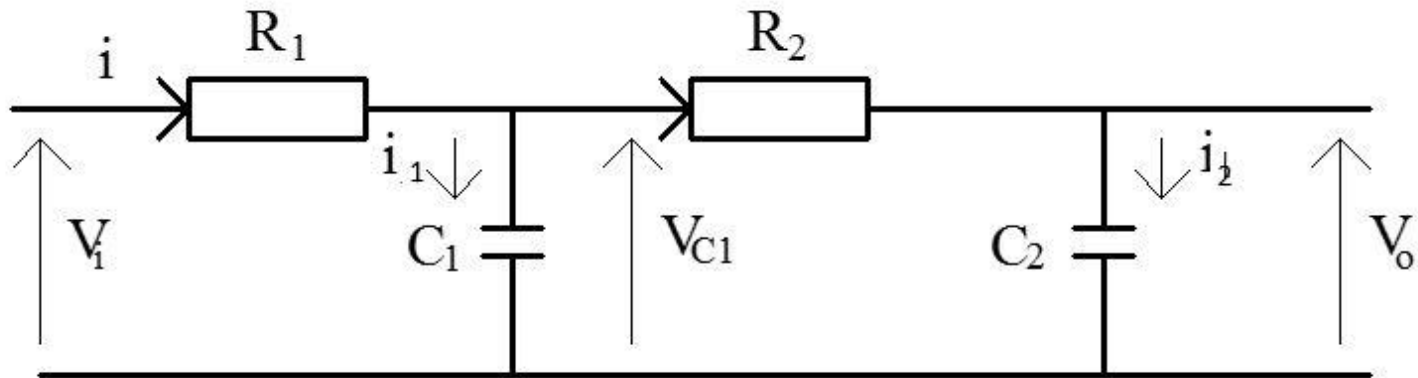


Transfer Function of a Cascaded Network

Example:



Steps: First, find the circuit equations
Then, find their Laplace Transforms
Then, Find the overall transfer function
Finally, build the block diagram



➡ First, find the circuit equations

$$i = \frac{V_i - V_{C1}}{R_1}$$

$$i_2 = \frac{V_{C1} - V_o}{R_2}$$

$$i = i_1 + i_2$$

$$V_{C1} = \frac{\int i_1 dt}{C_1}$$

$$V_o = \frac{\int i_2 dt}{C_2}$$

$$i = \frac{V_i - V_{C1}}{R_1}$$

$$i_2 = \frac{V_{C1} - V_o}{R_2}$$

$$i = i_1 + i_2$$

$$V_{C1} = \frac{\int i_1 dt}{C_1}$$

$$V_o = \frac{\int i_2 dt}{C_2}$$

➡ Then, find their Laplace Transforms

$$I(s) = \frac{V_I(s) - V_{C1}(s)}{R_1}$$

$$I_2(s) = \frac{V_{C1}(s) - V_o(s)}{R_2}$$

$$I(s) = I_1(s) + I_2(s)$$

$$V_{C1}(s) = \frac{I_1(s)}{C_1 s}$$

$$V_o(s) = \frac{I_2(s)}{C_2 s}$$

$$I(s) = \frac{V_I(s) - V_{C1}(s)}{R_1} \quad I_2(s) = \frac{V_{C1}(s) - V_o(s)}{R_2} \quad I(s) = I_1(s) + I_2(s)$$

$$V_{C1}(s) = \frac{I_1(s)}{C_1 s} \quad V_o(s) = \frac{I_2(s)}{C_2 s}$$

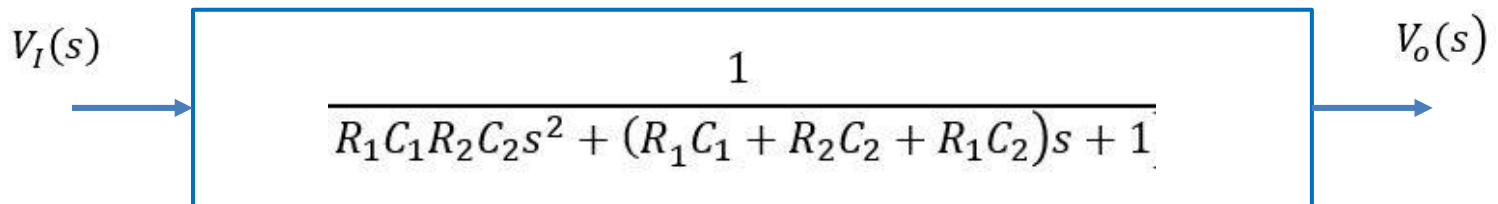
➡ Then, find the overall transfer function

$$\begin{array}{ccc}
 I(s) = I_1(s) + I_2(s) & & I_2(s) = \frac{V_{C1}(s) - V_o(s)}{R_2} \\
 \downarrow & & \downarrow \\
 \frac{V_I(s) - V_{C1}(s)}{R_1} = C_1 s V_{C1}(s) + C_2 s V_o(s) & \leftarrow & V_{C1}(s) = R_2 I_2(s) + V_o(s) \\
 & & = R_2 C_2 s V_o(s) + V_o(s) \\
 \downarrow & & \\
 V_I(s) = [R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1] V_o(s)
 \end{array}$$

➡ Finally, build the block diagram

$$V_I(s) = [R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1]V_o(s)$$

$$\frac{V_o(s)}{V_I(s)} = \frac{1}{[R_1 C_1 R_2 C_2 s^2 + (R_1 C_1 + R_2 C_2 + R_1 C_2)s + 1]}$$



END