

MA1300 Assignment # 2, Solutions

1. (P70, #15) Evaluate the limit $\lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$

Solution.

$$\lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3} = \lim_{t \rightarrow -3} \frac{(t+3)(t-3)}{(2t+1)(t+3)} = \lim_{t \rightarrow -3} \frac{t-3}{2t+1} = \frac{6}{5}.$$

2. (P70, #19) Evaluate the limit $\lim_{x \rightarrow -2} \frac{x+2}{x^3+8}$

Solution.

$$\lim_{x \rightarrow -2} \frac{x+2}{x^3+8} = \lim_{x \rightarrow -2} \frac{x+2}{(x+2)(x^2-2x+4)} = \lim_{x \rightarrow -2} \frac{1}{x^2-2x+4} = \frac{1}{12}.$$

3. (P70, #26) Evaluate the limit $\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2+t} \right)$

Solution.

$$\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2+t} \right) = \lim_{t \rightarrow 0} \frac{t+1-1}{t(t+1)} = \lim_{t \rightarrow 0} \frac{1}{t+1} = 1.$$

4. (P70, #36) Use the Squeeze Theorem to show that

$$\lim_{x \rightarrow 0} \sqrt{x^3+x^2} \sin \frac{\pi}{x} = 0.$$

Solution. Since $-\frac{1}{\sqrt{2}}|x| \leq -\sqrt{x^3+x^2} \leq \sqrt{x^3+x^2} \sin \frac{\pi}{x} \leq \sqrt{x^3+x^2} \leq \sqrt{\frac{3}{2}}|x|$ when $x \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, and by Example 7 in page 82 of the textbook, $\lim_{x \rightarrow 0} |x| = 0$, so

$$\lim_{x \rightarrow 0} -\frac{1}{\sqrt{2}}|x| = 0 = \lim_{x \rightarrow 0} \sqrt{\frac{3}{2}}|x|,$$

then the Squeeze Theorem thus implies

$$\lim_{x \rightarrow 0} \sqrt{x^3+x^2} \sin \frac{\pi}{x} = 0.$$

5. (P70, #38) If $2x \leq g(x) \leq x^4 - x^2 + 2$ for all x , evaluate $\lim_{x \rightarrow 1} g(x)$.

Solution. Since $\lim_{x \rightarrow 1} 2x = 2 = \lim_{x \rightarrow 1} (x^4 - x^2 + 2)$, the Squeeze Theorem implies $\lim_{x \rightarrow 1} g(x) = 2$.

6. (P70, #40) Prove that $\lim_{x \rightarrow 0^+} \sqrt{x}[1 + \sin^2(2\pi/x)] = 0$.

Solution. Since for $x \in (0, \infty)$, $0 \leq \sqrt{x}[1 + \sin^2(2\pi/x)] \leq 2\sqrt{x}$, and $\lim_{x \rightarrow 0^+} 0 = \lim_{x \rightarrow 0^+} 2\sqrt{x} = 0$, the Squeeze Theorem implies $\lim_{x \rightarrow 0^+} \sqrt{x}[1 + \sin^2(2\pi/x)] = 0$.

For Questions 7 ~ 9, find the limit, if it exists. If the limit does not exist, explain why.

7. (P70, #41) $\lim_{x \rightarrow 3} (2x + |x-3|)$

Solution. When $x > 3$, $2x + |x-3| = 3x-3$, we have $\lim_{x \rightarrow 3^+} 2x + |x-3| = 6$. When $x < 3$, $2x + |x-3| = x+3$, so $\lim_{x \rightarrow 3^-} 2x + |x-3| = 6$. Therefore $\lim_{x \rightarrow 3} 2x + |x-3| = 6$.

8. (P70, #42) $\lim_{x \rightarrow -6} \frac{2x+12}{|x+6|}$

Solution. When $x > -6$, $\frac{2x+12}{|x+6|} = \frac{2x+12}{x+6} = 2$, so $\lim_{x \rightarrow -6^+} \frac{2x+12}{|x+6|} = 2$. When $x < -6$, $\frac{2x+12}{|x+6|} = \frac{2x+12}{-x-6} = -2$, so $\lim_{x \rightarrow -6^-} \frac{2x+12}{|x+6|} = -2$. Therefore the limit $\lim_{x \rightarrow -6} \frac{2x+12}{|x+6|}$ does not exist.

9. (P70, #46) $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$

Solution. When $x < 0$, $\left(\frac{1}{x} - \frac{1}{|x|} \right) = \frac{2}{x}$, so $\lim_{x \rightarrow 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) = -\infty$.

10. (P70, #47) The *signum* (or *sign*) *function*, denoted by $\operatorname{sgn} x$, is defined by

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

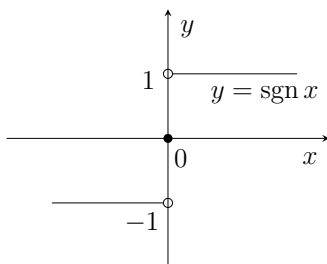
(a) Sketch the graph of this function.

(b) Find each of the following limits or explain why it does not exist.

(i) $\lim_{x \rightarrow 0^+} \operatorname{sgn} x$ (ii) $\lim_{x \rightarrow 0^-} \operatorname{sgn} x$ (iii) $\lim_{x \rightarrow 0} \operatorname{sgn} x$ (iv) $\lim_{x \rightarrow 0} |\operatorname{sgn} x|$

Solution.

(a)



(b) Since for $x > 0$, $\operatorname{sgn} x = 1$, we have $\lim_{x \rightarrow 0^+} \operatorname{sgn} x = \lim_{x \rightarrow 0^+} 1 = 1$. Similarly, $\lim_{x \rightarrow 0^-} \operatorname{sgn} x = -1$. So $\lim_{x \rightarrow 0} \operatorname{sgn} x$ does not exist. Since for $x \neq 0$, $|\operatorname{sgn} x| = 1$, so $\lim_{x \rightarrow 0} |\operatorname{sgn} x| = \lim_{x \rightarrow 0} 1 = 1$.

11. (P71, #54) In the theory of relativity, the Lorentz contraction formula

$$L = L_0 \sqrt{1 - v^2/c^2}$$

expresses the length L of an object as a function of its velocity v with respect to an observer, where L_0 is the length of the object at rest and c is the speed of light. Find $\lim_{v \rightarrow c^-} L$ and interpret the result. Why is a left-hand limit necessary?

Solution. When $v < c$ and v is arbitrarily close to c , $c^2 - v^2$ is arbitrarily close to 0, so is $\sqrt{c^2 - v^2}$. Therefore

$$\lim_{v \rightarrow c^-} L = \frac{L_0}{c} \lim_{v \rightarrow c^-} \sqrt{c^2 - v^2} = 0.$$

The limit shows that an object with length L_0 at rest will have its length approaching 0 as its speed approaches that of light. For the left-hand limit, physically, it is because no object can move faster than light, mathematically, it is because otherwise the defining formula $L_0\sqrt{1 - \frac{v^2}{c^2}}$ will be meaningless.

12. (P71, #59) If

$$f(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \rightarrow 0} f(x) = 0$.

Solution. Since $0 \leq f(x) \leq x^2$ and $\lim_{x \rightarrow 0} 0 = \lim_{x \rightarrow 0} x^2 = 0$, the Squeeze Theorem implies that $\lim_{x \rightarrow 0} f(x) = 0$.

13. (P71, #62) Evaluate $\lim_{x \rightarrow 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$.

Solution.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1} &= \lim_{x \rightarrow 2} \frac{(\sqrt{6-x} - 2)(\sqrt{3-x} + 1)}{(\sqrt{3-x} - 1)(\sqrt{3-x} + 1)} \\ &= \lim_{x \rightarrow 2} \frac{(2-x)(\sqrt{3-x} + 1)}{(2-x)(\sqrt{6-x} + 2)} = \lim_{x \rightarrow 2} \frac{\sqrt{3-x} + 1}{\sqrt{6-x} + 2} = \frac{1}{2}. \end{aligned}$$

14. (P81, #13)

(a) Find a number δ such that if $|x - 2| \leq \delta$, then $|4x - 8| < \varepsilon$, where $\varepsilon = 0.1$.

(b) Repeat part (a) with $\varepsilon = 0.01$.

Solution.

(a) Let $\delta = \frac{1}{80}$, then once $|x - 2| \leq \delta$, we have $|4x - 8| = 4|x - 2| \leq \frac{4}{80} < 0.1 = \varepsilon$.

(b) Let $\delta = \frac{1}{800}$, then once $|x - 2| \leq \delta$, we have $|4x - 8| = 4|x - 2| \leq \frac{4}{800} < 0.01 = \varepsilon$.

15. (P81, #39) If the function f is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

prove that $\lim_{x \rightarrow 0} f(x)$ does not exist.

Solution. Suppose $\lim_{x \rightarrow 0} f(x) = \Delta$. Let $\varepsilon = 1/4$, then for any $\delta > 0$, there exists a rational number $0 < x_\delta < \delta$, and an irrational number $0 < x'_\delta < \delta$, so $f(x_\delta) = 0$, $f(x'_\delta) = 1$, and $\varepsilon < 1 \leq |f(x_\delta) - \Delta| + |f(x'_\delta) - \Delta|$. Therefore $\lim_{x \rightarrow 0} f(x)$ does not exist.

16. (P81, #41) How close to -3 do we have to take x so that

$$\frac{1}{(x+3)^4} > 10,000$$

Solution. Let $|x - (-3)| < 0.1$ to give $(x+3)^4 < \frac{1}{10^4}$, which implies $\frac{1}{(x+3)^4} > 10,000$.

17. (P81, #42, Optional) Prove that $\lim_{x \rightarrow -3} \frac{1}{(x+3)^4} = \infty$.

Solution. For any $M > 0$, let $\delta = \sqrt[4]{\frac{1}{2M}}$, so whenever $|x - (-3)| \leq \delta$, we have $\frac{1}{(x+3)^4} \geq 2M > M$. So by definition, $\lim_{x \rightarrow -3} \frac{1}{(x+3)^4} = \infty$.