

EE6617: Tutorial 2

- 2.1 Suppose that a digital source generate bits 0 and 1 with unequal probability $P(0) = q$ and $P(1) = 1 - q$. The bits are transmitted over an additive Gaussian noise channel. Let $P(j|k)$ denote the probability that $j \in \{0, 1\}$ is received when $k \in \{0, 1\}$ is transmitted. Hence, we have $P(j = 1|k = 0) = P(j = 0|k = 1) = p$ and $P(0|0) = P(1|1) = 1 - p$.
- (a) What is the probability that 0 was transmitted when 1 is received?
 - (b) What is the probability of bit error?
- 2.2 The CDF and PDF of a random variable x ($-\infty \leq x \leq \infty$) satisfy

$$p(x) = \frac{d\Pr(\bar{x} \leq x)}{dx}.$$

Show that

- (a) Given $p(x) \geq 0$, CDF is monotonically increasing on x
 - (b) $\int_{-\infty}^{\infty} p(x)dx = 1$
- 2.3 (a) Is independent random variables x and y also uncorrelated? How about the reverse?
- (b) If $x[n]$ and $y[m]$ are uncorrelated, i.e., $C_{xy}(m, n) = 0$, is it true that $R_{xy}(m, n) = 0$? If not, in which condition the statement is true?
- 2.4 Suppose $T(\mathbf{x}) = \sum_{n=0}^{N-1} x[n]$ with i.i.d $\mathbf{x}$ and $x \sim \mathcal{N}(\mu, \sigma^2)$
- (a) Show $\text{var}(T(\mathbf{x})) = N\sigma^2$
 - (b) Show $\text{var}(S(\mathbf{x})) = \sigma^2/N$ for $S(\mathbf{x}) = \frac{1}{N}T(\mathbf{x})$

Sol. 2.1 (a) By Bayes' theorem

$$P(k|j) = \frac{P(k, j)}{P(j)} = \frac{P(j|k)P(k)}{P(j|0)P(0) + P(j|1)P(1)}.$$

Hence, $P(k = 0|j = 1)$ is

$$\begin{aligned} P(k = 0|j = 1) &= \frac{P(j = 1|k = 0)P(k = 0)}{P(j = 1|k = 0)P(k = 0) + P(j = 1|k = 1)P(k = 1)} \\ &= \frac{pq}{pq + (1 - p)(1 - q)}. \end{aligned}$$

(b) The probability error is

$$P_e = P(j = 1|k = 0)P(k = 0) + P(j = 0|k = 1)P(k = 1) = pq + p(1 - q) = p.$$

Sol. 2.2 (a) By the definition of CDF, $\Pr(\bar{x} \leq x) = \int_{-\infty}^x p(t)dt$, we have

$$\Pr(\bar{x} \leq x) = \int_{-\infty}^x p(t)dt \leq \int_{-\infty}^{x+\epsilon} p(t)dt = \Pr(\bar{x} \leq x + \epsilon)$$

for arbitrary $\epsilon \geq 0$. This establishes that the CDF $\Pr(\bar{x} \leq x)$ is a monotonically increasing function on x .

(b) By definition

$$\int_{-\infty}^{\infty} p(x)dx = \int_{-\infty}^{\infty} d\Pr(\bar{x} \leq x) = \Pr(\bar{x} \leq x) \Big|_{-\infty}^{\infty}$$

Since $\Pr(\bar{x} \leq x)$ is monotonically increasing function on x and $0 \leq \Pr(\bar{x} \leq x) \leq 1$, we have

$$\Pr(\bar{x} \leq x) \Big|_{-\infty}^{\infty} = \Pr(\bar{x} \leq \infty) - \Pr(\bar{x} \leq -\infty) = 1 - 0 = 1$$

Sol. 2.3 (a) Independence implies $R_{xy}(m, n) = E[x[m]y[n]] = E[x[m]]E[y[n]]$, which readily verifies

$$C_{xy}(m, n) = R_{xy}(m, n) - E[x[m]]E[y[n]] = 0.$$

Hence, uncorrelatedness implies independence. In general, the reverse is not true.

(b) From (a), we have $R_{xy}(m, n) = E[x[m]]E[y[n]]$ and if $x[m]$ and $y[n]$ are zero mean then $R_{xy}(m, n) = 0$

Sol. 2.4 (a) $E[T(\mathbf{x})] = N\mu$

$$\begin{aligned} \text{var}(T(\mathbf{x})) &= E[(T(\mathbf{x}) - N\mu)^2] = E\left[\left(\sum_{n=0}^{N-1} x[n] - N\mu\right)^2\right] \\ &= E\left[\left(\sum_{n=0}^{N-1} (x[n] - \mu)\right)^2\right] = E\left[\left(\sum_{n=0}^{N-1} y[n]\right)^2\right] \end{aligned}$$

where $y[n] = x[n] - \mu \sim \mathcal{N}(0, \sigma^2)$ and i.i.d. Then,

$$\text{var}(T(\mathbf{x})) = E \left[\left(\sum_{n=0}^{N-1} y[n] \right)^2 \right] = N\sigma^2$$

and this follows from the fact that $E[y[n]y[m]] = E[y[n]]E[y[m]] = 0$, $m \neq n$.

(b) $E[S(\mathbf{x})] = \mu$

$$\begin{aligned} \text{var}(S(\mathbf{x})) &= E \left[\left(\frac{1}{N} T(\mathbf{x}) - \mu \right)^2 \right] = E \left[\left(\frac{1}{N} \sum_{n=0}^{N-1} x[n] - \mu \right)^2 \right] \\ &= E \left[\left(\frac{1}{N} \sum_{n=0}^{N-1} (x[n] - \mu) \right)^2 \right] = \frac{1}{N^2} E \left[\left(\sum_{n=0}^{N-1} y[n] \right)^2 \right] \\ &\quad \text{with } y[n] = x[n] - \mu \\ &= \frac{\sigma^2}{N} \end{aligned}$$

or simply $\text{var}(S(\mathbf{x})) = \text{var} \left(\frac{1}{N} T(\mathbf{x}) \right) = \frac{1}{N^2} \text{var}(T(\mathbf{x})) = \frac{\sigma^2}{N}$