

MA 1201 Semester B 2018/19

Assignment 1 — Due at 5 pm, 21/2/2019 (Thursday) in tutorial classes

Instructions:

- Please show your work. Unsupported answers will receive **NO** credits.
- Make sure you write down the correct lecture session (A/B/C/D/E/F/G/H) you have registered for, together with your full name and student ID on the front page of your answer script. Hand your solution to your TA in tutorial class before the deadline.
- In case your tutorial class is scheduled on days other than Thursday, you may choose to hand your solution to your TA in one of the tutorial classes scheduled on Thursday.
- **NO** late homework will be accepted. Homework submitted to wrong tutorial sessions will **NOT** be graded and will receive **0 POINTS**.

Recall that the vector equation for a line L passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\vec{r}(t) = \vec{r}_0 + t\vec{v}, \quad -\infty < t < \infty,$$

where $\vec{r}$ is the position vector of a point $P(x, y, z)$ on L and $\vec{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$. In component form, the vector equation is equivalent to three scalar equations:

$$\begin{cases} x = x_0 + tv_1 \\ y = y_0 + tv_2 \\ z = z_0 + tv_3 \end{cases}.$$

1. (10 points) Find the point of intersection of the lines

$$\begin{cases} x = t \\ y = -t + 2 \\ z = t + 1 \end{cases} \quad \text{and} \quad \begin{cases} x = 2s + 2 \\ y = s + 3 \\ z = 5s + 6 \end{cases},$$

and then find the equation of the plane determined by these lines.

Solution. To find the intersection, set

$$t = 2s + 2, \quad -t + 2 = s + 3, \quad t + 1 = 5s + 6.$$

This is a 3×2 (over-determined) linear system:

$$\begin{aligned} -2s + t &= 2 \\ -s - t &= 1, \\ -5s + t &= 5 \end{aligned}$$

which can be shown to be consistent (i.e. admits at least one solution), and which can be solved to yield

$$t = 0, \quad s = -1.$$

(The method for solving general linear systems will be discussed more systematically in Chapter 6.) The point of intersection is thus

$$P_0 = (t, -t + 2, t + 1)|_{t=0} = (2s + 2, s + 3, 5s + 6)|_{s=-1} = (0, 2, 1).$$

To find the equation of the plane determined by the lines, note first that the direction vectors of the lines are given by

$$\vec{a} = \langle 1, -1, 1 \rangle, \quad \vec{b} = \langle 2, 1, 5 \rangle.$$

Thus a normal vector of the plane is (see page 12–14 of the supplementary notes for the determinant formula for computing vector products)

$$\begin{aligned} \vec{n} = \vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 1 \\ 2 & 1 & 5 \end{vmatrix} = \vec{i} \begin{vmatrix} -1 & 1 \\ 1 & 5 \end{vmatrix} - \vec{j} \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} \\ &= [(-1) \cdot 5 - 1 \cdot 1] \vec{i} - [1 \cdot 5 - 2 \cdot 1] \vec{j} + [1 \cdot 1 - 2 \cdot (-1)] \vec{k} = \langle -6, -3, 3 \rangle, \end{aligned}$$

and the equation of the plane is given by

$$(-6)(x-0) + (-3)(y-2) + 3(z-1) = 0, \quad \text{or} \quad -6x - 3y + 3z = -3.$$

2. (15 points) Find the distance from the line

$$\begin{cases} x = t + 2 \\ y = t + 1 \\ z = -\frac{1}{2}(t + 1) \end{cases}$$

to the plane $x + 2y + 6z = 10$. (Hint: Explain first why the line is parallel with the plane (the normal vector of the plane is perpendicular to the direction vector of the line). Then pick an arbitrary point A on the line and an arbitrary point B on the plane. Project the vector $\vec{AB}$ (orthogonally) onto the normal vector of the plane, and find the length of the projected vector.)

Solution. Note first that the line is parallel with the plane, since the normal vector of the plane $\vec{n} = \langle 1, 2, 6 \rangle$ is perpendicular to the direction vector of the line $\vec{a} = \langle 1, 1, -\frac{1}{2} \rangle$:

$$\vec{n} \cdot \vec{a} = \langle 1, 2, 6 \rangle \cdot \langle 1, 1, -\frac{1}{2} \rangle = 1 \cdot 1 + 2 \cdot 1 + 6 \cdot (-\frac{1}{2}) = 0.$$

Let

$$A = (t + 2, t + 1, -\frac{1}{2}(t + 1))|_{t=-1} = (1, 0, 0)$$

be a point on the line and

$$B = (10 - 2y - 6z, y, z)|_{y=0, z=0} = (10, 0, 0)$$

be a point on the plane. The distance between the line and the plane is then precisely $|\text{proj}_{\vec{n}} \vec{AB}|$, the magnitude of the (orthogonal) projection of the vector

$$\vec{AB} = B - A = \langle 10 - 1, 0 - 0, 0 - 0 \rangle = \langle 9, 0, 0 \rangle$$

onto the normal vector $\vec{n}$ (see page 8–9 of the supplementary notes for a simplified formula for computing vector projections):

$$d = |\text{proj}_{\vec{n}} \vec{AB}| = \left| \frac{\vec{AB} \cdot \vec{n}}{|\vec{n}|} \right| = \left| \frac{\langle 9, 0, 0 \rangle \cdot \langle 1, 2, 6 \rangle}{|\langle 1, 2, 6 \rangle|} \right| = \left| \frac{9 \cdot 1 + 0 \cdot 2 + 0 \cdot 6}{\sqrt{1^2 + 2^2 + 6^2}} \right| = \frac{9}{\sqrt{41}}.$$

3. (50 points) Evaluate the following indefinite integrals.

(a) (15 points) $\int \frac{x+1}{2x^2-4x+9} dx.$

Motivation. The “trouble maker” is the fraction $1/(2x^2 - 4x + 9) = (2x^2 - 4x + 9)^{-1}$, which has to be simplified. But if

$$u = 2x^2 - 4x + 9,$$

then

$$du = (2x^2 - 4x + 9)' dx = (4x - 4) dx = 4(x - 1) dx,$$

i.e. an extra factor of $(x - 1)$ (= a constant multiple of u') is needed to transform the differential dx . This suggests the decomposition of the numerator:

$$x + 1 = (x - 1 + 1) + 1 = (x - 1) + 2.$$

Solution. Write

$$\int \frac{x+1}{2x^2-4x+9} dx = \int \left(\frac{x-1}{2x^2-4x+9} + \frac{2}{2x^2-4x+9} \right) dx = \int \frac{x-1}{2x^2-4x+9} dx + \int \frac{2}{2x^2-4x+9} dx.$$

For the first integral, the substitution

$$u = 2x^2 - 4x + 9,$$

$$du = 4(x - 1) dx, \quad \text{or} \quad (x - 1) dx = \frac{1}{4} du,$$

yields

$$\begin{aligned} \int \frac{x-1}{2x^2-4x+9} dx &= \int \frac{1}{2x^2-4x+9} \cdot (x-1) dx \\ &= \int \frac{1}{u} \cdot \left(\frac{1}{4} du \right) = \frac{1}{4} \int \frac{1}{u} du = \frac{1}{4} \ln|u| + C = \frac{1}{4} \ln|2x^2 - 4x + 9| + C. \end{aligned}$$

For the second integral, the fact that the quadratic polynomial $2x^2 - 4x + 9$ is irreducible ($\Delta = (-4)^2 - 4 \cdot 2 \cdot 9 < 0$) implies that

$$\begin{aligned} \int \frac{2}{2x^2-4x+9} dx &= \int \frac{2}{2(x^2-2x)+9} dx = \int \frac{2}{2(x^2-2 \cdot x \cdot 1 + 1^2 - 1^2) + 9} dx \\ &= \int \frac{2}{2(x-1)^2+7} dx = \frac{2}{7} \int \frac{1}{(2/7)(x-1)^2+1} dx = \frac{2}{7} \cdot \frac{\tan^{-1}[\sqrt{2/7}(x-1)]}{\sqrt{2/7}} + C. \end{aligned}$$

Thus

$$\int \frac{x+1}{2x^2-4x+9} dx = \frac{1}{4} \ln|2x^2 - 4x + 9| + \sqrt{2/7} \tan^{-1}[\sqrt{2/7}(x-1)] + C.$$

(b) (15 points) $\int \frac{3x^2+4}{\sqrt{2x+1}} dx.$

Motivation. The “trouble maker” is the fraction $1/\sqrt{2x+1} = (2x+1)^{-1/2}$, which has to be simplified.

Solution. Let

$$u = 2x + 1, \quad \text{or} \quad x = \frac{1}{2}(u - 1),$$

$$du = (2x + 1)' dx = 2 dx, \quad \text{or} \quad dx = \frac{1}{2} du.$$

Then

$$\begin{aligned}
 \int \frac{3x^2+4}{\sqrt{2x+1}} dx &= \int \frac{1}{\sqrt{2x+1}} \cdot (3x^2+4) dx = \int (2x+1)^{-1/2} (3x^2+4) dx \\
 &= \int u^{-1/2} \left(3 \cdot \frac{1}{4} (u-1)^2 + 4 \right) \left(\frac{1}{2} du \right) = \frac{1}{2} \int u^{-1/2} \left(\frac{3}{4} u^2 - \frac{3}{2} u + \frac{19}{4} \right) du \\
 &= \frac{1}{2} \int \left(\frac{3}{4} u^{3/2} - \frac{3}{2} u^{1/2} + \frac{19}{4} u^{-1/2} \right) du = \frac{1}{2} \left(\frac{3}{4} \cdot \frac{2}{5} u^{5/2} - \frac{3}{2} \cdot \frac{2}{3} u^{3/2} + \frac{19}{4} \cdot 2 u^{1/2} \right) + C \\
 &= \frac{3}{20} (2x+1)^{5/2} - \frac{1}{2} (2x+1)^{3/2} + \frac{19}{4} (2x+1)^{1/2} + C.
 \end{aligned}$$

(c) (10 points) $\int \frac{1}{x} \sqrt{x^2-1} dx$.

Motivation. The “trouble maker” is the radical $\sqrt{x^2-1} = (x^2-1)^{1/2}$, which has to be simplified. But if

$$u = x^2 - 1,$$

then

$$du = (x^2 - 1)' dx = 2x dx,$$

i.e. an extra factor of x (= a constant multiple of u') is needed to transform the differential dx . This suggests the decomposition of the fraction $1/x$:

$$\frac{1}{x} = \frac{1}{x} \cdot \frac{x}{x} = \frac{1}{x} \cdot \left(\frac{1}{x} \cdot x \right) = \left(\frac{1}{x} \cdot \frac{1}{x} \right) \cdot x = \frac{1}{x^2} \cdot x.$$

Solution. Let

$$\begin{aligned}
 u &= x^2 - 1, & \text{or} & & x^2 &= u + 1, \\
 du &= 2x dx, & \text{or} & & x dx &= \frac{1}{2} du.
 \end{aligned}$$

Then

$$\begin{aligned}
 \int \frac{1}{x} \sqrt{x^2-1} dx &= \int (x^2-1)^{1/2} \cdot \frac{1}{x} dx = \int (x^2-1)^{1/2} \cdot \frac{1}{x^2} \cdot x dx \\
 &= \int u^{1/2} \cdot \frac{1}{u+1} \cdot \left(\frac{1}{2} du \right) = \frac{1}{2} \int \frac{u^{1/2}}{u+1} du.
 \end{aligned}$$

To proceed, introduce another substitution

$$\begin{aligned}
 v &= u^{1/2}, & \text{or} & & u &= v^2, \\
 du &= (v^2)' dv = 2v dv.
 \end{aligned}$$

It then follows that

$$\begin{aligned}
 \frac{1}{2} \int \frac{u^{1/2}}{u+1} du &= \frac{1}{2} \int \frac{v}{v^2+1} \cdot (2v dv) = \int \frac{v^2}{v^2+1} dv = \int \left(\frac{v^2+1}{v^2+1} - \frac{1}{v^2+1} \right) dv \\
 &= \int 1 dv - \int \frac{1}{v^2+1} dv = v - \tan^{-1} v + C = \sqrt{x^2-1} - \tan^{-1} \sqrt{x^2-1} + C.
 \end{aligned}$$

Remark. 1. The two substitutions $u = x^2 - 1$ and $v = u^{1/2}$ could have been combined into a single one

$$v = u^{1/2} = \sqrt{x^2-1}.$$

In practice, however, it is usually prudent to try substituting for the expression under the square root first, instead of substituting for the entire square root function.

2. The second substitution illustrates the general procedure for substitution discussed on page 76–78 of the supplementary notes, where the old variable u is solved from the equation of substitution $v = u^{1/2}$ in terms of the new variable v . Once $u = v^2$ is expressed as a function of v , $du = (v^2)' dv$ can be obtained simply by differentiation. Compare this with the following computation:

$$v = u^{1/2} \implies dv = (u^{1/2})' du = \frac{1}{2} u^{-1/2} du \implies du = 2u^{1/2} dv = 2v dv.$$

(d) (10 points) $\int \frac{1}{1+e^{2x}} dx.$

Motivation. The “trouble maker” is the fraction $1/(1+e^{2x}) = (1+e^{2x})^{-1}$, which has to be simplified.

Solution. Let

$$u = 1 + e^{2x}, \quad \text{or} \quad e^{2x} = u - 1,$$

$$du = (1 + e^{2x})' dx = 2e^{2x} dx = 2(u - 1) dx, \quad \text{or} \quad dx = \frac{1}{2(u - 1)} du.$$

Then

$$\begin{aligned} \int \frac{1}{1+e^{2x}} dx &= \int \frac{1}{u} \cdot \left(\frac{1}{2(u-1)} du \right) = \frac{1}{2} \int \frac{1}{u(u-1)} du = \frac{1}{2} \int \left(\frac{1}{u-1} - \frac{1}{u} \right) du \\ &= \frac{1}{2} \left(\int \frac{1}{u-1} du - \int \frac{1}{u} du \right) = \frac{1}{2} (\ln|u-1| - \ln|u|) + C = \frac{1}{2} [\ln e^{2x} - \ln(1+e^{2x})] + C. \end{aligned}$$

4. (25 points) Evaluate the following definite integrals.

(a) (10 points) $\int_1^4 e^{|x-2|} dx.$

Motivation. Since, by definition,

$$|x-2| = \begin{cases} x-2, & \text{if } x-2 \geq 0 \\ -(x-2), & \text{if } x-2 < 0 \end{cases} = \begin{cases} x-2, & \text{if } x \geq 2 \\ 2-x, & \text{if } x < 2 \end{cases},$$

it's necessary to partition the interval of integration $[1, 4]$ at $x = 2$ into two parts, $[1, 2]$ and $[2, 4]$.

Solution.

$$\begin{aligned} \int_1^4 e^{|x-2|} dx &= \int_1^2 e^{|x-2|} dx + \int_2^4 e^{|x-2|} dx = \int_1^2 e^{2-x} dx + \int_2^4 e^{x-2} dx \\ &= -e^{2-x} \Big|_1^2 + e^{x-2} \Big|_2^4 = -[e^{2-2} - e^{2-1}] + [e^{4-2} - e^{2-2}] = e^2 + e - 2. \end{aligned}$$

(b) (15 points) $\int_{-\pi/3}^{\pi/3} (|x| + \tan x)^2 dx.$ (Hint: No integration by parts is needed.)

Motivation. The integral is defined on a symmetric interval $[-\pi/3, \pi/3]$, and the functions being integrated have special symmetry properties ($|x|$ is even, and $\tan x$ is odd). Thus it's a good idea to try exploiting these symmetry properties before even expanding the absolute value function $|x|$.

Solution. Write

$$\begin{aligned}\int_{-\pi/3}^{\pi/3} (|x| + \tan x)^2 dx &= \int_{-\pi/3}^{\pi/3} (|x|^2 + 2|x| \tan x + \tan^2 x) dx \\ &= \int_{-\pi/3}^{\pi/3} x^2 dx + 2 \int_{-\pi/3}^{\pi/3} |x| \tan x dx + \int_{-\pi/3}^{\pi/3} \tan^2 x dx.\end{aligned}$$

For the first integral, direct integration yields

$$\int_{-\pi/3}^{\pi/3} x^2 dx = 2 \int_0^{\pi/3} x^2 dx = 2 \cdot \frac{1}{3} x^3 \Big|_0^{\pi/3} = \frac{2}{3} [(\pi/3)^3 - 0^3] = \frac{2}{81} \pi^3.$$

For the second integral, the fact that $|x| \tan x = \text{even} \cdot \text{odd}$ is an odd function implies that

$$\int_{-\pi/3}^{\pi/3} |x| \tan x dx = 0.$$

For the third integral, an application of the Pythagorean identity $1 + \tan^2 x = \sec^2 x$ yields

$$\begin{aligned}\int_{-\pi/3}^{\pi/3} \tan^2 x dx &= 2 \int_0^{\pi/3} \tan^2 x dx = 2 \int_0^{\pi/3} (\sec^2 x - 1) dx \\ &= 2(\tan x - x) \Big|_0^{\pi/3} = 2 \left\{ [\tan(\pi/3) - \pi/3] - (\tan 0 - 0) \right\} = 2(\sqrt{3} - \pi/3).\end{aligned}$$

In summary,

$$\int_{-\pi/3}^{\pi/3} (|x| + \tan x)^2 dx = \frac{2}{81} \pi^3 + 2(\sqrt{3} - \pi/3) = \frac{2}{81} \pi^3 - \frac{2}{3} \pi + 2\sqrt{3}.$$

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