

EF2452: Mathematics for Economics and Finance

Lecture 7

Part I of II

Department of Economics and Finance
City University of Hong Kong
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Main Topics

- Constrained optimization
- Minimizing total cost of production
- The cost function as the solution to a constrained optimization problem
- The method of Lagrange multipliers

Constrained Optimization

- We are now going to study problems that involve constrained optimization.
- There will be a function that needs to be maximized or minimized (called the **objective function**).
- There will be a condition that needs to be satisfied by the variables (known as the **constraint**).

Cost Minimization: Example

- Suppose $v = 10$ and $w = 8$, the production function is $5kl$, and the firm wishes to produce 1600 units of output.
- We have to solve the problem

$$\text{minimize } 10k + 8l \quad \text{subject to } 5kl = 1600$$

- We can simplify the problem by using the constraint to eliminate one variable, say l . Because $5kl = 1600$, we have

$$l = \frac{320}{k}$$

- Then, the problem is reduced to minimizing

$$10k + 8\left(\frac{320}{k}\right)$$

Cost Minimization: Example

- Setting the first derivative (w.r.t. k) equal to zero gives the first-order condition

$$10 - \frac{2560}{k^2} = 0 \implies 10k^2 = 2560 \implies k = 16$$

$$\text{and } l = \frac{320}{k} = 20$$

- The minimum cost is

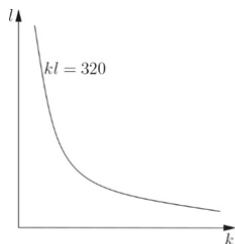
$$C = 10k + 8l = (10)(16) + (8)(20) = 320$$

- Check that the second-order condition for a minimum is satisfied (the second derivate is positive):

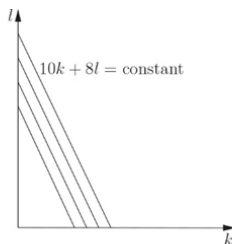
$$\frac{(2)(2560)}{k^3} > 0$$

Cost Minimization: Graphically

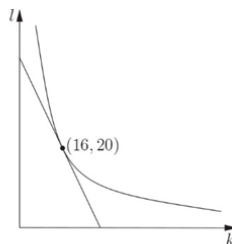
- (a) shows the constraint, the isoquant $5kl = 1600$, or $kl = 320$. The problem is to find the point on this isoquant for which the cost of $10k + 8l$ is the least.
- The lines in (b) are typical contours $10k + 8l = \text{constant}$. Each line corresponds to a total cost value, with lines towards the origin being associated with lower employment of capital and labor and, therefore, lower cost.



(a)



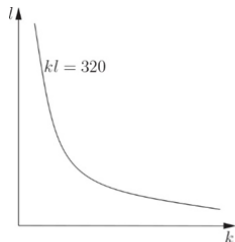
(b)



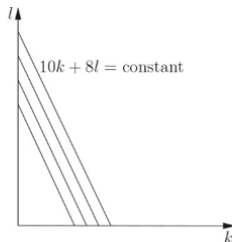
(c)

Cost Minimization: Graphically

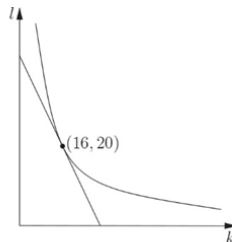
- The point on the isoquant $kl = 320$ for which $10k + 8l$ is the least is determined by the contour nearest the origin which meets the isoquant. This contour is a tangent to the isoquant.



(a)



(b)



(c)

Lagrange Multiplier Method

- Often, it is not possible to eliminate one variable and use the previous method to solve the constrained optimization. A more general method to solve constrained optimization is called the method of Lagrange multipliers. Given a constrained optimization problem

$$\text{maximize or minimize } F(x, y) \quad \text{subject to } G(x, y) = 0$$

- We form the Lagrangean

$$\mathcal{L}(x, y, \lambda) = F(x, y) - \lambda G(x, y)$$

- λ is a new variable known as the Lagrange multiplier.

Lagrange Multiplier Method

- Then, the solution to the constrained optimization problem can be found by solving the equations

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial F}{\partial x} - \lambda \frac{\partial G}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial F}{\partial y} - \lambda \frac{\partial G}{\partial y} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = G(x, y) = 0$$

- Note that these are only the first-order conditions for optimization. There are second-order conditions (SOCs) for a maximum or a minimum; however, for our course, we will not discuss these SOCs as they are more involved in this case of constrained optimization using the Lagrange multiplier method.

An Example

- In the example we discussed earlier, we have $f(k, l) = 10k + 8l$ and the constraint is $G(k, l) = 5kl - 1600$, so the Lagrangean is

$$\mathcal{L} = 10k + 8l - \lambda(5kl - 1600)$$

- The optimal values of k and l are determined by the equations

$$\frac{\partial \mathcal{L}}{\partial k} = 10 - 5\lambda l = 0$$

$$\frac{\partial \mathcal{L}}{\partial l} = 8 - 5\lambda k = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 5kl - 1600 = 0$$

An Example

- Using the first two equations, we have (eliminating λ)

$$\frac{10}{8} = \frac{l}{k} \implies l = \frac{5k}{4}$$

- Substituting the expression of l in the third equation, we have

$$(5k) \frac{5k}{4} = 1600 \implies 25k^2 = 6400 \implies k = 16$$

- And,

$$l = \frac{5k}{4} = 20$$

Cost Function

- The cost function (that we have been discussing in our last few lectures) can be obtained as the solution to a constrained optimization problem.
- We consider the general problem of minimizing total capital and labor costs, given that a prescribed output must be produced. This means that the resulting cost function will depend on the firm's production function and the costs per unit for capital and labor.
- In the last problem, we required the production level to be 1600 units. The answer to that problem, the minimum cost of producing 1600 units, is just $C(1600)$, the cost function $C(q)$ evaluated at $q = 1600$.

Cost Function

- Now, the cost function is the solution to the following constrained optimization exercise (cost minimization):

$$\text{minimize } vk + wl \quad \text{subject to } q(k, l) = q^*$$

- For the example where $q(k, l) = 5kl$, $v = 10$, and $w = 8$, we have

$$\text{minimize } 10k + 8l \quad \text{subject to } 5kl = q^*$$

- The Lagrangean is

$$\mathcal{L} = 10k + 8l - \lambda(5kl - q^*)$$

Cost Function

- The optimal values of k and l are determined by the equations

$$\frac{\partial \mathcal{L}}{\partial k} = 10 - 5\lambda l = 0$$

$$\frac{\partial \mathcal{L}}{\partial l} = 8 - 5\lambda k = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 5kl - q^* = 0$$

- Using the first two equations, we have (eliminating λ)

$$\frac{10}{8} = \frac{l}{k} \implies l = \frac{5k}{4}$$

- Substituting the expression of l in the third equation, we have

$$(5k) \frac{5k}{4} = q^* \implies k^2 = \frac{4}{25} q^* \implies k = \frac{2}{5} \sqrt{q^*}$$

Cost Function

- Therefore,

$$l = \frac{5k}{4} = \left(\frac{5}{4}\right)\left(\frac{2}{5}\right)\sqrt{q^*} = \frac{1}{2}\sqrt{q^*}$$

- So, the minimum cost of producing q^* units of output is

$$C(q^*) = 10k + 8l = (10)\frac{2}{5}\sqrt{q^*} + (8)\frac{1}{2}\sqrt{q^*} = 8\sqrt{q^*}$$

- Since q^* here represents any given quantity, it follows that the cost function is

$$C(q) = 8\sqrt{q}$$

References

Following chapters of the textbook:

21 Constrained Optimization