

Block B Unit 1 outline

- AC circuit components
 - > capacitors
 - > inductors
- Key concepts when dealing with AC
 - > Sinusoids (frequency, amplitude, phase)
 - > Instantaneous and Average power
 - > Phasors and Complex numbers
 - > Impedance and Admittance

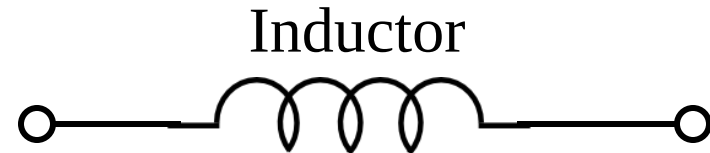
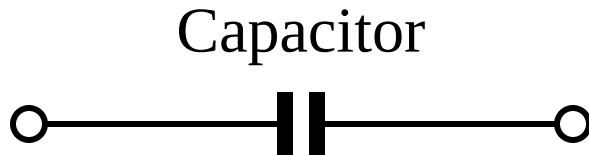
G. Rizzoni, “Fundamentals of EE” Chapters 4.1, 4.2, 4.4, 7.1

Energy Dissipation and Storage

KEYPOINT: As ideal components, unlike the resistor, **capacitors and inductors store energy**. The resistor on the other hand only dissipates or consumes energy.



Dissipates Energy

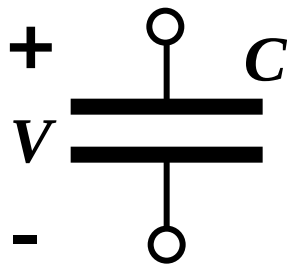


Store Energy

Capacitor and Capacitance

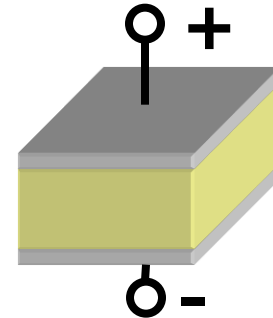
Capacitor

- A capacitor comprises 2 parallel conducting plates separated by an insulator (or dielectric)
- A capacitor can store energy in the electric field when a voltage is applied across the plates



$$\begin{aligned} Q &= CV \\ C &= Q/V \end{aligned}$$

$$W = \frac{1}{2} CV^2$$

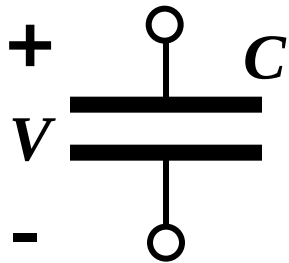


Capacitance

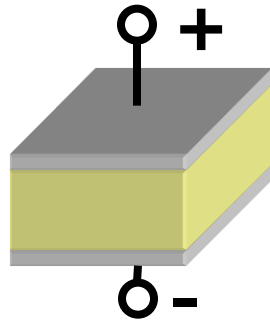
- Capacitance is a measure of the capacity of the device to store charge
- Commonly symbolized by the letter (C)
- Measured by the unit Farad (F)
- Related to charge and voltage by: $Q = CV$, where V is the voltage across the capacitor and Q is the charge stored

Capacitors store energy

A capacitor can store energy in the electric field when a voltage is applied across the plates

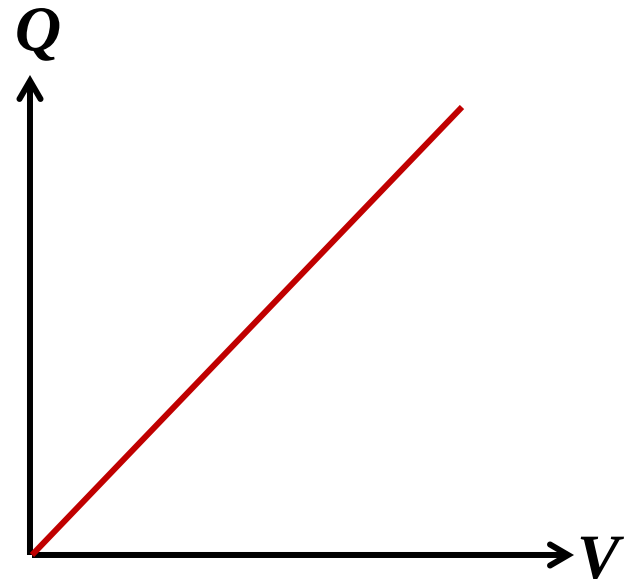


$$Q = CV$$
$$C = Q/V$$



Energy stored:

$$W = \frac{1}{2} CV^2$$



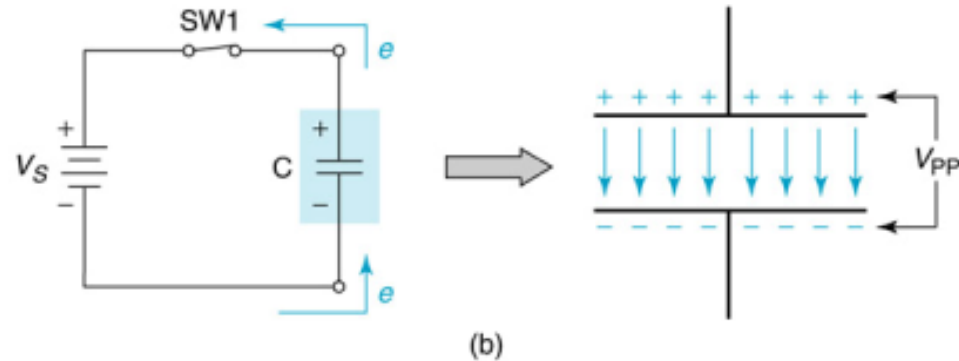
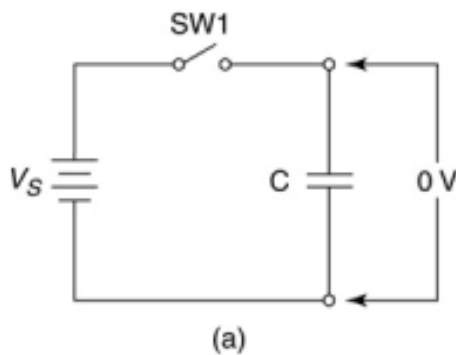
What does the slope of the curve represent here?

Area under curve
= Energy stored

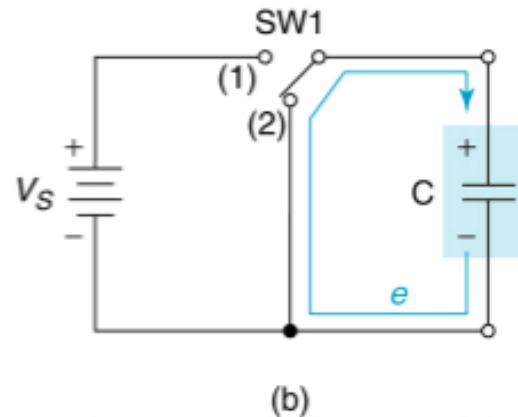
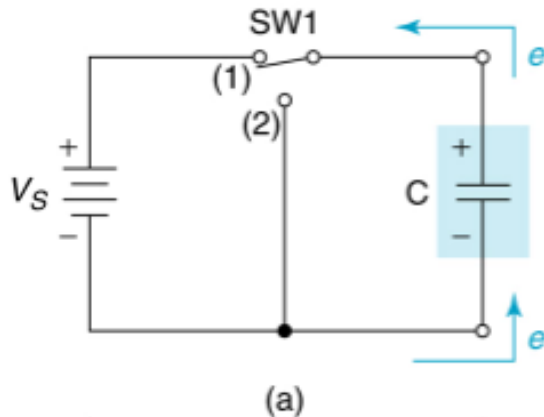
Capacitor charge and discharge

- Charge develops as voltage is applied
- Electric field stores energy

Charging occurs when a voltage difference is applied. Voltage across the capacitor is initially zero in this figure.

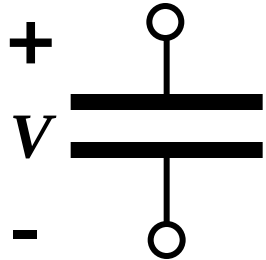


Discharging when the voltage difference is removed. In the figures below, the voltage across the capacitor is switched to zero.



I-V relation in a capacitor

If the voltage across the capacitor is time-varying, this gives rise to a time-varying charge on the capacitor voltage to give time varying charge:



$$Q(t) = CV(t)$$

Differentiating with respect to time (t) we obtain the current:

$$i(t) = C \frac{dV(t)}{dt}$$

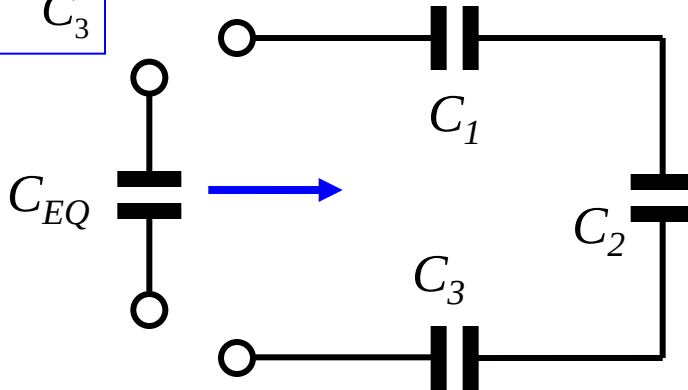
**Definitely a must
to memorize!**

Inversely, we can likewise express the capacitor voltage in terms of its current by integrating the capacitor current:

$$v_c(t) = \frac{1}{C} \int_{-\infty}^t i_c(t) dt$$

Capacitors in Series and Parallel

$$\frac{1}{C_{EQ}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$



The Proof:

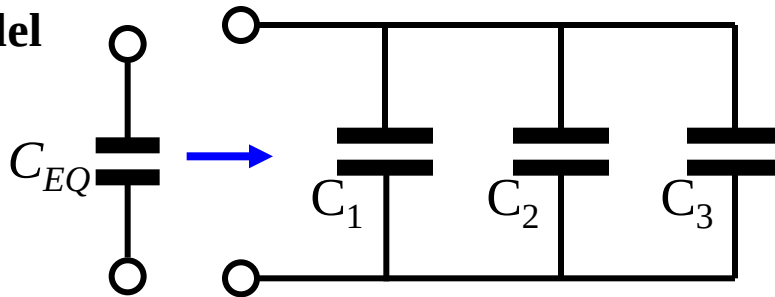
Applying KVL: $v(t) = v_1 + v_2 + v_3$

$$\frac{1}{C_{EQ}} \int i_c dt = \frac{1}{C_1} \int i_c dt + \frac{1}{C_2} \int i_c dt + \frac{1}{C_3} \int i_c dt$$

$$\frac{1}{C_{EQ}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

In Series

In Parallel



$$C_{EQ} = C_1 + C_2 + C_3$$

The Proof:

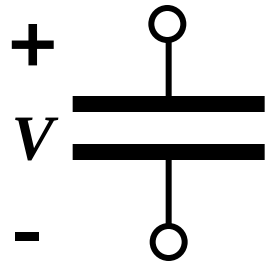
Applying KCL: $i_c = i_1 + i_2 + i_3$

$$C_{EQ} \frac{dv(t)}{dt} = C_1 \frac{dv(t)}{dt} + C_2 \frac{dv(t)}{dt} + C_3 \frac{dv(t)}{dt}$$

$$C_{EQ} = C_1 + C_2 + C_3$$

DC and AC response in Capacitor

Basic equation from the beginning



$$Q(t) = CV(t)$$

$$i(t) = C \frac{dV(t)}{dt}$$

If the voltage is reversed:

Notice that Q has reversed too

Q is been changed $\Rightarrow$ Current

If voltage is constant with time $\Rightarrow dV/dt = 0 \Rightarrow I = 0$ (No current)

No voltage change $\Rightarrow$ No current

If voltage changes with time $\Rightarrow dV/dt \neq 0 \Rightarrow I \neq 0$ (There is a current)

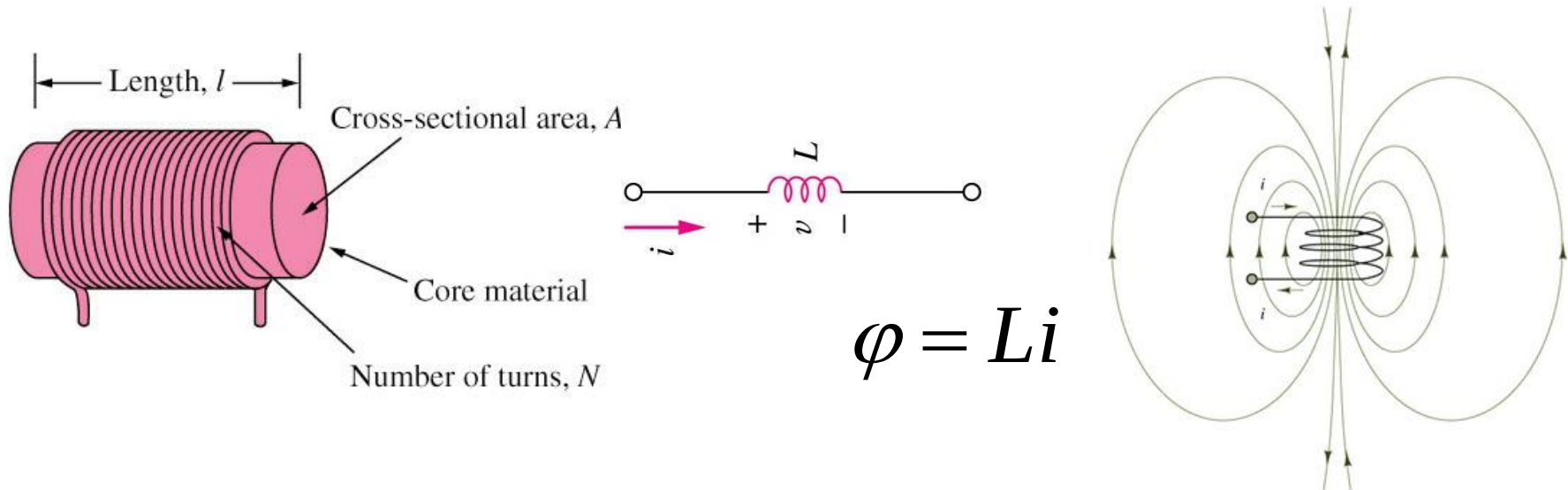
Voltage change $\Rightarrow$ Current

Capacitor response to DC & AC

- If applied voltage is DC
 - > Insulating dielectric blocks the current from flowing through
 - > Plates will charge up
 - > At DC, capacitor blocks current from flowing through
- If applied voltage is AC
 - > Charge on the plates also likewise varies in time with the AC
 - > Capacitor therefore does not act as an open circuit in the presence of an AC
 - > In AC, capacitor allows current to pass through

Inductors and Inductance

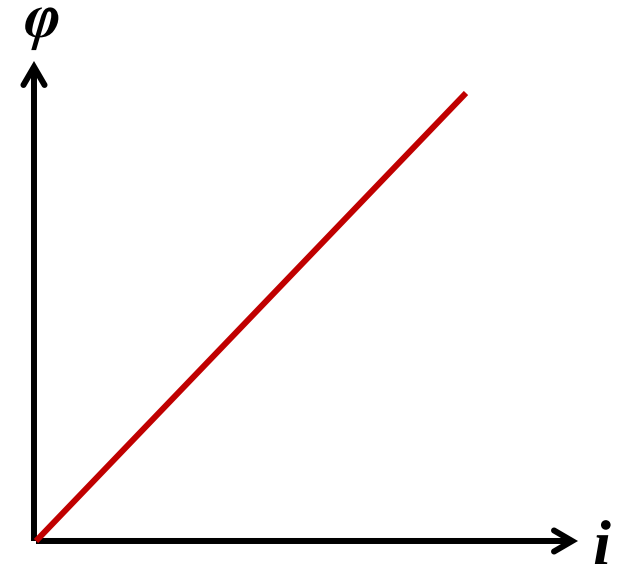
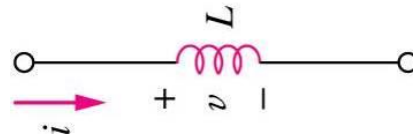
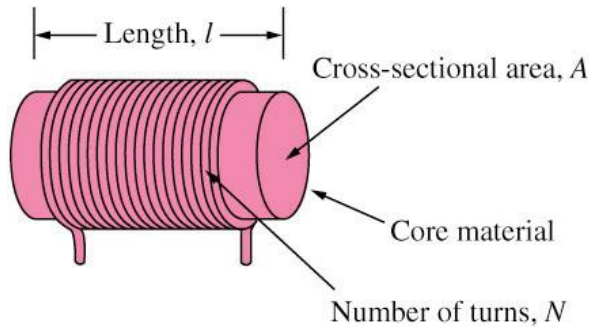
- Elements that have the ability to store energy in a **magnetic field**
- Inductor is typically made by winding a coil around a core



- Passing a current through an inductor produces a magnetic flux ϕ that is related to the inductance (L)
- Inductance is commonly symbolized by the letter (L)
- Unit for Inductance is the Henry (H)

Inductors store energy

An inductor can store energy in the magnetic field when a current is running through the coil



$$\phi = Li$$

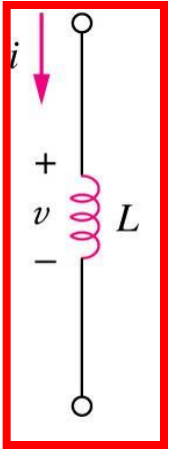
Energy stored:

$$W = \frac{1}{2} Li^2$$

What does the slope of the curve represent here?

Area under curve
= Energy stored

I-V relation in an Inductor



From Faraday's Law: $v(t) = \frac{d\phi(t)}{dt}$

But the flux is given by: $\Phi = L i$

$$v(t) = L \frac{di(t)}{dt}$$

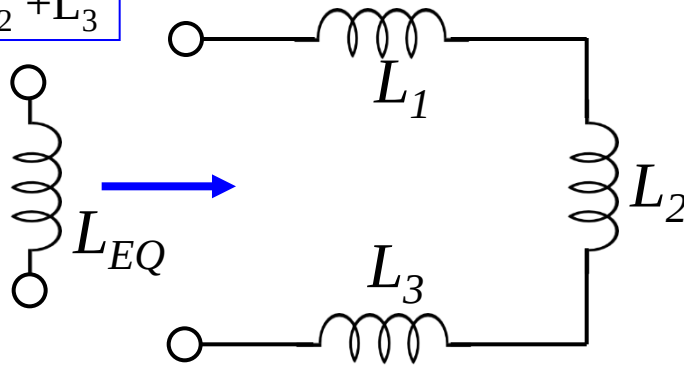
**Definitely a must
to memorize!**

Inversely, we can likewise express the inductor current in terms of the voltage across it by integrating the inductor voltage:

$$i_L(t) = \frac{1}{L} \int_{-\infty}^t v_L(t) dt$$

Inductors in Series and Parallel

$$L_{EQ} = L_1 + L_2 + L_3$$



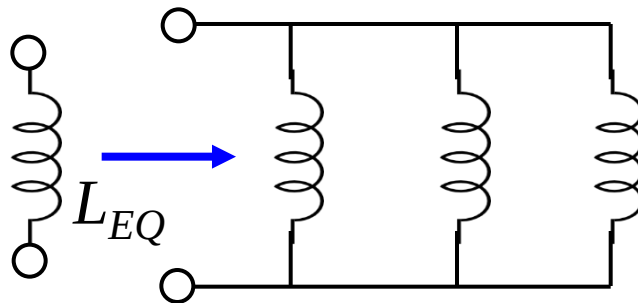
Proof

Applying KVL: $v(t) = v_1 + v_2 + v_3$

$$L_{EQ} \frac{di(t)}{dt} = L_1 \frac{di(t)}{dt} + L_2 \frac{di(t)}{dt} + L_3 \frac{di(t)}{dt}$$
$$L_{EQ} = L_1 + L_2 + L_3$$

In Series

In Parallel



Proof

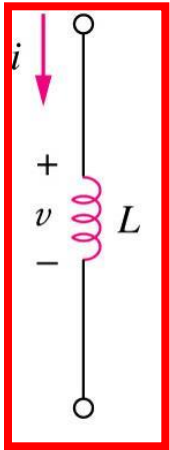
Apply KCL: $i = i_1 + i_2 + i_3$

$$\frac{1}{L_{EQ}} \int v(t) dt = \frac{1}{L_1} \int v(t) dt + \frac{1}{L_2} \int v(t) dt + \frac{1}{L_3} \int v(t) dt$$

$$\frac{1}{L_{EQ}} = \frac{1}{L_1} + \frac{1}{L_2} + \frac{1}{L_3}$$

DC and AC response in Inductor

Basic equation from the beginning



$$\phi(t) = L i(t)$$

$$V(t) = L \frac{di(t)}{dt}$$

If the current is reversed, then the direction of the flux is reversed as well

⇒ Voltage (due to Faraday's law)

If current is constant with time ⇒ $di/dt = 0$ ⇒ $V = 0$ (No voltage)

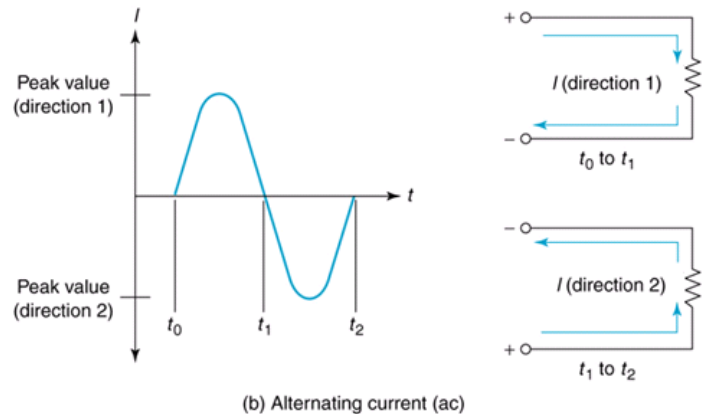
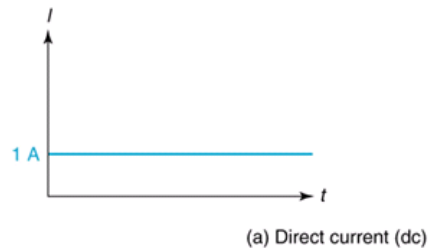
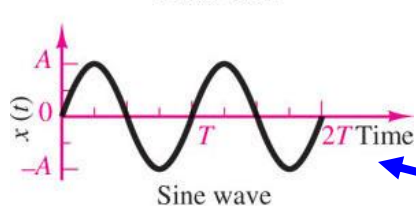
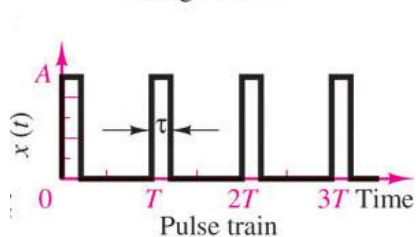
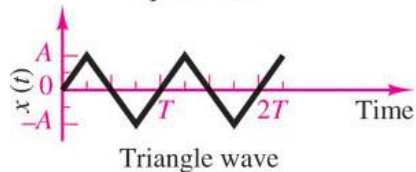
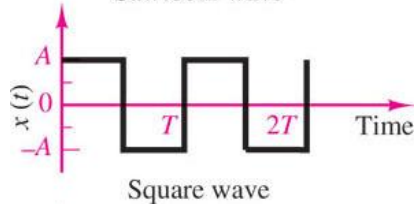
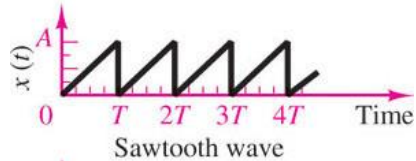
No current change ⇒ No current

If current changes with time ⇒ $dV/dt \neq 0$ ⇒ $I \neq 0$ (There is voltage)

Current change ⇒ Current

AC and Periodic Signals

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- Period signals are an important class of time-varying signals
- A periodic signal $x(t)$ satisfies the equation: $x(t) = x(t + nT)$, where T is the period

In this course, we will focus mainly on **sinusoids**

Sinusoids

A generalized sinusoid is defined as

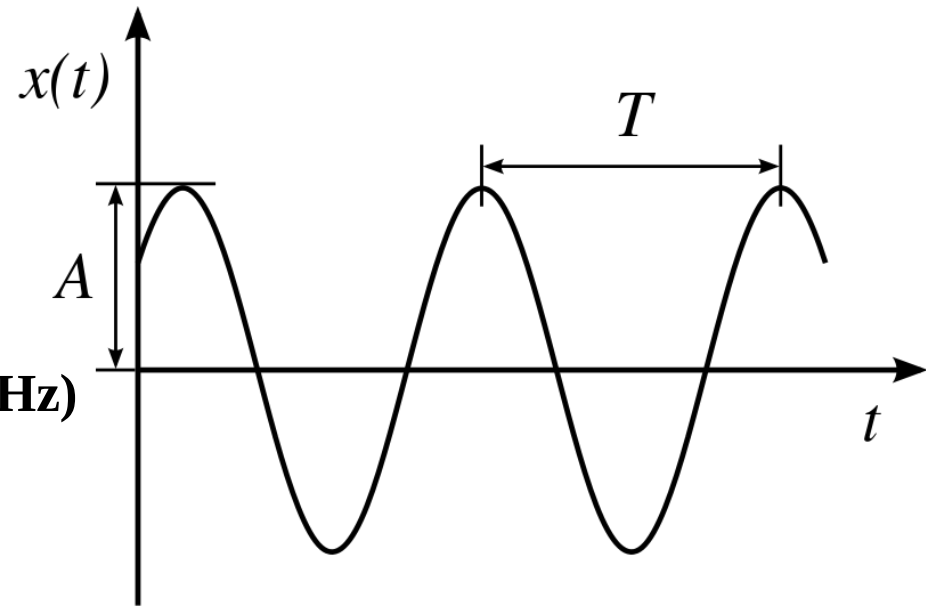
$$x(t) = A \cos(\omega t + \theta)$$

A = amplitude

ω = radian frequency = $2\pi f$ (rad/s)

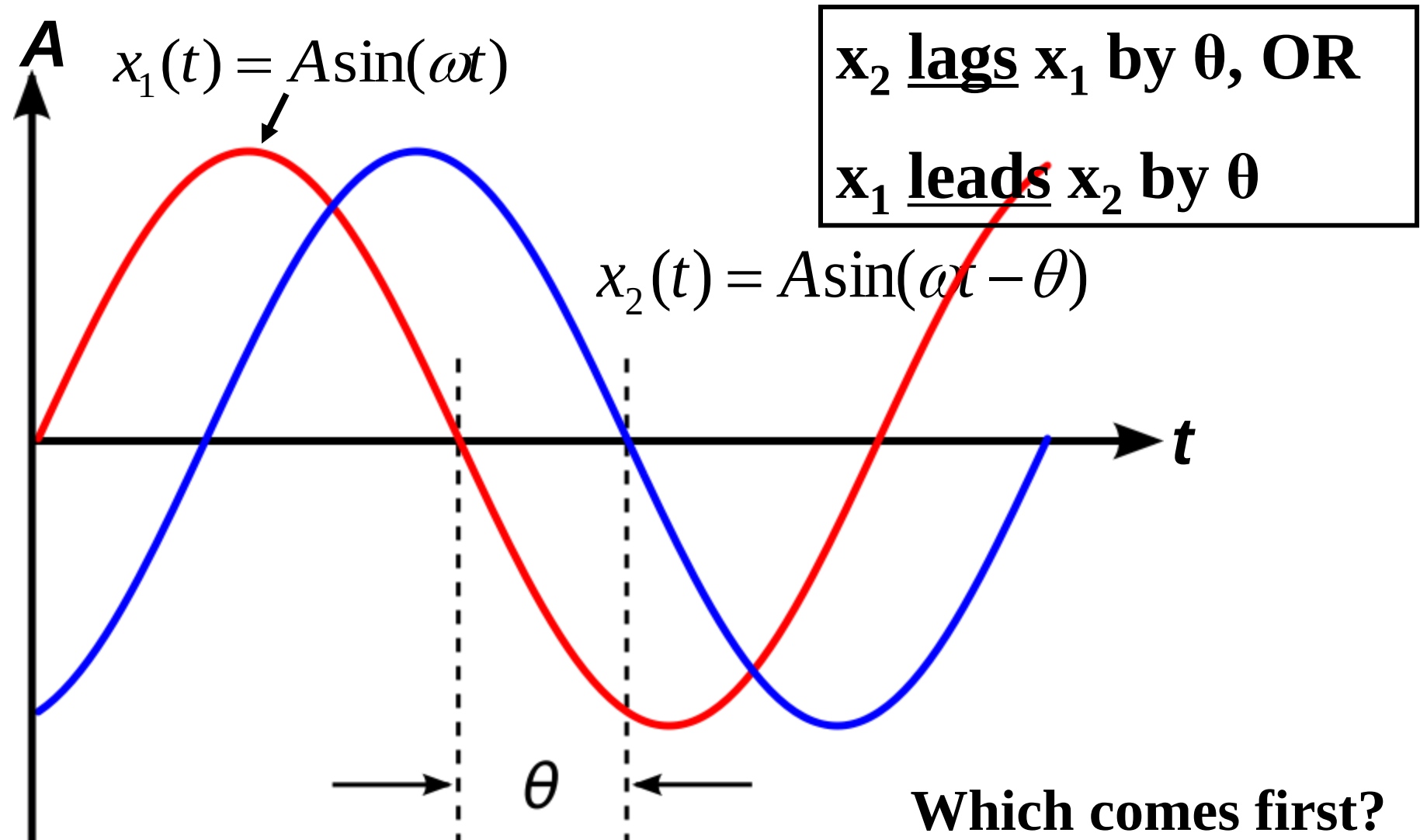
f = natural frequency = $1/T$ (cycles/s or Hz)

θ = phase = $2\pi(\Delta t/T)$ = rad



We can immediately see that an AC signal has a lot more key features than a DC signal. For a DC signal, the magnitude alone is a sufficient quantitative description. But in the case of sinusoidal AC signals, concepts like frequency and phase also need to be considered.

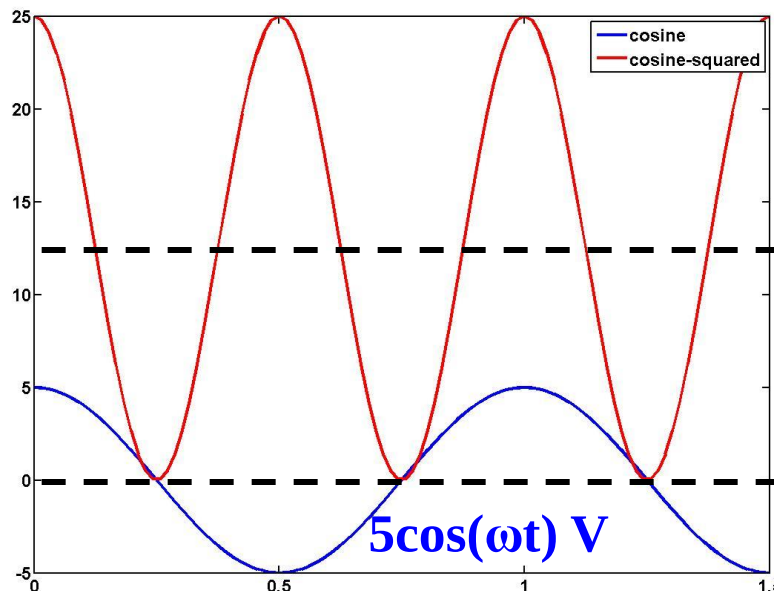
Sinusoids: Phase



Power dissipation in AC

Although the average voltage across a resistor with a sinusoidal AC voltage across it is zero, note that the average power dissipated is not zero. Since $P = V^2/R$, we should thus consider the **square of the voltage** across the resistor.

The following graph shows the voltage across a 1Ω resistor in blue and the corresponding square of this voltage in red. The red curve therefore shows the **INSTANTANEOUS POWER**.



The instantaneous power is clearly sinusoidal with a DC offset:
 $V^2(t) = 12.5\cos(\omega t) + 12.5$

Mean of $V_2 = 12.5$

But the average power is 12.5W

Mean = 0

Power dissipation in AC

Find the mean of the square
Integrate over one period

$$\langle v^2 \rangle = \frac{1}{T} \int_0^T v^2(t) dt = \frac{1}{T} \int_0^T [V_{peak} \sin(\omega t)]^2 dt$$

$$= \frac{V_{peak}^2}{T} \int_0^T \sin^2(\omega t) dt$$

$$= \frac{V_{peak}^2}{T} \int_0^T \frac{1}{2} [1 - \cos(2\omega t)] dt$$

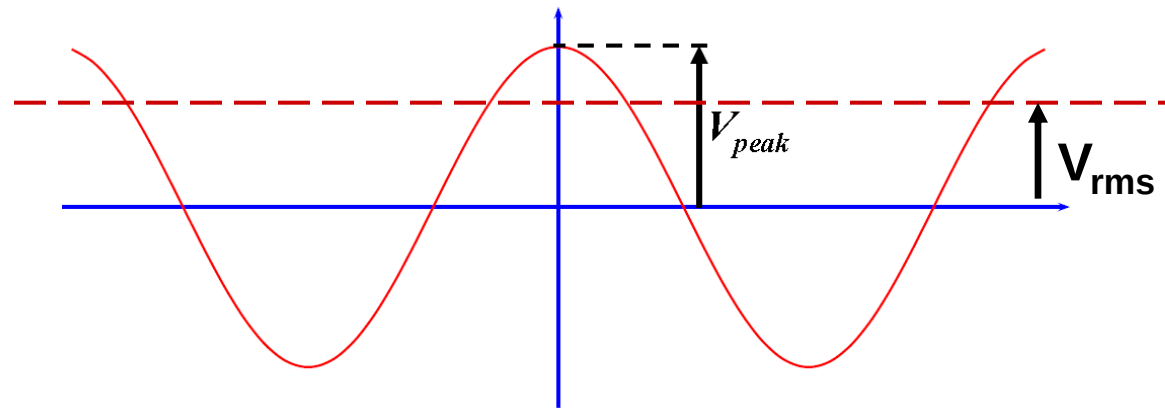
$$= \frac{V_{peak}^2}{2}$$

$$V_{rms} = \frac{1}{\sqrt{2}} V_{peak}$$

Therefore power dissipated in the resistor is

$$P = \frac{V_{peak}^2}{2R} = \frac{V_{rms}^2}{R}$$

The RMS (root mean square) is the equivalent DC value that causes the same average power to be dissipated by the resistor



Power dissipation example

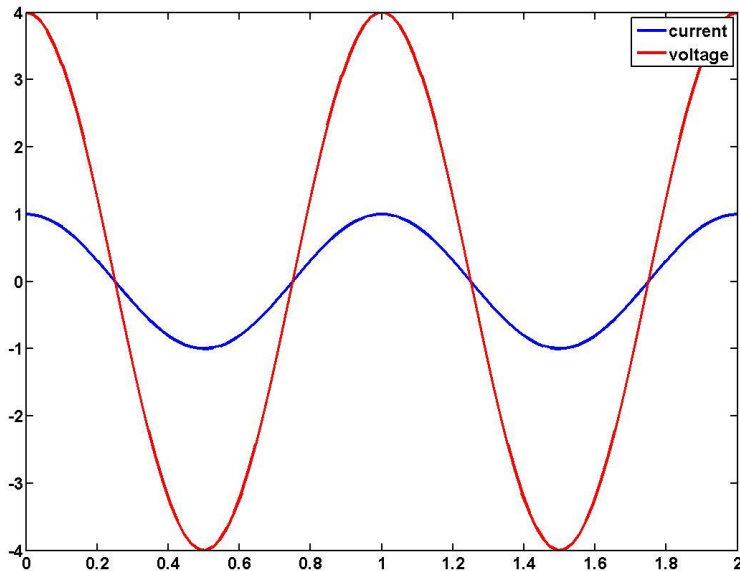
Find the power dissipated in a 1Ω resistor if the voltage drop across it is:

1) $2(\sin 100t)$ V

2) $5 + 3(\sin 100t)$ V

3) $4 + 2(\sin 50t) + 3(\sin 100t)$ V

I-V relationship in a resistor



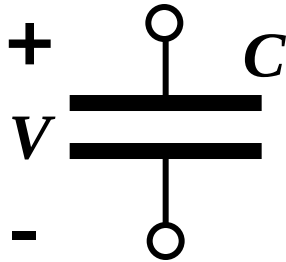
This plot shows the relationship between the current and voltage in a 4Ω resistor when the signal is an AC sinusoid.

It can be seen that there is no phase difference between the voltage and current.

This is because R is simply given by the ratio of voltage to current and there is therefore no phase shift between the two, but simply a scaling in the amplitude.

I-V relationship in a capacitor

$$I = C(dV/dt)$$



If the voltage across a capacitor is:

$$V(t) = V\cos(\omega t)$$

Then the current through it will be:

$$I(t) = C (dV/dt)$$

$$= \omega CV\sin(\omega t)$$

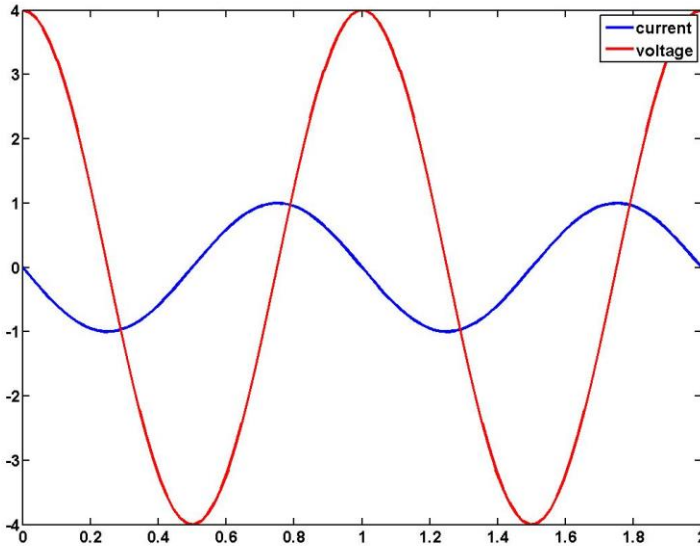
$$= \omega CV\cos(\omega t + 90^\circ)$$

Therefore:

$$I(t) = -\omega CV\sin(\omega t) = \omega CV\cos(\omega t + 90^\circ)$$

Shift by 90°

I-V relationship in a capacitor



If the voltage across a capacitor is:

$$V(t) = V \cos(\omega t)$$

Then the current through it will be:

$$I(t) = \omega C V \cos(\omega t + 90^\circ)$$

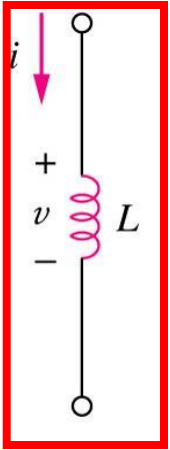
We can see therefore that in a capacitor:

Current leads voltage $\pi/2$ $\frac{V}{I} = \frac{1}{\omega C} \angle -\frac{\pi}{2}$

From this, a number of key points should be highlighted:

- 1) As mentioned, the current goes through a phase shift **relative** to the voltage (in a resistor, there is no phase shift between the current and voltage)
- 2) The ratio of voltage to current depends not only on the **capacitance**, but the **frequency** of the sinusoid as well (in a resistor, the ratio between voltage and current is simply R and thus independent of frequency)

I-V relationship in an inductor



$$v(t) = L \frac{di(t)}{dt}$$

If the current through a capacitor is:

$$i(t) = I \cos(\omega t)$$

Then the current through it will be:

$$v(t) = L (di/dt)$$

$$= -\omega L I \sin(\omega t)$$

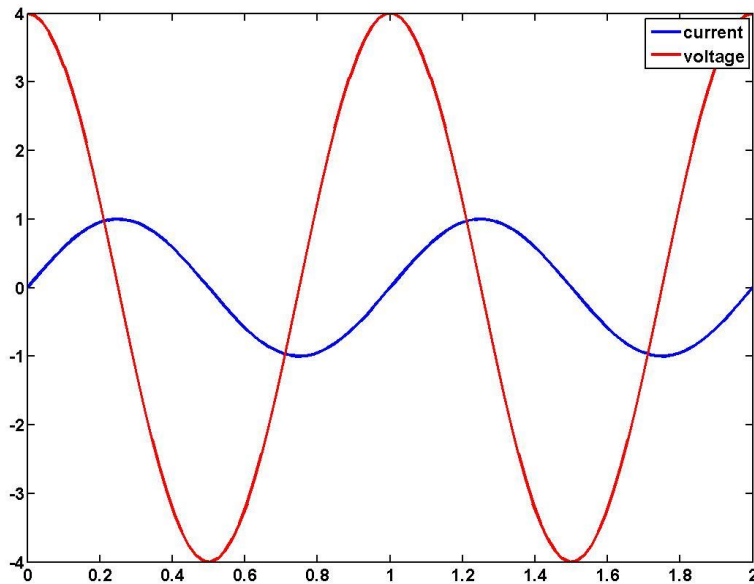
$$= \omega L I \cos(\omega t + 90^\circ)$$

Therefore:

$$v(t) = -\omega L I \sin(\omega t) = \omega L I \cos(\omega t + 90^\circ)$$

Shift by 90°

I-V relationship in an inductor



If the current through an inductor is:

$$I(t) = I \cos(\omega t)$$

Then the voltage across it will be:

$$V(t) = \omega L I \cos(\omega t + 90^\circ)$$

We can see therefore that in an inductor:

Current lags voltage $\pi/2$ $\frac{V}{I} = \omega L \angle 90^\circ$

The same key points can be observed from here once again:

- 1) As mentioned, the current goes through a phase shift **relative** to the voltage (in a resistor, there is no phase shift between the current and voltage)
- 2) The ratio of voltage to current depends not only on the **inductance**, but the **frequency** of the sinusoid as well (in a resistor, the ratio between voltage and current is simply R and thus independent of frequency)

Impedance and Resistance

- Definition of resistance: $R = V/I$
 - > R is simply a ratio value
- With AC components, this definition of V/I is made more general
 - > We call the ratio of V/I as the impedance
 - > Impedance would contain (1) the ratio value of V/I , and (2) the relative phase difference between V and I

Phasors and Complex numbers

- In DC signal analysis, only the magnitudes of the signals are important
- In AC signal analysis, both the magnitude **as well as** phase are important
- Use phasors and complex numbers to do this (show both magnitude and phase)

Magnitude $\longrightarrow A \angle \theta \longleftarrow$ Phase

Euler's identity

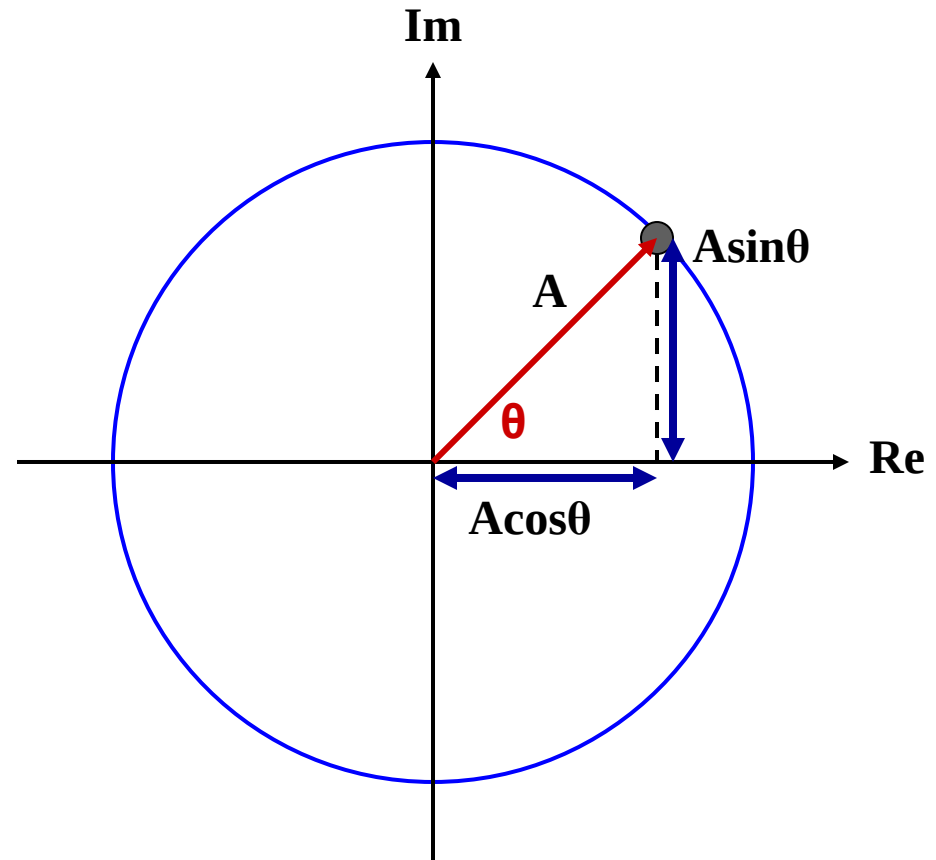
Complex Number

$$Ae^{j\theta} = A(\cos\theta + j\sin\theta)$$

In electrical engineering, we use 'j' to denote the imaginary part instead of 'i' so as not to confuse it with current.

Real part: $A\cos\theta$

Imaginary part: $A\sin\theta$



Phasor

$$A \angle \theta$$

In this course, we use phasors (and complex numbers) as a power tool to do analysis on AC circuits. It helps us to take care of both the phase and magnitude at the same time.

Impedance

Impedance is defined as the relationship of the voltage across an element over the current flowing through it. It describes both the ratio of their amplitudes as well as the phase relationship

For a **Capacitor**, the impedance is given by:

$$\frac{V}{I} = \frac{1}{j\omega C} = \frac{1}{\omega C} \angle -90^\circ$$

Derivation:

$$\text{Let } V = V_{\max} \exp(j\omega t)$$

$$\text{Given that } I = C(dV/dt)$$

$$\begin{aligned} \text{Therefore: } I &= C \frac{d}{dt} \{V_{\max} \exp(j\omega t)\} \\ &= j\omega C V_{\max} \exp(j\omega t) \\ &= j\omega C V \end{aligned}$$

For an **Inductor**, the impedance is given by:

$$\frac{V}{I} = j\omega L = \omega L \angle 90^\circ$$

Derivation:

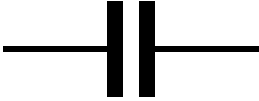
$$\text{Let } I = I_{\max} \exp(j\omega t)$$

$$\text{Given that } V = L(dI/dt)$$

$$\begin{aligned} \text{Therefore: } V &= L \frac{d}{dt} \{I_{\max} \exp(j\omega t)\} \\ &= j\omega L I_{\max} \exp(j\omega t) \\ &= j\omega L I \end{aligned}$$

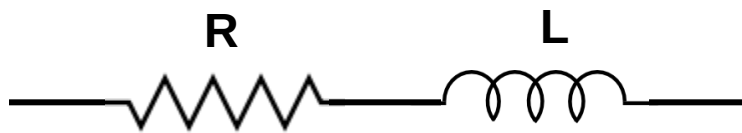
Impedance Summary

Memorize this

Element	Symbol	Impedance	I-V relation
Resistor		R	In phase
Inductor		$j\omega L$	I lags V by
Capacitor		$1/j\omega C$	I leads V by

“Complex” Impedance

- So far, examples of impedances are either purely real or purely imaginary
- Consider the following combination



Since in R & L are in series,
total impedance is: $Z = R + j\omega L$

General form: $Z = R + jX$

Real part is called
the **AC resistance**

Imaginary part is
called the **reactance**

We can combine impedance values like in DC circuits but now treating values as complex numbers and taking into account the effect of frequency.

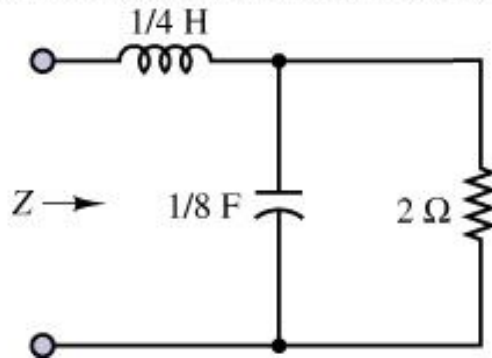
Impedance: example 1

Problem 4.61

Given: $\omega = 4 \text{ rad/s}$

Work out the impedance (Z) seen across the terminals

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Impedance of L:

$$X_L = j\omega L = j(4)(1/4) = j$$

Impedance of C:

$$X_C = 1/j\omega C = 1/[j(4)(1/8)] = -j2$$

Impedance: example 1 solution

Impedance of C and parallel with R: $Z_{RC} = \frac{X_C R}{R + X_C}$

$$= \frac{(-j2)(2)}{2 - j2}$$
$$= 1 - j$$

Z_{RC} is in series with impedance of L: $Z = Z_{RC} + X_L$

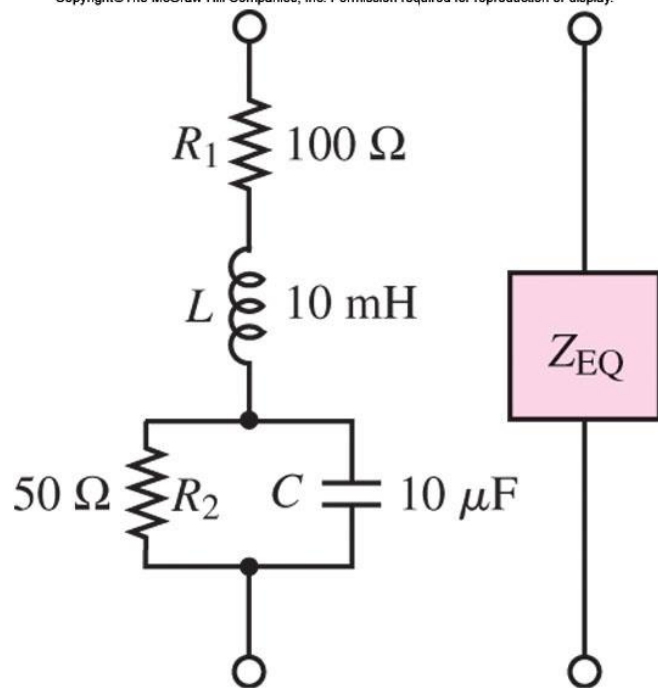
$$= 1 - j + j$$
$$= 1\Omega$$

Impedance: example 2

Example 4.14 of Rizzoni

Find the equivalent resistance (Z_{EQ}) and admittance seen across the terminals

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Impedance of L:

$$X_L = j\omega L = j(10^4)(10^{-2}) = j100$$

Impedance of C:

$$X_C = 1/j\omega C = 1/[j(10^4)(10^{-5})] = -j10$$

Given: $\omega = 10^4\text{ rad/s}$

Impedance: example 2 solution

Impedance of C parallel R_2 :

$$\begin{aligned}Z_{RC} &= \frac{X_C R}{R + X_C} \\&= \frac{(-j10)(50)}{50 - j10} \\&= \frac{500}{10 + j50} \\&= \frac{50}{1 + j5}\end{aligned}$$

$$\begin{aligned}Z_{EQ} &= Z_{RC} + Z_{RL} \\&= \frac{50}{1 + j5} + 100(1 + j) \\&= 101.92 + j90.385 \\&= 136.2 \angle 41.57^\circ \Omega\end{aligned}$$

$$\begin{aligned}Y_{EQ} &= 1/Z_{EQ} = (1/136.2) \angle -41.6^\circ S \\&= 7.342 \angle -41.6^\circ mS\end{aligned}$$

Impedance of L series R_1 :

$$\begin{aligned}Z_{RL} &= X_L + R \\&= 100 + j100 \\&= 100(1 + j)\end{aligned}$$

Admittance

Admittance is simply the **inverse** of the **impedance**: $Y = 1/Z$

Admittance likewise comprises both real and imaginary parts

$$Y = G + jB$$

Real part is called the **AC conductance**

Imaginary part is called the **susceptance**

$$Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{1}{R + jX} \left(\frac{R - jX}{R - jX} \right) \\ = \frac{R - jX}{R^2 + X^2}$$

$$G = \frac{R}{R^2 + X^2}$$

$$B = -\frac{X}{R^2 + X^2}$$

Example on Admittance

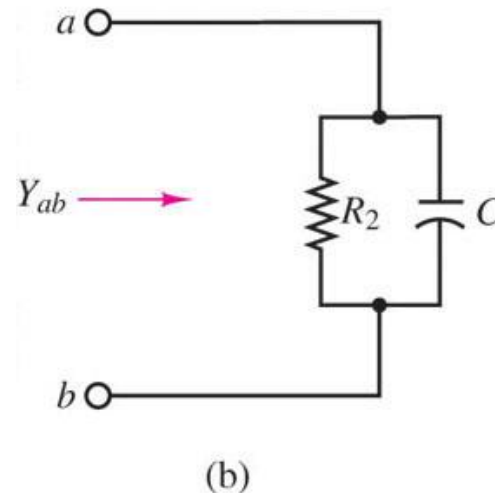
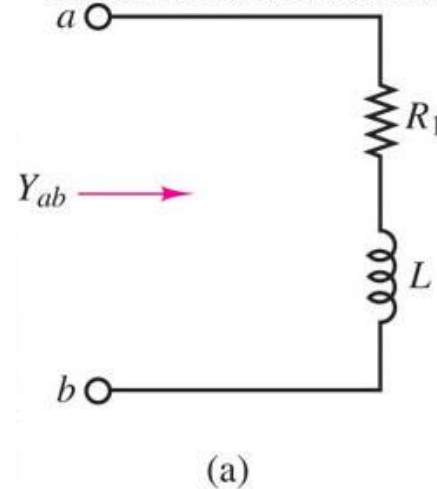
$$Z_{ab} = R_1 + j\omega L$$

$$Y_{ab} = 1/Z_{ab} = 1/(R_1 + j\omega L)$$

$$Y_{ab} = \frac{R_1}{R_1^2 + (\omega L)^2} - \frac{j\omega L}{R_1^2 + (\omega L)^2}$$

$$Y_{ab} = 1/Z_{ab} = 1/R_2 + j\omega C$$

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Summary

- Capacitors: Passes current at high frequency but blocks DC
- Inductor: Acts as short circuit at DC and block at high frequency
- Impedance introduced as a more general form to describe V/I
- Impedance depends on frequency
- Represent as these as phasors so we can add up impedances just like resistances in DC