

EE3114 Systems and Control

Tutorial - 4

Semester B, 2017-18

Review: ODE

Second-Order ODE: $a\ddot{y} + b\dot{y} + c = 0 \quad (a \neq 0)$

Characteristic Equation: $as^2 + bs + c = 0$

$b^2 - 4ac > 0$:

Two distinct real roots: $r_1 \neq r_2 \rightarrow y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

$b^2 - 4ac < 0$:

Two complex conjugate roots: $r_{1,2} = \sigma \pm j\omega_d$

$\rightarrow y(t) = C_0 e^{\sigma t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$

$b^2 - 4ac = 0$:

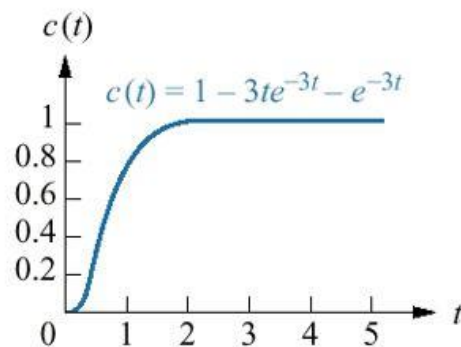
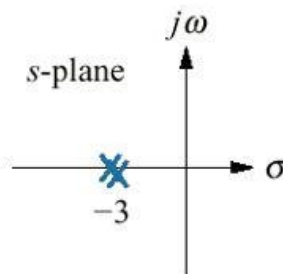
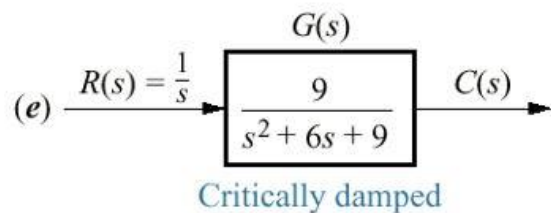
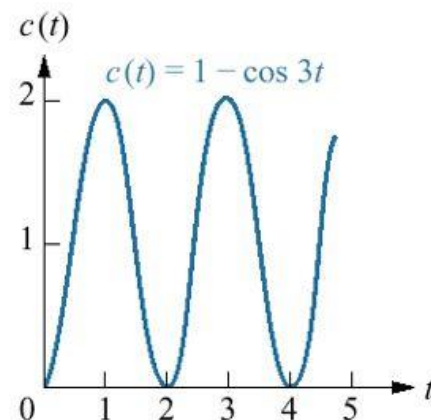
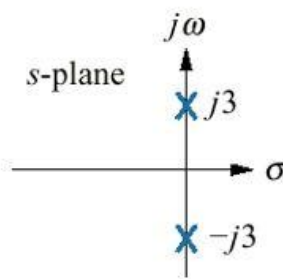
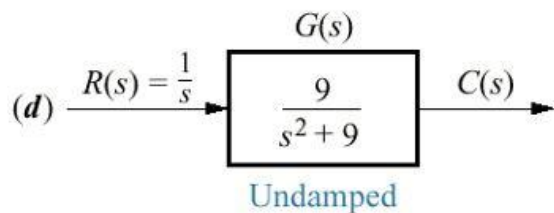
Two same real roots: $r_1 = r_2 = r \rightarrow y(t) = C_1 e^{rt} + C_2 t e^{rt}$

Open-Loop System Stability

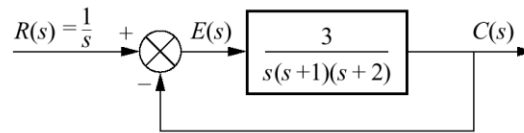
System	Pole-zero Plot	Response
(a) $R(s) = \frac{1}{s}$ $\rightarrow$ $G(s) = \frac{b}{s^2 + as + b}$ $\rightarrow$ $C(s)$ General		
(b) $R(s) = \frac{1}{s}$ $\rightarrow$ $G(s) = \frac{9}{s^2 + 9s + 9}$ $\rightarrow$ $C(s)$ Overdamped		$c(t) = 1 + 0.171e^{-7.854t} - 1.171e^{-1.146t}$
(c) $R(s) = \frac{1}{s}$ $\rightarrow$ $G(s) = \frac{9}{s^2 + 2s + 9}$ $\rightarrow$ $C(s)$ Underdamped		$c(t) = 1 - e^{-t}(\cos\sqrt{8}t + \frac{\sqrt{8}}{8} \sin\sqrt{8}t)$ $= 1 - 1.06e^{-t} \cos(\sqrt{8}t - 19.47^\circ)$

Open-Loop System Stability

Marginally stable
(poles are on the imaginary axis)

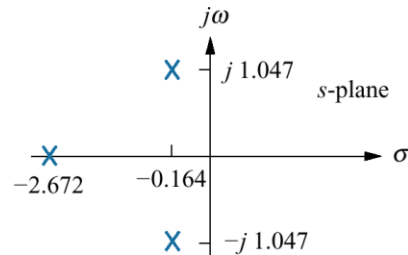


Closed-Loop System Stability

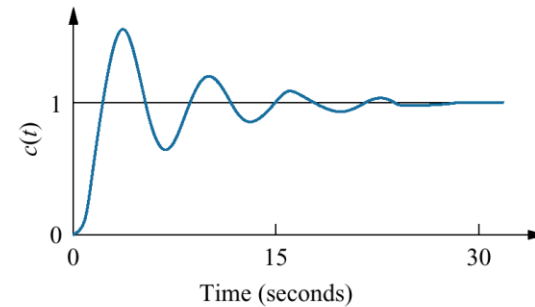


Stable system

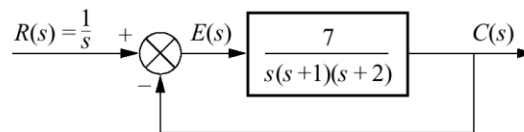
(Asymptotically)
Stable →



Stable system's
closed-loop poles
(not to scale)

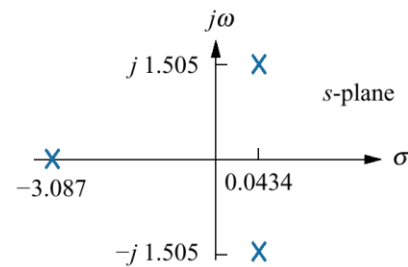


(a)

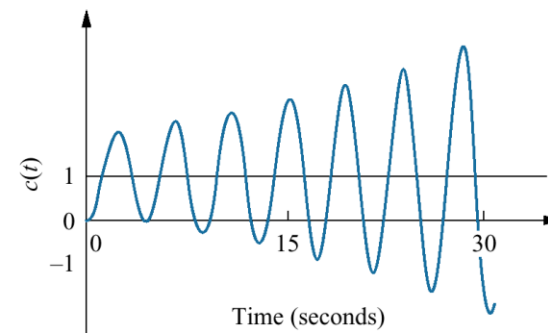


Unstable system

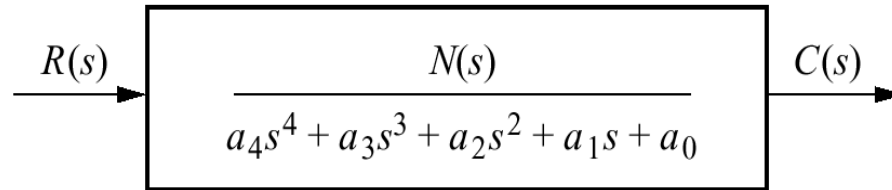
Unstable →



Unstable system's
closed-loop poles
(not to scale)



(b)



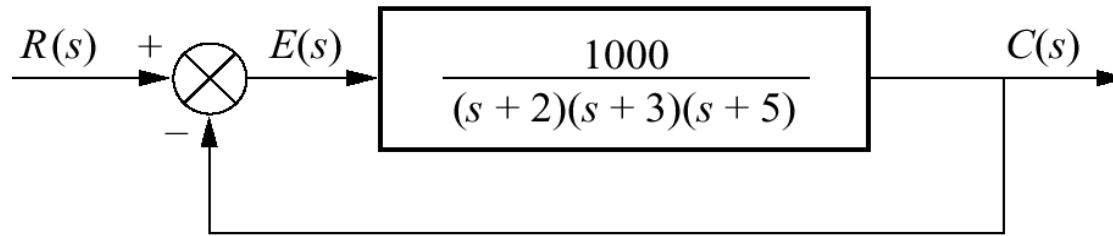
Routh Table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

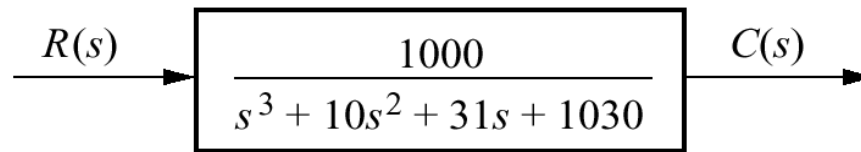
s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$-\frac{\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$-\frac{\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$-\frac{\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$-\frac{\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$-\frac{\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$-\frac{\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$-\frac{\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

↑ If no sign changes → stable

Example:



(a)



(b)

$$\begin{aligned} (b) \quad \Rightarrow \quad \frac{G(s)}{1+G(s)} &= \frac{\frac{1000}{(s+2)(s+3)(s+5)}}{1 + \frac{1000}{(s+2)(s+3)(s+5)}} \\ &= \frac{1000}{(s+2)(s+3)(s+5) + 1000} = \frac{1000}{s^3 + 10s^2 + 31s + 1030} \end{aligned}$$

$$H(s) = \frac{1000}{s^3 + 10s^2 + 31s + 1030}$$

s^3	1	31	0
s^2	~~10~~ 1	~~1030~~ 103	0
s^1	$-\frac{\begin{vmatrix} 1 & 31 \\ 1 & 103 \end{vmatrix}}{1} = -72$	$-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$	$-\frac{\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix}}{1} = 0$
s^0	$-\frac{\begin{vmatrix} 1 & 103 \\ -72 & 0 \end{vmatrix}}{-72} = 103$	$-\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$	$-\frac{\begin{vmatrix} 1 & 0 \\ -72 & 0 \end{vmatrix}}{-72} = 0$



Two sign changes → two unstable poles
(on RHP)

Example:

$$G(s) = \frac{10}{s^5 + 2s^4 + 3s^3 + 6s^2 + 5s + 3}$$

	s^5	1	3	5
No sign change →	s^4	2	6	3
	s^3	$\cancel{0} \ \epsilon \ (> 0)$	$\frac{7}{2}$	0
Sign change →	s^2	$\frac{6\epsilon - 7}{\epsilon}$	3	0
Sign change →	s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	0	0
No sign change →	s^0	3	0	0

Label	First Column	$\epsilon = +$	$\epsilon = -$
s^5	1	+	+
s^4	2	+	+
s^3	$\cancel{0} \ \epsilon$	+	-
s^2	$\frac{6\epsilon - 7}{\epsilon}$	-	+
s^1	$\frac{42\epsilon - 49 - 6\epsilon^2}{12\epsilon - 14}$	+	+
s^0	3	+	+

Two sign changes → Unstable (2 poles on RHP)

Example:

$$D(s) = 3s^5 + 5s^4 + 6s^3 + 3s^2 + 2s + 1$$

s^5	3	6	2
s^4	5	3	1
s^3	4.2	1.4	
s^2	1.33	1	
s^1	-1.75		
s^0	1		



Two sign changes → Unstable (2 poles on RHP)

Example:

$$D(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$$

Solution:

s^5	1	11	28	
s^4 :	5	23	12	
s^3 :	6.4	25.6	0	
s^2 :	3	12		$\leftarrow Q'(s) = 3s^2 + 12$
~~s:~~	~~0~~	~~0~~		
NEW	s :	6	0	$\leftarrow \frac{dQ(s)}{ds} = 6s$
	s^0 :	12		

No sign changes in first column $\rightarrow$ all roots have negative real parts except for a pair on the imaginary axis: roots of $Q(s) = \pm j2$

Note: original roots are at $-3, \pm j2, -1, -1$

$$\begin{aligned}
 D(s) &= Q(s)\tilde{D}(s) = (3s^2 + 12) \left(\frac{s^3}{3} + \frac{5s^2}{3} + \frac{7s}{3} + 1 \right) \\
 &= (s^2 + 4)(s^3 + 5s^2 + 7s + 3) = (s + j2)(s - j2)(s + 3)(s + 1)^2
 \end{aligned}$$

Example:

$$D(s) = s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

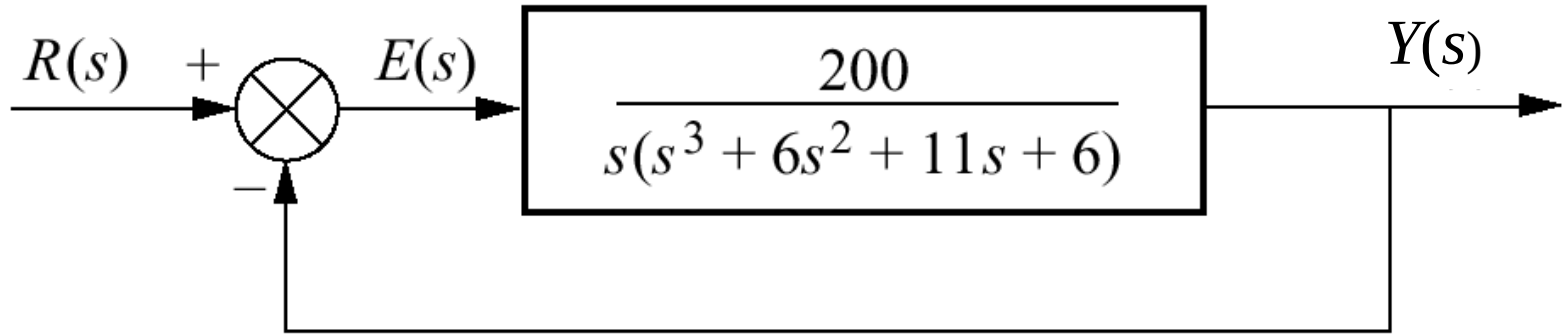
Routh Table

s^5	1	24	-25	
s^4	2	48	-50	
s^3	0	0	0	$Q = 2s^4 + 48s^2 - 50$
s^3	8	96		$Q' = 8s^3 + 96s$
s^2	24	-50		
s^1	112.7	0		
s^0	-50			

One sign change → Unstable

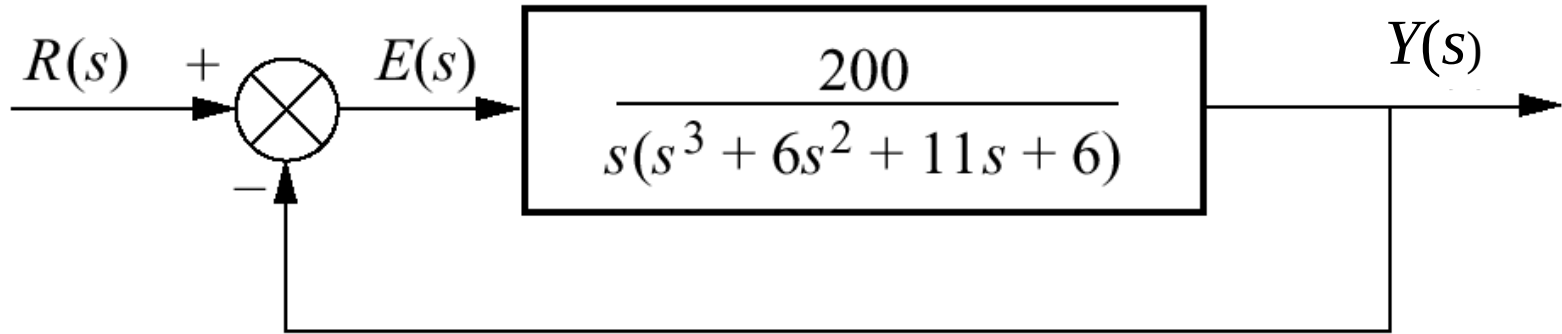
$$\begin{aligned} D(s) &= s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 \\ &= (2s^4 + 48s^2 - 50)(s/2 + 1) \end{aligned}$$

Exercise 1:



(1) Closed-loop Transfer Function = ? (5 minutes)

Exercise 1:



(1) Closed-loop Transfer Function = ?

Solution:

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{G(s)}{1 + G(s)} = \frac{\frac{200}{s(s^3 + 6s^2 + 11s + 6)}}{1 + \frac{200}{s(s^3 + 6s^2 + 11s + 6)}} \\ &= \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}\end{aligned}$$

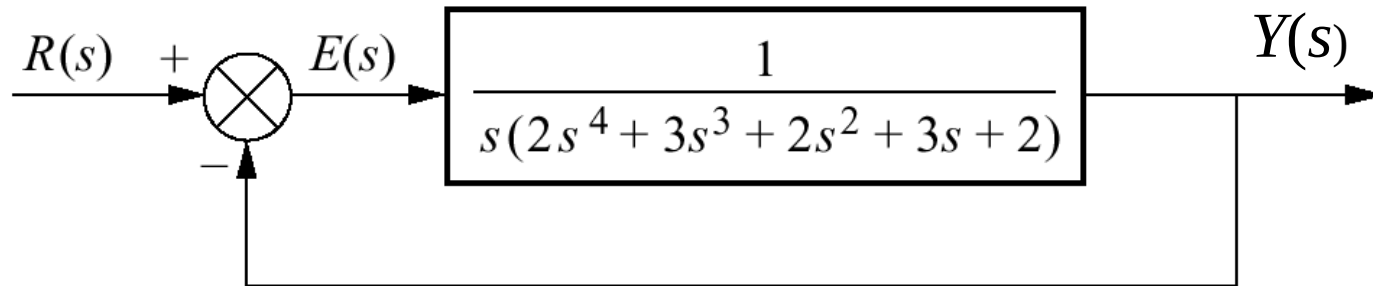
(2) Determine its stability using Rough Table (5 minutes)

(2) Determine its stability using Rough Table (Solution)

s^4	1	11	200
s^3	~~6~~ 1	~~6~~ 1	
s^2	~~10~~ 1	~~200~~ 20	
s^1	-19		
s^0	20		

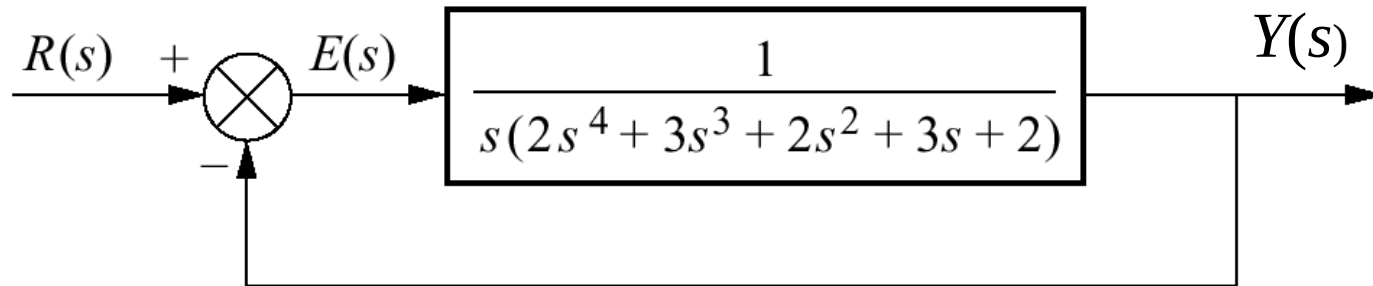
Two sign changes → unstable (with 2 poles on RHP)

Exercise 2:



(1) Closed-loop Transfer Function = ? (5 minutes)

Exercise 2:



(1) Closed-loop Transfer Function = ?

Solution:

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{G(s)}{1 + G(s)} = \frac{\frac{1}{s(2s^4 + 3s^3 + 2s^2 + 3s + 2)}}{1 + \frac{1}{s(2s^4 + 3s^3 + 2s^2 + 3s + 2)}} \\ &= \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}\end{aligned}$$

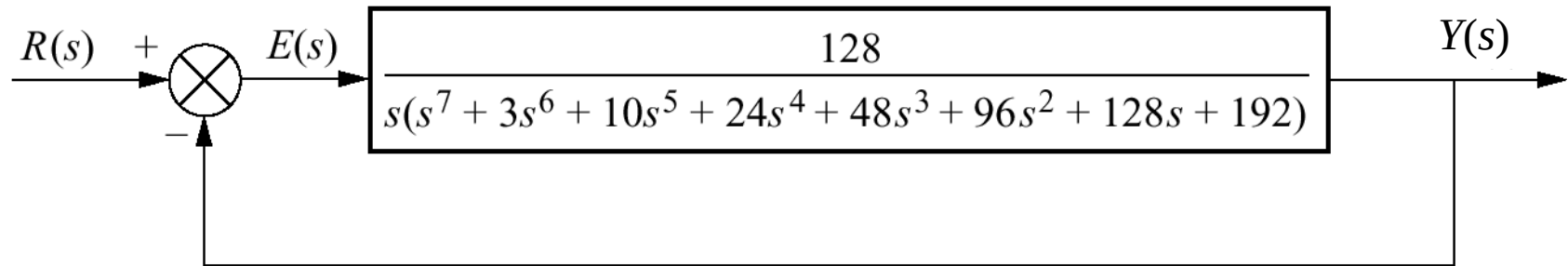
(2) Determine its stability using Rough Table (5 minutes)

(2) Determine its stability using Rough Table (Solution)

s^5	2	2	2
s^4	3	3	1
s^3	$\cancel{0} \ \epsilon$	$\frac{4}{3}$	
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$		
s^0	1		

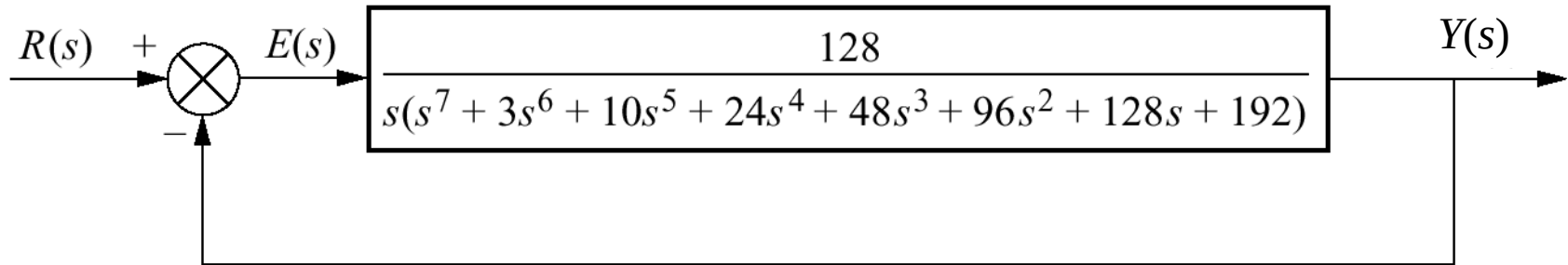
Two sign changes $\rightarrow$ unstable (with 2 poles on RHP)

Exercise 3:



(1) Closed-loop Transfer Function = ? (5 minutes)

Exercise 3:



(1) Closed-loop Transfer Function = ? (Solution)

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{G(s)}{1+G(s)} = \frac{\frac{128}{s(s^7 + 3s^6 + 10s^5 + 24s^4 + 48s^3 + 96s^2 + 128s + 192)}}{1 + \frac{128}{s(s^7 + 3s^6 + 10s^5 + 24s^4 + 48s^3 + 96s^2 + 128s + 192)}} \\ &= \frac{1}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128}\end{aligned}$$

(2) Determine its stability using Rough Table (10 minutes)

(2) Determine its stability using Rough Table (Solution)

s^8	1	10	48	128	128
s^7	~~3~~ 1	~~24~~ 8	~~96~~ 32	~~192~~ 64	
s^6	~~2~~ 1	~~16~~ 8	~~64~~ 32	~~128~~ 64	
s^5	~~0~~ ~~6~~ 3	~~0~~ ~~32~~ 16	~~0~~ ~~64~~ 32	~~0~~ ~~0~~ 0	
s^4	$\frac{8}{3}$ 1	$\frac{64}{3}$ 8	~~64~~ 24		
s^3	~~-8~~ -1	~~-40~~ -5			
s^2	~~3~~ 1	~~24~~ 8			
s^1	3				
s^0	8				

Two sign changes → unstable

$$Q(s) = (s^6 + 8s^4 + 32s^2 + 64)(s^2 + 3s + 2)$$

END