

Lecture 3: Numerical Errors

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Motivation

What are **numerical methods**?

Numerical methods are techniques for performing **mathematical operations** using computers fast and accurately!

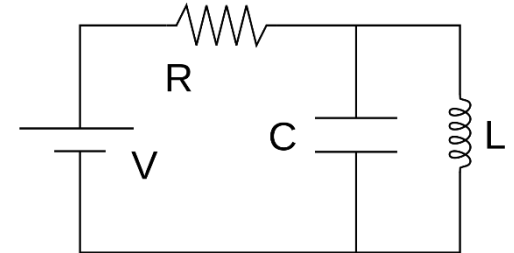
Examples: solving linear equations, integration, differentiation, ...

$$\int$$

$$\frac{dy}{dx}$$

$$Ax=b$$

These operations come up in many engineering applications!



Motivation

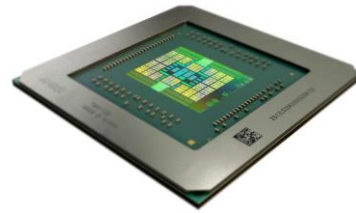
Why are numerical methods important for engineers?

Motivation

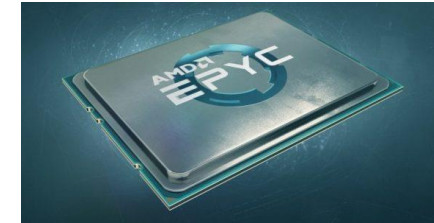
Example: IC design



Stratix 10 GX 10M
More than 43 billion transistors

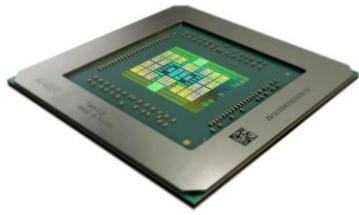


Navi 10
more than 10 billion transistors

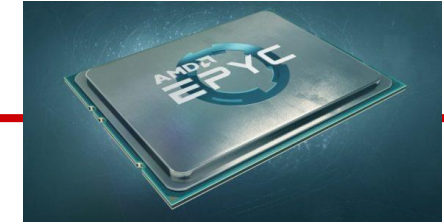


AMD Epyc Rome
Around 10 billion transistors

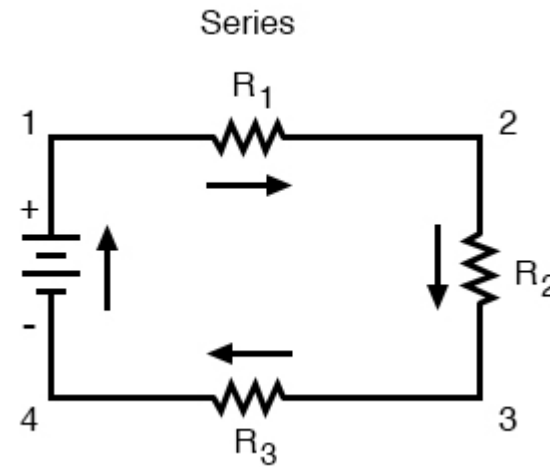
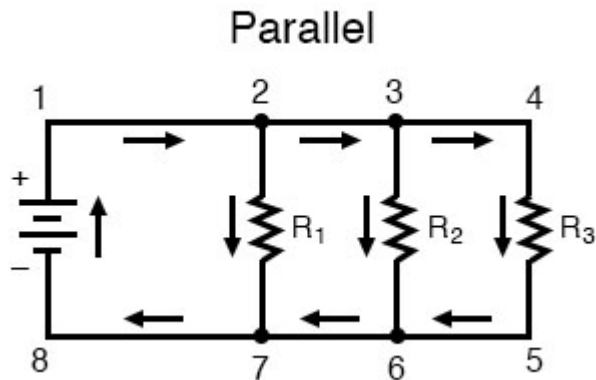
Many **engineering design** problems are **large-scale**!



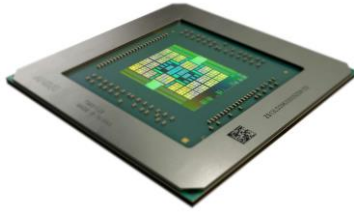
Motivation



Closed-form solutions only exist in simple cases.



Imagine **manually** finding the voltages and currents of a circuit with billions of components!



Motivation



We know how to solve circuits using **current and voltage** laws.

Imagine designing and analyzing a circuit with 40 billion components!

Manual design and analysis are **impossible**!

Computers can perform calculations much faster than us! (few billion operations per second!)

We have to **use computers** for the design and analysis!

Hence, we need **numerical methods**!

Motivation

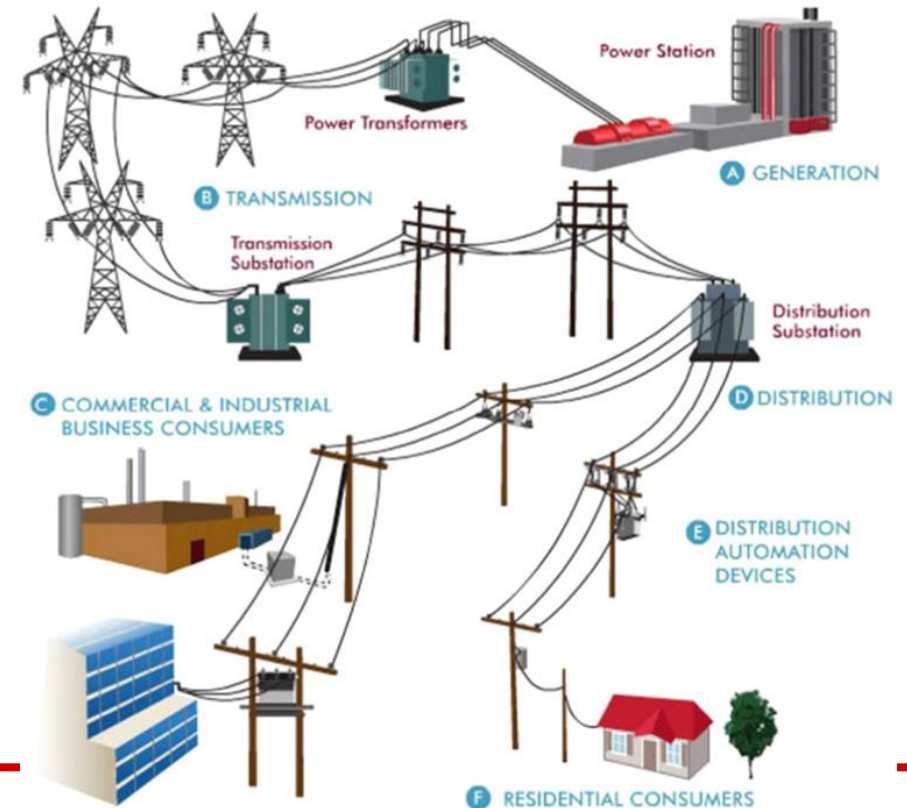
Example: **electricity grid!**

Thousands of generators!

Millions of consumers!

Manual analysis and design are almost **impossible!**

We need computers for the **design and analysis!**





Motivation

Many engineering systems are designed using **computers**.

Theory describes the laws of physics.

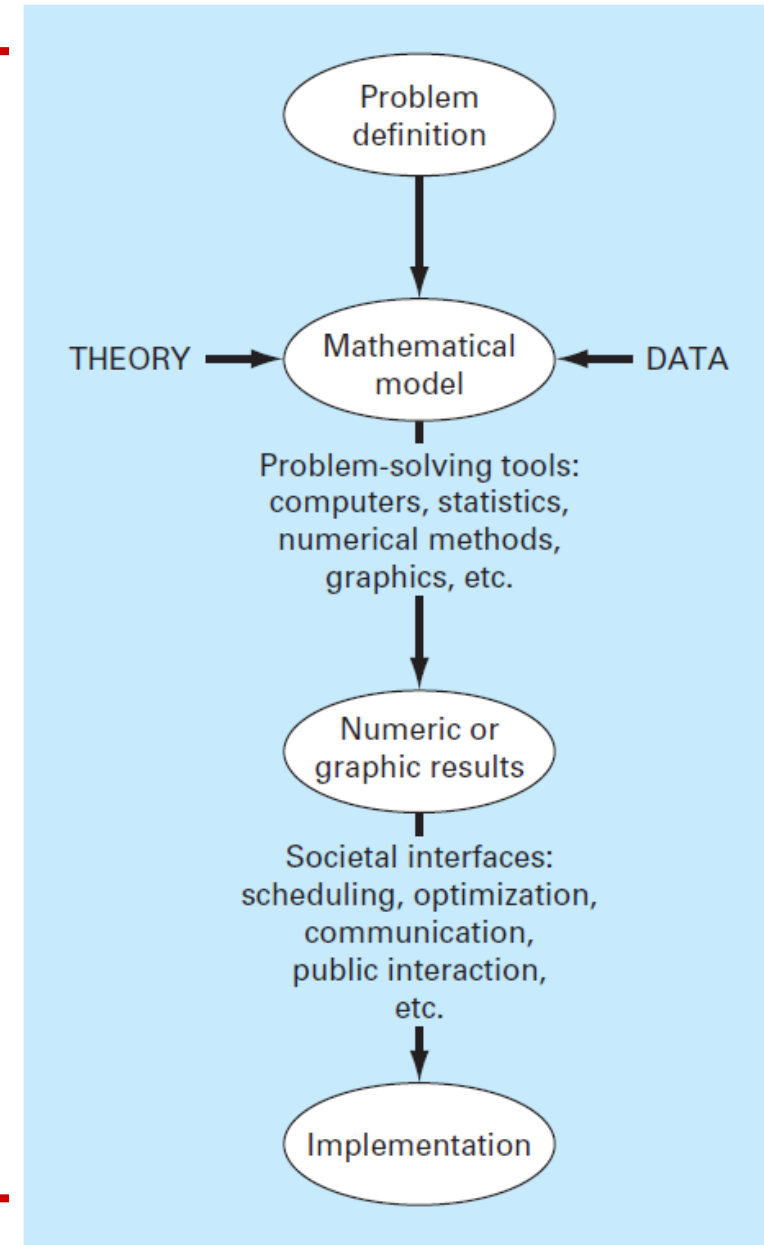
Example: Kirchhoff voltage and current laws.

Data are the parameters of the system.

Example: **values** of resistors, capacitors and inductors.

Analysis will be performed using **numerical methods**.

Example: Finding the values of voltages and currents.



Error Analysis

This lecture is about the **error analysis** of **numerical computations**.

Why is **error analysis** important?

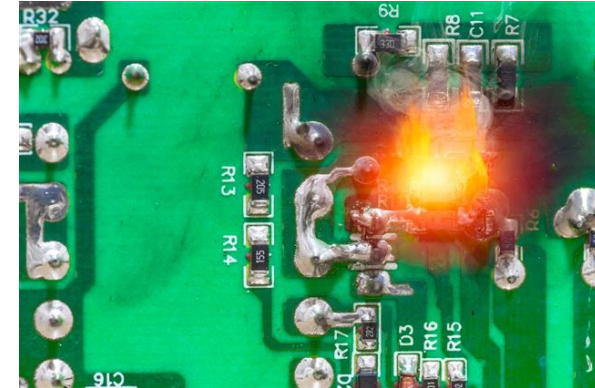
Error Analysis

Performing numerical calculations may introduce **errors**, e.g.,

round-off error

truncation error

We need to be aware of errors. Otherwise, our design may fail!



Error Analysis



Computers have limited **memory capacities**!

Imagine you want to store π on a computer ($\pi = 3.1415926535 \dots$)

Even if we use **all the computers**, we cannot store π !

We need to round it off. For example, store **3.1415** for π .

However, this is an approximation and introduces a **round-off error**.

$$\pi - 3.1415 = 0.0000926535 \dots$$

Error Analysis

Computers can only perform **addition**!



Basic operations, e.g., multiplication, division, are done using addition.

Example: $4 \times 6 = 6 + 6 + 6 + 6$!

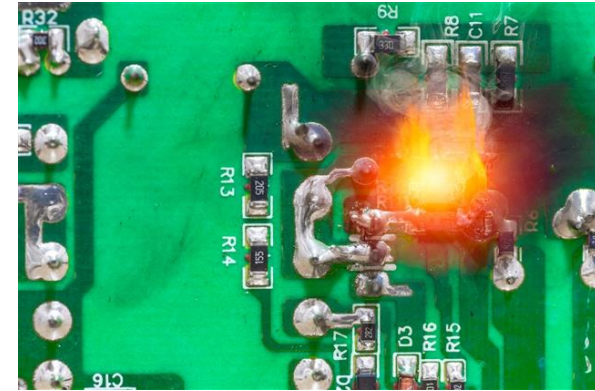
Some complicated operations cannot be **exactly** done by addition, e.g., e^x , $\log(x)$.

We need to approximate these operations using basic operations that computers can perform.

Approximation introduces **truncation error**!

Error Analysis

What may happen if we ignore the error?



Error Analysis

In this lecture, we learn more about the
round-off and **truncation** errors.

Round-off Error

Floating-point Number System

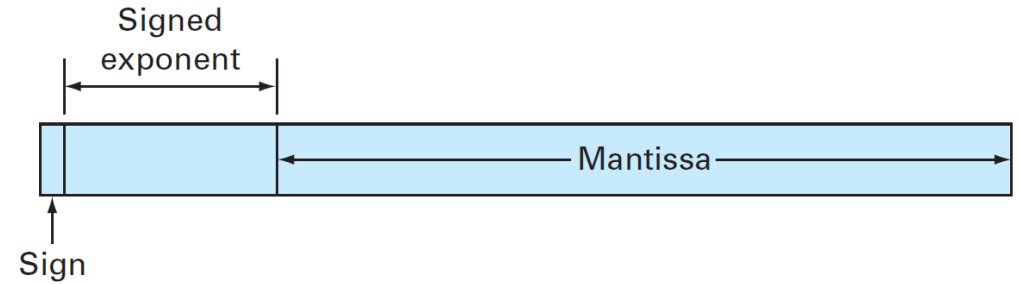
How do computers store numbers?



Remember, some numbers cannot be stored on computers! (e.g., π)

Floating-point is a popular way to store numbers on a computer.

Floating-point Number System

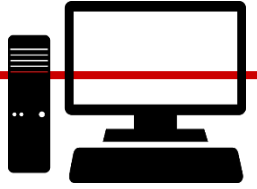


The number is expressed as:

1. a sign (positive or negative).
2. a fractional part, called a **mantissa or significand**
3. an integer part, called an **exponent or characteristic**

$$\text{mantissa} \nearrow m 10^{\text{exponent} \leftarrow e}$$

Floating-point Number System



To find the floating-point representation of x simply write it as

Decimal point

Zero!

n decimal places

exponent

$x = 0.a_1 \dots a_n 10^e$

mantissa

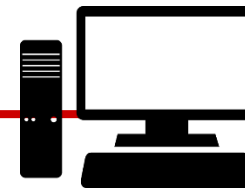
- e (exponent) is an integer.
- a_1 should be non-zero!
- First digit to the left of decimal point is zero.

Example



Find the floating-point representation of 152853.5047.

Example



Find the floating-point representation of 152853.5047.

10 decimal places
←————→
Answer: 0.1528535047×10^6

$$x = 0.a_1 \dots a_n 10^e$$

$$\begin{aligned} \text{Solution: } 152853.5047 &= \frac{152853.5047}{10^6} \times 10^6 \\ &= 0.1528535047 \times 10^6 \end{aligned}$$

Mantissa: 0.1528535047

Exponent: 6

Example



Find the floating-point representation of 152853.5047, with 5 decimal places.

Example



Find the floating-point representation of 152853.5047, with 5 decimal places.

5 decimal places

Answer: $0.\overline{15285} \times 10^6$

$$x = 0.a_1 \dots a_n 10^e$$

Mantissa: 0.15285

Exponent: 6

Error= 3.5047 (round-off error)

Example



Find the floating-point representation of 0.029411765.

Example



Find the floating-point representation of 0.029411765.

$$x = 0.a_1 \dots a_n 10^e$$

Answer $0.29411765 \times 10^{-1}$

Solution: $0.029411765 = 0.029411765 \times 10^1 \times 10^{-1}$
 $= 0.29411765 \times 10^{-1}$

Mantissa: 0.29411765

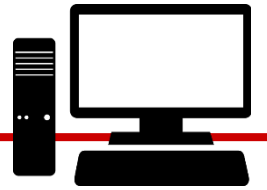
Exponent: -1

Example



Find the floating-point representation of 0.029411765 , with 4 decimal places. What is the error?

Example



Find the floating-point representation of 0.029411765 , with 4 decimal places. What is the error?

$$x = 0.a_1 \dots a_n 10^e$$

Answer 0.2941×10^{-1} , error: 0.000001765

Mantissa: 0.2941

Exponent: -1

- Not 0.0294×10^0 error: 0.000011765
- Remember to remove the leading zeros. Why?

General Rule



If the **first digit dropped** is more than **4**, we **add 1** to the **last digit kept** (in the floating point representation).

Example: Find the floating-point representation of 152856.1, with 5 decimal places.

Answer:

$$\begin{aligned} 152856 &= \frac{152856.1}{10^6} \times 10^6 \\ &= 0.1528561 \times 10^6 \end{aligned}$$

$$x = 0.a_1 \dots a_n 10^e$$

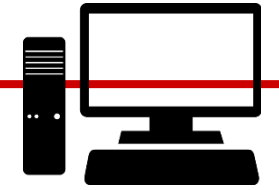
Mantissa: 0.15286

Exponent: 6

Last digit kept

First digit dropped

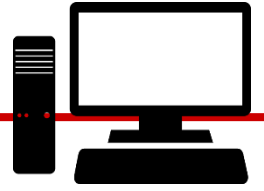
Definition of Error



- We often approximate the variables!
- The true value of a variable will be different from its approximation.
- Exact value of error, i.e. true error, E_t = true value – approximation
- Relative error = $\frac{E_t}{\text{true value}}$
- True percent relative error

$$\epsilon_t = \frac{E_t}{\text{true value}} 100\%$$

Example



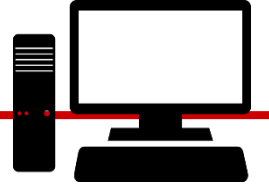
- Suppose that you have the task of measuring the lengths of a bridge and a rivet and come up with 9999 cm and 9 cm, respectively. If the **true values** are 10,000 cm and 10 cm, respectively, compute (a) the true error and (b) the true percent relative error for each case.



rivet



Example



- The error of measuring the bridge is $E_t = 10000 - 9999 = 1$ cm.
- The error of measuring the rivet is $E_t = 10 - 9 = 1$ cm.
- The percent relative error for the bridge is $\epsilon_t = \frac{1}{10000} 100\% = 0.01\%$.
- The percent relative error for the rivet is $\epsilon_t = \frac{1}{10} 100\% = 10\%$.
- Conclusion: The bridge measurement is more accurate! (relative error is more important than the absolute error!)

Truncation Error

Taylor's Approximation

- Remember, we cannot compute every function using computers.
- How about to approximate them using polynomials?

$$c_0x^n + c_1x^{n-1} + \cdots + c_{n-1}x + c_n$$

- Why polynomials?

They only involve $+$, $\times$.

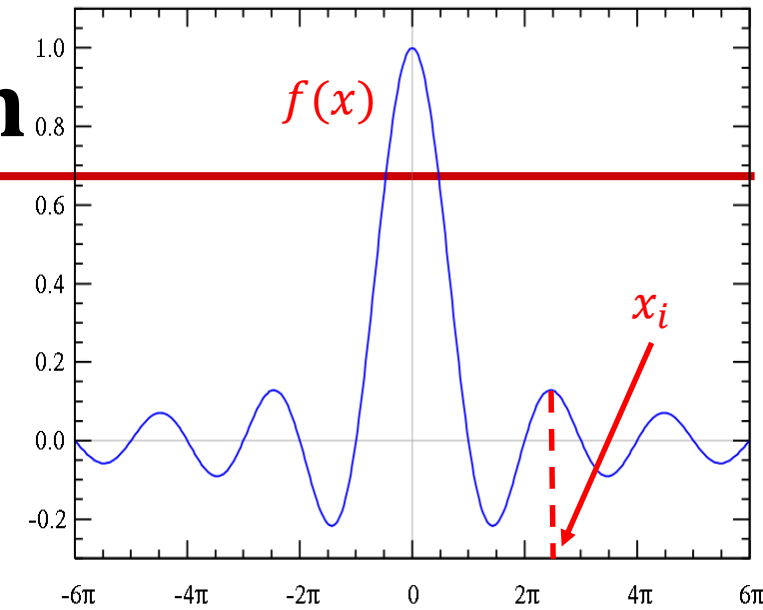
They can be calculated using computers.

$$2x^2 + 1x + 5$$
$$2x^3 + 8x^2 + 4x + 2$$

Answer: Use Taylor's Approximation!

Taylor's Approximation

How to approximate Taylor's approximation of $f(x)$?



- Select a point x_i .
- Compute $f(x_i), f'(x_i), f''(x_i), \dots, f^n(x_i)$ ($f^i(x_i)$: i -th derivative of $f(x)$ at x_i)
- Form the following polynomial

$$f(x_i) + \frac{f'(x_i)}{1!}(x - x_i) + \frac{f''(x_i)}{2!}(x - x_i)^2 + \dots + \frac{f^n(x_i)}{n!}(x - x_i)^n$$

This is called n -th order Taylor's Approximation of $f(x)$ around x_i .

Taylor's Approximation

$$f(x_i) + \frac{f'(x_i)}{1!} (x - x_i) + \frac{f''(x_i)}{2!} (x - x_i)^2 + \cdots + \frac{f^n(x_i)}{n!} (x - x_i)^n + \cdots$$

Zeroth order Taylor's Approximation: $f(x_i)$.

First order Taylor's Approximation: $f(x_i) + \frac{f'(x_i)}{1!} (x - x_i)$.

Second order Taylor's Approximation: $f(x_i) + \frac{f'(x_i)}{1!} (x - x_i) + \frac{f''(x_i)}{2!} (x - x_i)^2$.

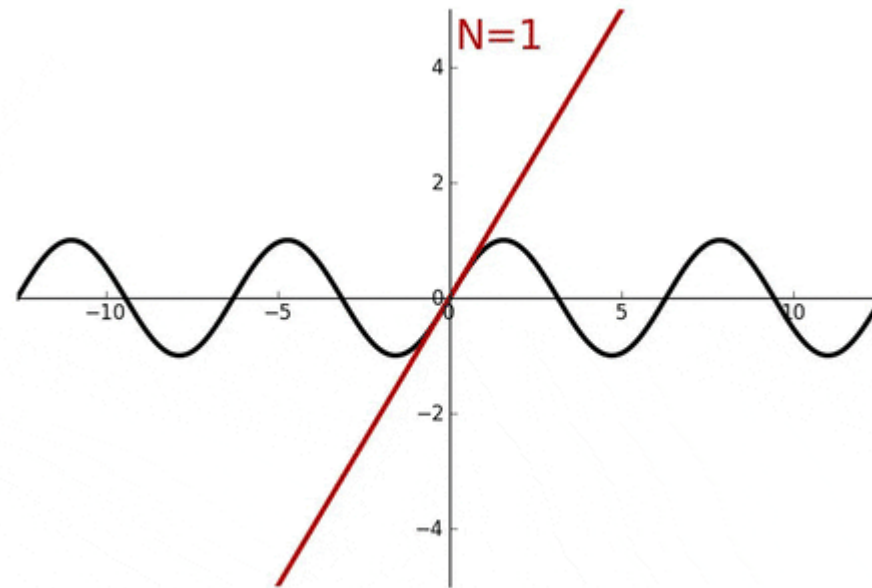
n-th order Taylor's Approximation:

$$f(x_i) + \frac{f'(x_i)}{1!} (x - x_i) + \frac{f''(x_i)}{2!} (x - x_i)^2 + \cdots + \frac{f^n(x_i)}{n!} (x - x_i)^n$$

Recall: $n! = 1 \times 2 \times \cdots \times n$.

Example

What does happen as we increase n ?

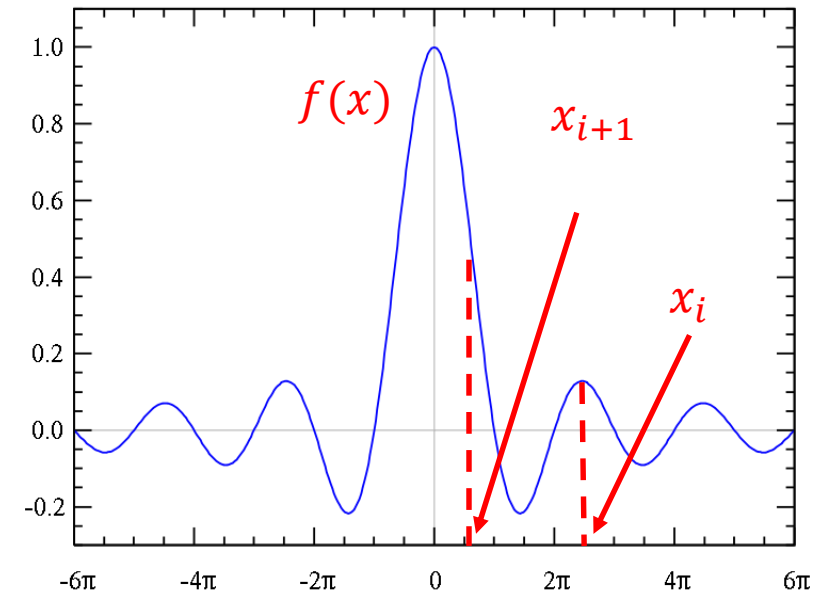


Taylor's approximation of a function around zero.

Application of Taylor's Approximation

We can approximate the value of f at other points using its Taylor's approximation at x_i .

Simply replace x_{i+1} for x in the Taylor's approximation of $f(x)$ around x_i .



Application of Taylor's Approximation

Select a point x_{i+1} .

The value of f at x_{i+1} can be approximated by

$$f(x_{i+1}) \cong f(x_i) + \frac{f'(x_i)}{1!} (x_{i+1} - x_i) + \frac{f''(x_i)}{2!} (x_{i+1} - x_i)^2 + \cdots + \frac{f^n(x_i)}{n!} (x_{i+1} - x_i)^n$$

Sometimes, we simplify the notation by defining $h = x_{i+1} - x_i$ as the **step size**.

$$f(x_{i+1}) \cong f(x_i) + \frac{f'(x_i)}{1!} (h) + \frac{f''(x_i)}{2!} (h)^2 + \cdots + \frac{f^n(x_i)}{n!} (h)^n$$

Example

Find the n-th order Taylor's expansion of $f(x) = e^x$ around $x_i = 0$.

$$f(x_i) + \frac{f'(x_i)}{1!} (x - x_i) + \frac{f''(x_i)}{2!} (x - x_i)^2 + \cdots + \frac{f^n(x_i)}{n!} (x - x_i)^n$$

Example

Compute the derivatives of $f(x) = e^x$

$$f'(x) = e^x$$

$$f''(x) = e^x$$

$$f^3(x) = e^x$$

$$f^n(x) = e^x$$

We have $f(0) = 1, f^n(0) = 1$.

$$f(x_i) + \frac{f'(x_i)}{1!} (x - x_i) + \frac{f''(x_i)}{2!} (x - x_i)^2 + \cdots + \frac{f^n(x_i)}{n!} (x - x_i)^n$$

The n-th order Taylor approximation around zero will be

$$1 + x + \frac{1}{2!} x^2 + \cdots + \frac{1}{n!} x^n$$

Example

Use the Taylor's approximation of $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$ around $x_i = 0$ to estimate (approximate) the value of function at $x_{i+1} = 1$.

Example

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

Compute the derivatives of $f(x)$

$$f'(x) = -0.4x^3 - 0.45x^2 - 1x - 0.25$$

$$f''(x) = -1.2x^2 - 0.9x - 1$$

$$f^3(x) = -2.4x^1 - 0.9$$

$$f^4(x) = -2.4$$

Example

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$x_i = 0, x_{i+1} = 1, h = 1.$$

$$f(x_{i+1}) \cong f(x_i) + \frac{f'(x_i)}{1!} (h) + \frac{f''(x_i)}{2!} (h)^2 + \dots + \frac{f^n(x_i)}{n!} (h)^n$$

Zeroth order approximation

$$f(x_{i+1}) \simeq 1.2$$

$$f'(0) = -0.4(0.0)^3 - 0.45(0.0)^2 - 1.0(0.0) - 0.25 = -0.25$$

First order approximation

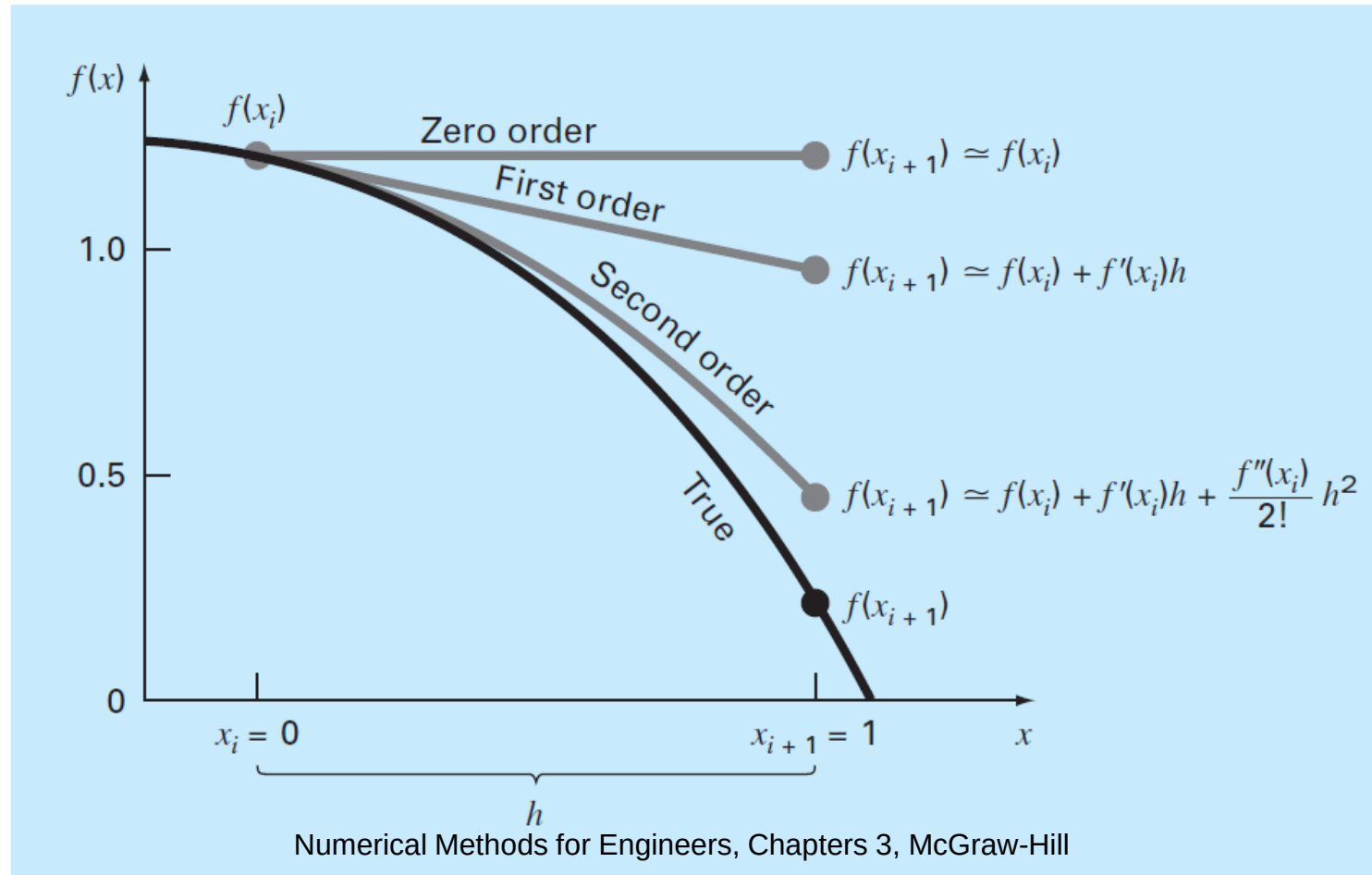
$$f(x_{i+1}) \simeq 1.2 - 0.25h$$

$$f''(0) = -1.2(0.0)^2 - 0.9(0.0) - 1.0 = -1.0$$

Second order approximation

$$f(x_{i+1}) \simeq 1.2 - 0.25h - 0.5h^2$$

Example



Conclusion: The approximation becomes more accurate as we increase n (the order).

Example

Problem Statement. Use Taylor series expansions with $n = 0$ to 6 to approximate $f(x) = \cos x$ at $x_{i+1} = \pi/3$ on the basis of the value of $f(x)$ and its derivatives at $x_i = \pi/4$. Note that this means that $h = \pi/3 - \pi/4 = \pi/12$.

Example

$$f(x) = \cos x$$

Compute the derivatives of $f(x)$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$$f^3(x) = \sin x$$

$$f^4(x) = \cos x$$

Example

$$f(x) = \cos x, x_i = \pi/4, x_{i+1} = \pi/3, h = \pi/12.$$

$$f(x_{i+1}) \cong f(x_i) + \frac{f'(x_i)}{1!} (h) + \frac{f''(x_i)}{2!} (h)^2 + \dots + \frac{f^n(x_i)}{n!} (h)^n$$

Zeroth order approximation $f\left(\frac{\pi}{3}\right) \cong \cos\left(\frac{\pi}{4}\right) = 0.707106781$

First order approximation $f\left(\frac{\pi}{3}\right) \cong \cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\left(\frac{\pi}{12}\right) = 0.521986659$

second order approximation $f\left(\frac{\pi}{3}\right) \cong \underbrace{\cos\left(\frac{\pi}{4}\right)}_f \underbrace{- \sin\left(\frac{\pi}{4}\right)}_{f'(\pi/4)} \underbrace{\left(\frac{\pi}{12}\right)}_h - \underbrace{\frac{\cos(\pi/4)}{2}}_{f''(\pi/4)/2} \underbrace{\left(\frac{\pi}{12}\right)^2}_{h^2} = 0.497754491$

Example

TABLE 4.1 Taylor series approximation of $f(x) = \cos x$ at $x_{i+1} = \pi/3$ using a base point of $\pi/4$. Values are shown for various orders (n) of approximation.

Order n	$f^{(n)}(x)$	$f(\pi/3)$	ϵ_f
0	$\cos x$	0.707106781	-41.4
1	$-\sin x$	0.521986659	-4.4
2	$-\cos x$	0.497754491	0.449
3	$\sin x$	0.499869147	2.62×10^{-2}
4	$\cos x$	0.500007551	-1.51×10^{-3}
5	$-\sin x$	0.500000304	-6.08×10^{-5}
6	$-\cos x$	0.499999988	2.44×10^{-6}

- **Conclusion:** The approximation becomes more accurate as we increase the order.

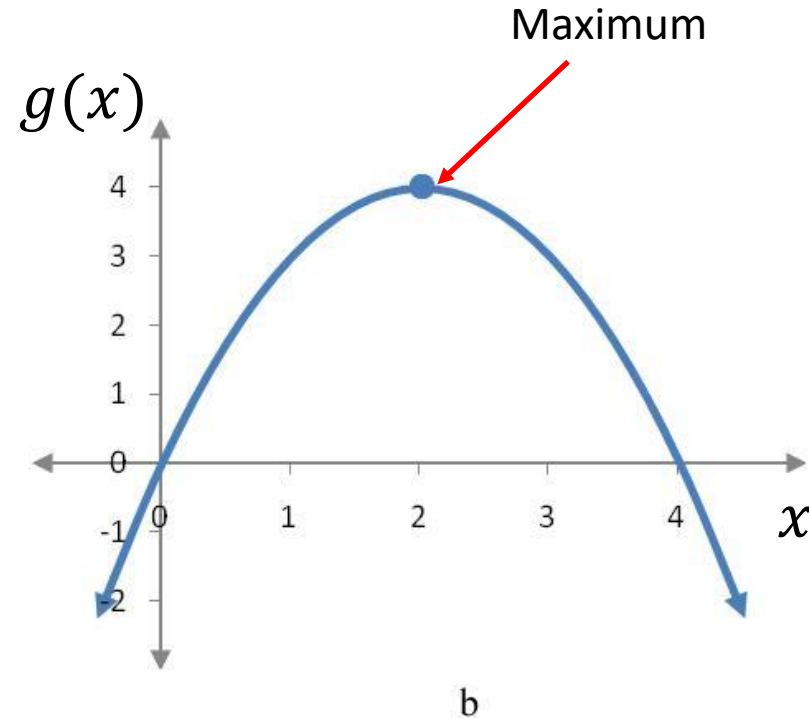
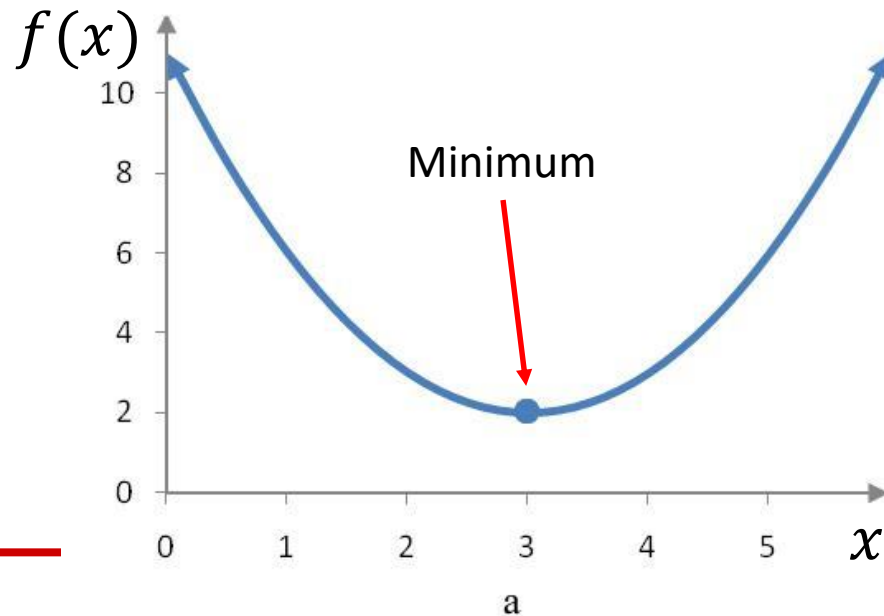
Numerical Differentiation

Objective: to numerically find the derivative of f at the point x_i .

- Why is differentiation important?

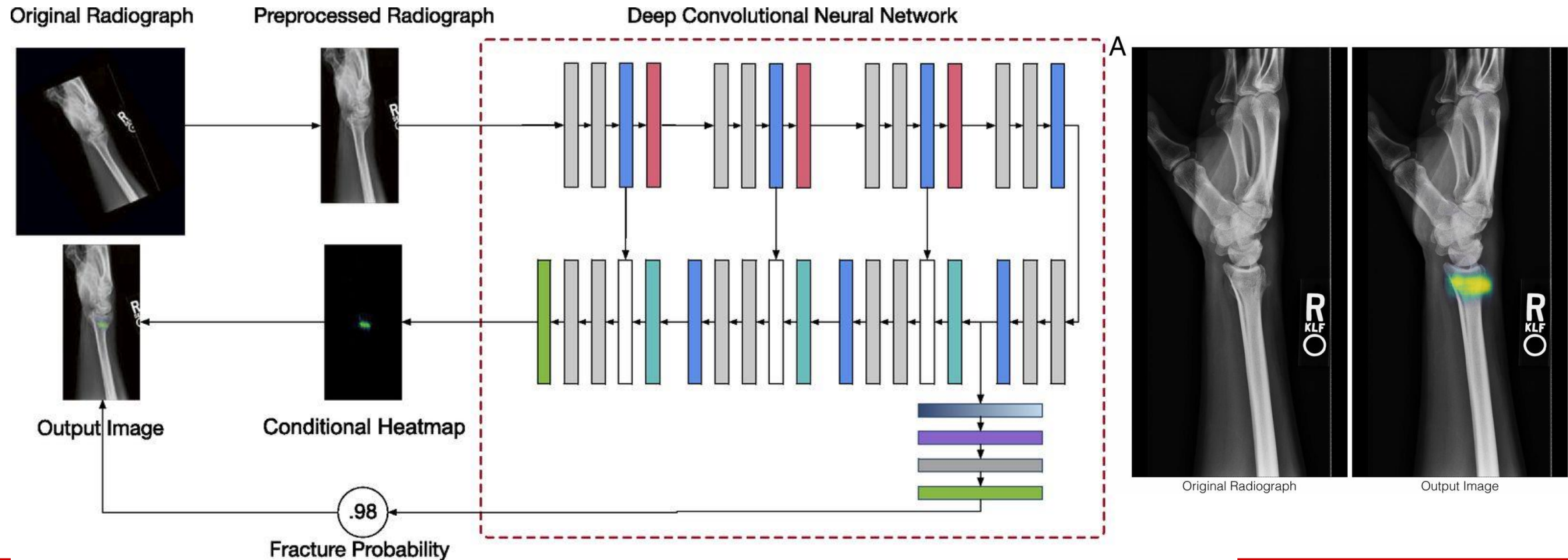
Motivation: Optimization

- **Optimization**: finding the maximum or minimum of a function!
- **Maximum (minimum)** of a function can be found by **differentiation**.
- Many applications in **AI, deep learning**, etc.

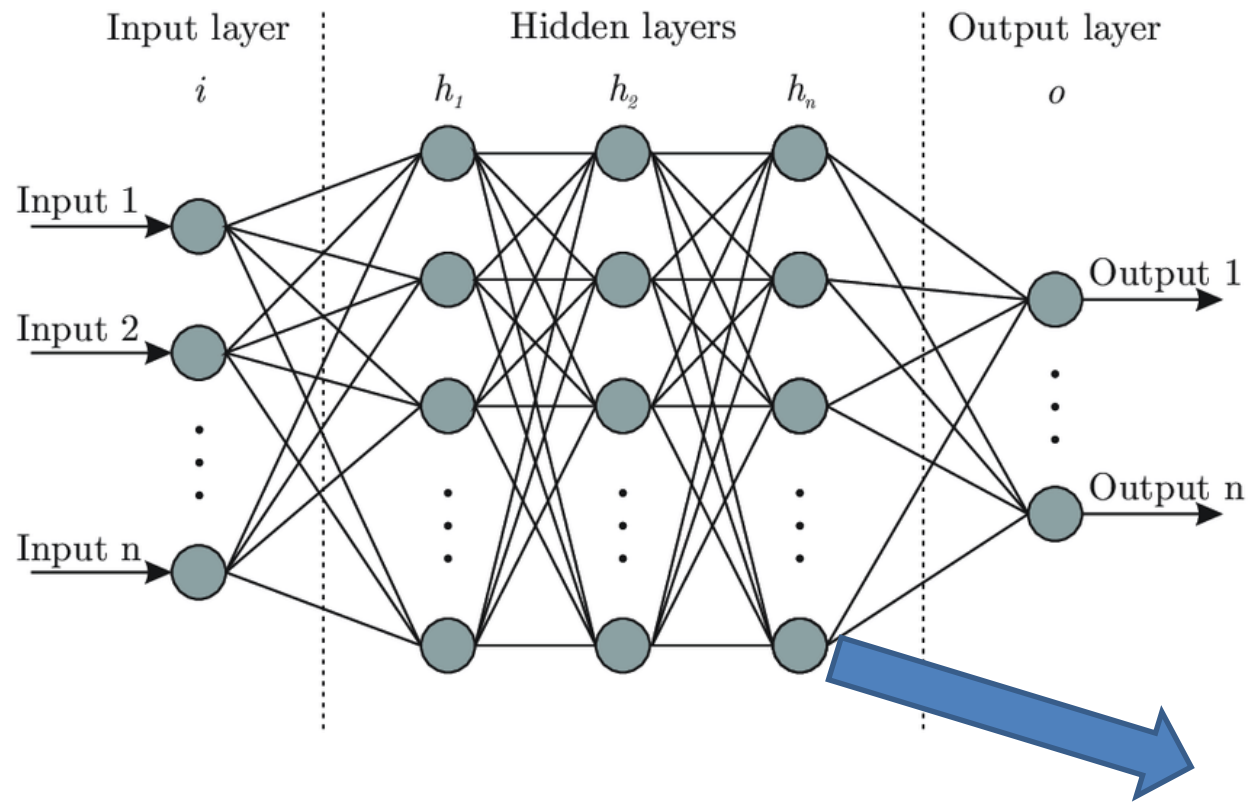


Motivation: Neural Networks

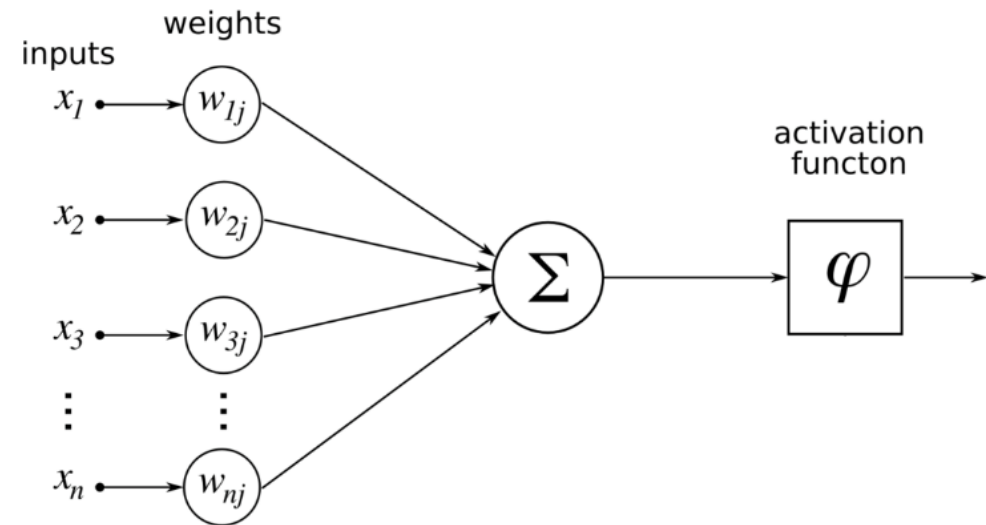
- Neural networks are often designed using **differentiation**.
- Example: **Detecting Bone Fracture and Injuries**



Motivation: Neural Networks



Weights are found by **differentiation**.



Numerical Differentiation

We will study three methods for numerical differentiation:

1. Forward difference method.
2. Backward difference method.
3. Centered difference method.

Forward Difference Method

Find the derivative of f at x_i .

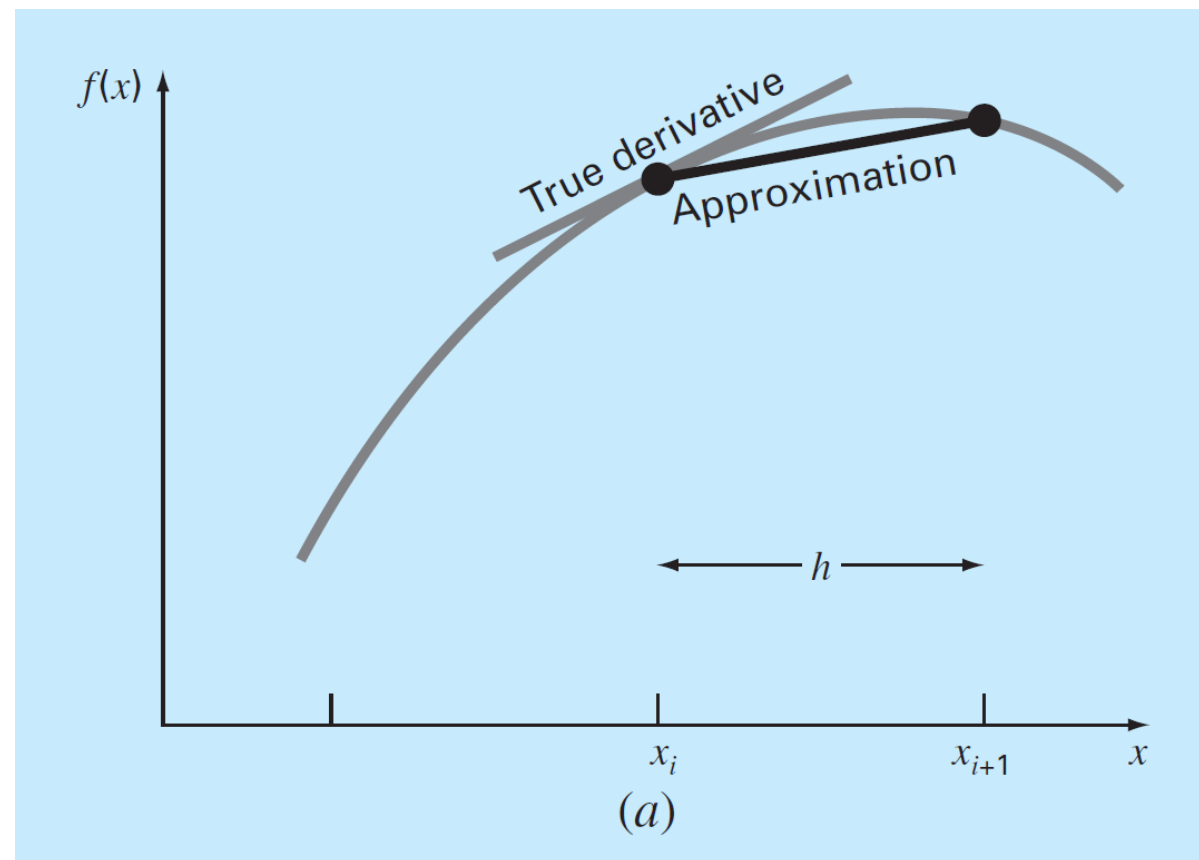
1. Select a point x_{i+1} located after x_i .

2. Approximate $f'(x_i)$ with

$$f'(x_i) \cong \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

- Why it works?
- Remember the first order Taylor's approximation

$$f(x_{i+1}) \cong f(x_i) + \frac{f'(x_i)}{1!} (x_{i+1} - x_i)$$



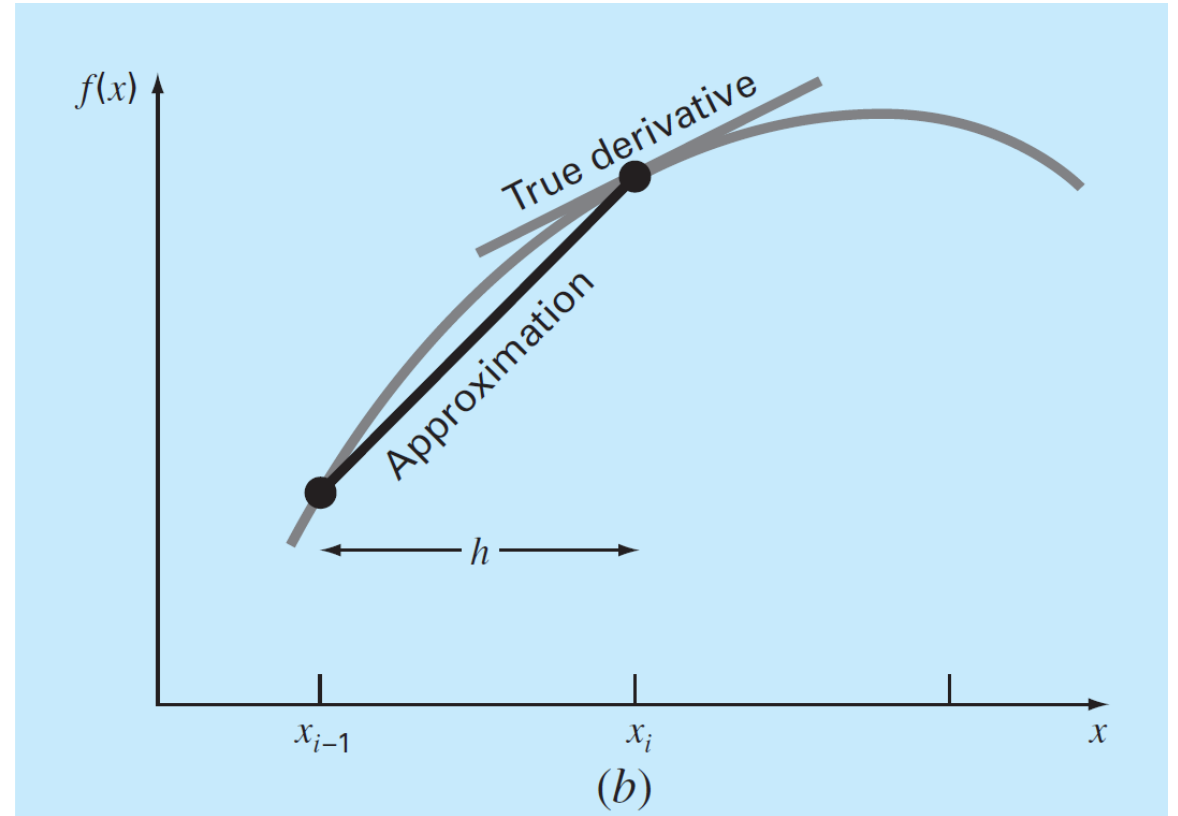
Numerical Methods for Engineers, Chapters 3, McGraw-Hill

Backward Difference Method

Find the derivative of f at x_i .

1. Select a point x_{i-1} located behind x_i .
2. Approximate $f'(x_i)$ with

$$f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$



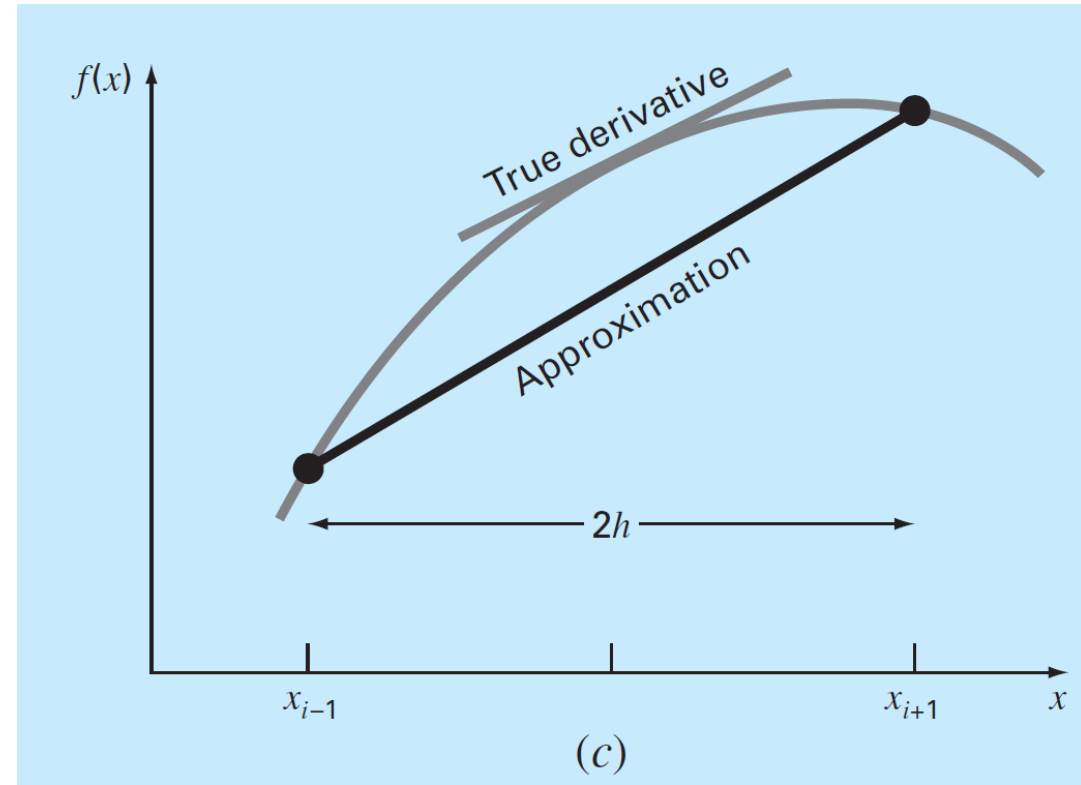
Numerical Methods for Engineers, Chapters 3, McGraw-Hill

Centered Difference Method

Find the derivative of f at x_i .

1. Select a point x_{i-1} located behind x_i .
2. Select a point x_{i+1} located after x_i .
3. Approximate $f'(x_i)$ with

$$f'(x_i) \cong \frac{f(x_{i+1}) - f(x_{i-1}))}{x_{i+1} - x_{i-1}}$$



Numerical Methods for Engineers, Chapters 3, McGraw-Hill

- The centered difference approximation is often **more accurate** than other two approximations.

Example

Use the forward, backward and centered methods to approximate the first derivative of

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

at $x = 0.5$ with the step-size $h = 0.5$. Repeat the result for the step-size $h = 0.25$.

Example

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

$$x_i = 0.5$$

step-size $h = 0.5$.

$$x_{i-1} = 0 \quad f(x_{i-1}) = 1.2$$

$$x_i = 0.5 \quad f(x_i) = 0.925$$

$$x_{i+1} = 1.0 \quad f(x_{i+1}) = 0.2$$

Forward method $\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \quad f'(0.5) \cong \frac{0.2 - 0.925}{0.5} = -1.45 \quad |\varepsilon_t| = 58.9\%$

Backward method $\frac{f(x_i) - f(x_{i-1}))}{x_i - x_{i-1}} \quad f'(0.5) \cong \frac{0.925 - 1.2}{0.5} = -0.55 \quad |\varepsilon_t| = 39.7\%$

Centered method $\frac{f(x_{i+1}) - f(x_{i-1}))}{x_{i+1} - x_{i-1}} \quad f'(0.5) \cong \frac{0.2 - 1.2}{1.0} = -1.0 \quad |\varepsilon_t| = 9.6\%$

Example

For $h=0.25$,

$$\begin{array}{ll} x_{i-1} = 0.25 & f(x_{i-1}) = 1.10351563 \\ x_i = 0.5 & f(x_i) = 0.925 \\ x_{i+1} = 0.75 & f(x_{i+1}) = 0.63632813 \end{array}$$

Forward method $\frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$ $f'(0.5) \cong \frac{0.63632813 - 0.925}{0.25} = -1.155$ $|\epsilon_t| = 26.5\%$

Backward method $\frac{f(x_i) - f(x_{i-1}))}{x_i - x_{i-1}}$ $f'(0.5) \cong \frac{0.925 - 1.10351563}{0.25} = -0.714$ $|\epsilon_t| = 21.7\%$

Centered method $\frac{f(x_{i+1}) - f(x_{i-1}))}{x_{i+1} - x_{i-1}}$ $f'(0.5) \cong \frac{0.63632813 - 1.10351563}{0.5} = -0.934$ $|\epsilon_t| = 2.4\%$

Conclusions:

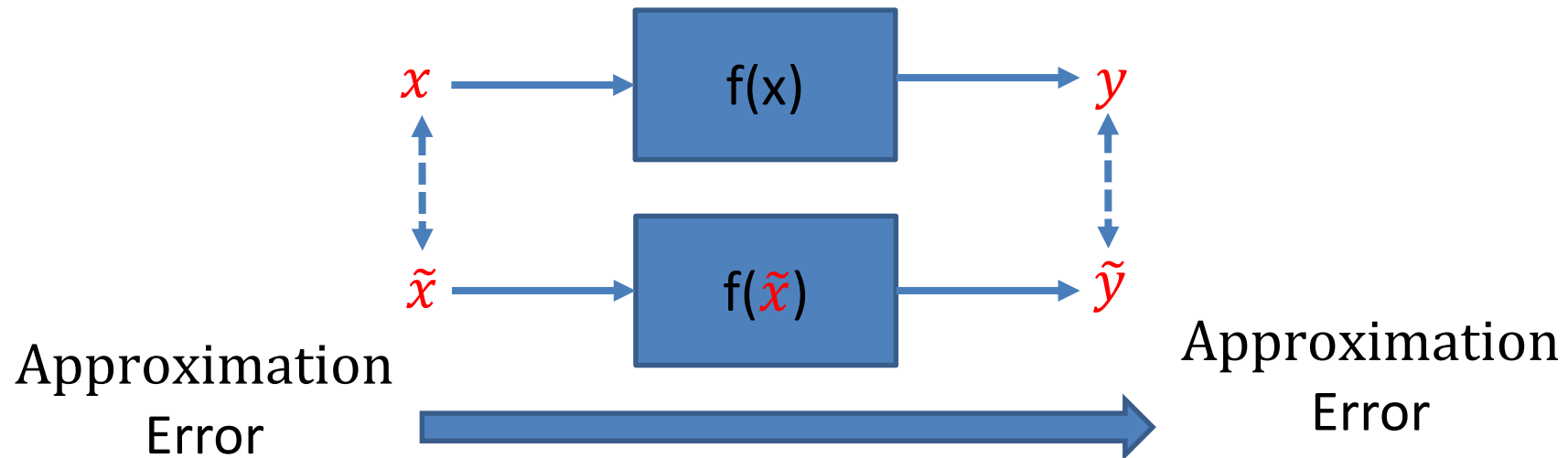
Centered method is more accurate.

The accuracy improves by decreasing the step-size (h).

Error Propagation



- Consider a computer that approximates π with 3.1415 in the computations.
- Suppose, we need to compute $\log(\pi)$. So, we get $\log(\pi) \cong \log 3.1415$.
- All these computations have error because 3.1415 is not exactly equal to π !



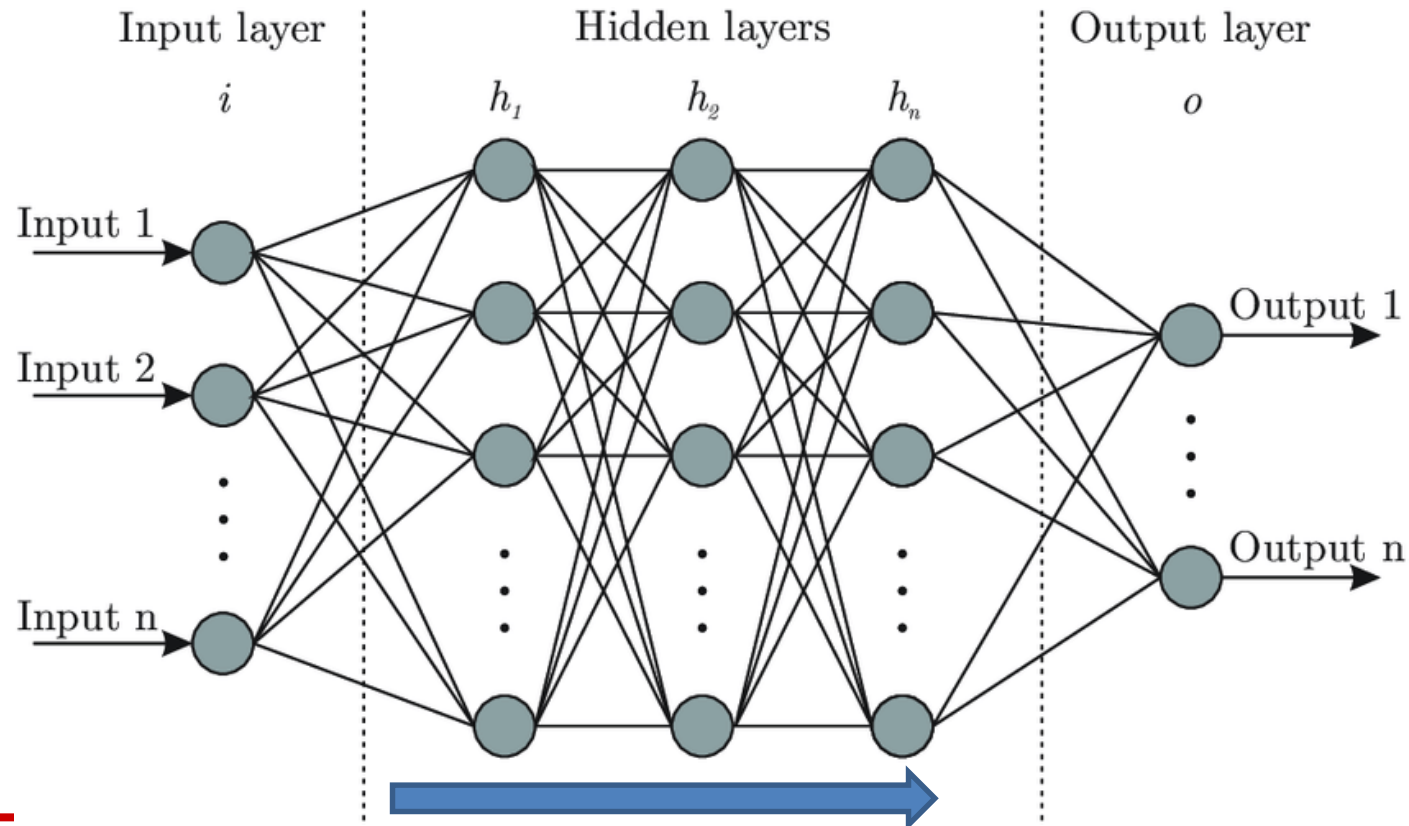
- This is called error propagation!

Error Propagation

Why is error propagation important?

Error Propagation

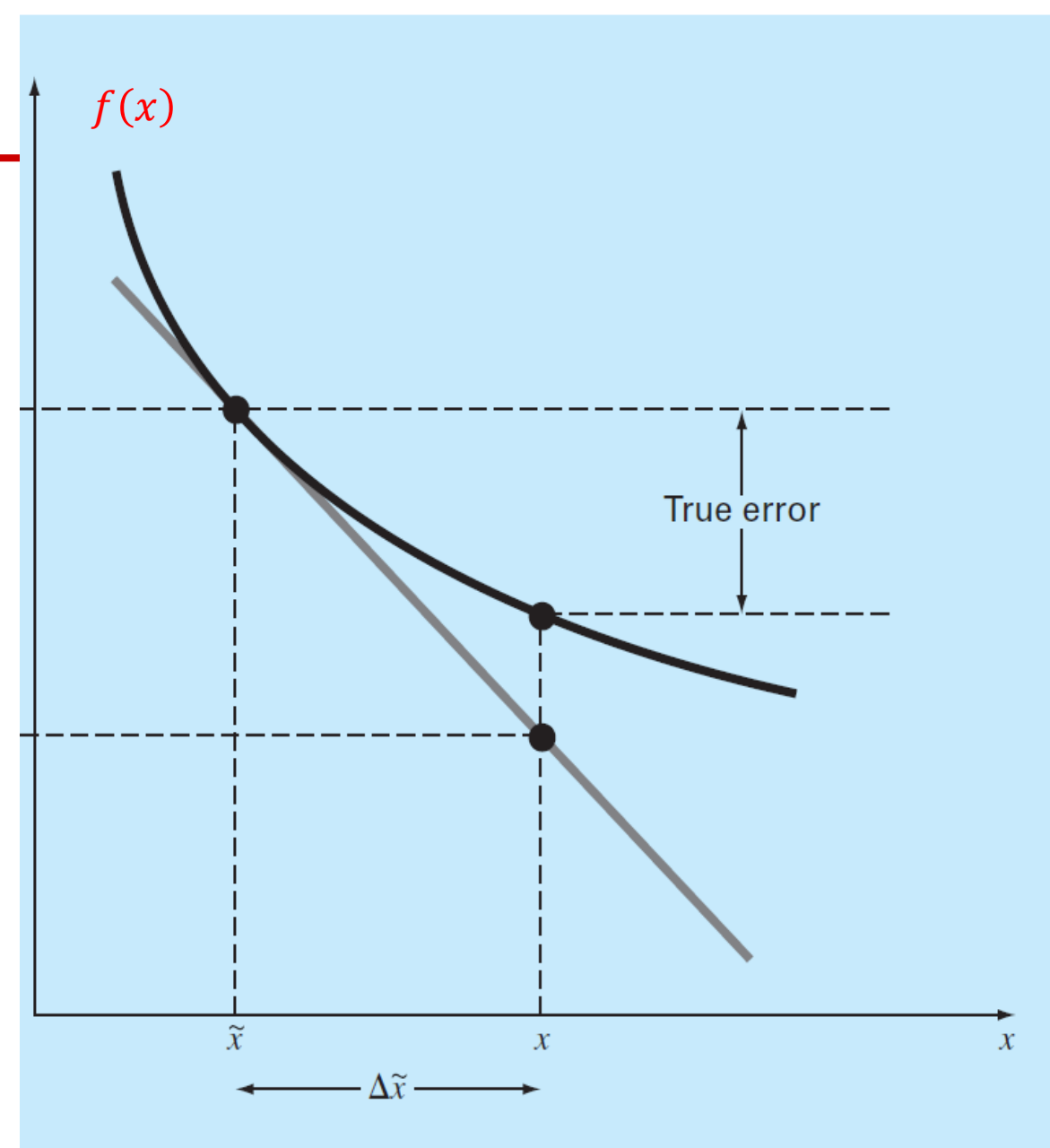
In many engineering applications, computations are done in multiple stages.



Computational errors propagate from one layer to another layer.

Error Propagation

- Suppose $\tilde{x}$ is the **approximation** of x .
- We want to **approximate** $f(x)$ using $f(\tilde{x})$.
- What is the approximation error $f(x) - f(\tilde{x})$?



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Error Propagation

- The approximation of error in $f(\tilde{x})$ is

$$f(x) - f(\tilde{x}) \cong f'(\tilde{x})(x - \tilde{x}) \quad (1)$$

- Sometimes we use $\Delta\tilde{x} = (x - \tilde{x})$ and $\Delta f = f(x) - f(\tilde{x})$ to write the error as

$$\Delta f \cong f'(\tilde{x})\Delta\tilde{x} \quad (2)$$

Remember (1) and (2) are the same.

Example

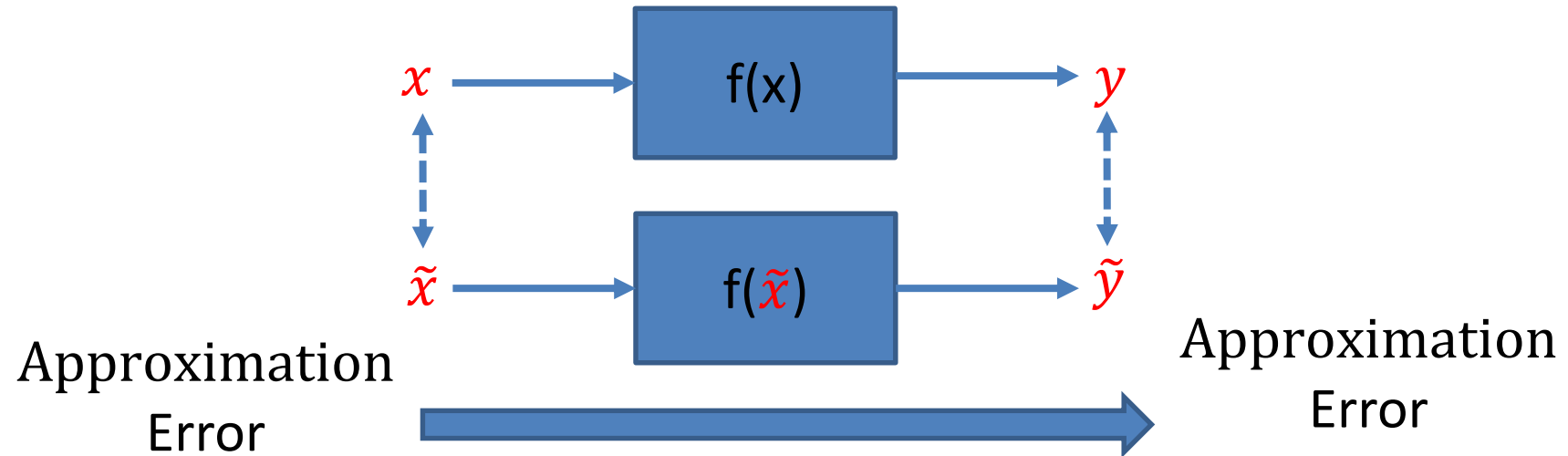
Problem Statement. Given a value of $\tilde{x} = 2.5$ with an error of $\Delta\tilde{x} = 0.01$, estimate the resulting error in the function $f(x) = x^3$.

Example

Problem Statement. Given a value of $\tilde{x} = 2.5$ with an error of $\Delta\tilde{x} = 0.01$, estimate the resulting error in the function $f(x) = x^3$.

$$\Delta f \cong f'(\tilde{x})(x - \tilde{x}) \qquad \Delta f(\tilde{x}) \cong 3(2.5)^2(0.01) = 0.1875 \qquad (\text{Approximate error})$$
$$f'(x) = 3x^2$$

Condition Number



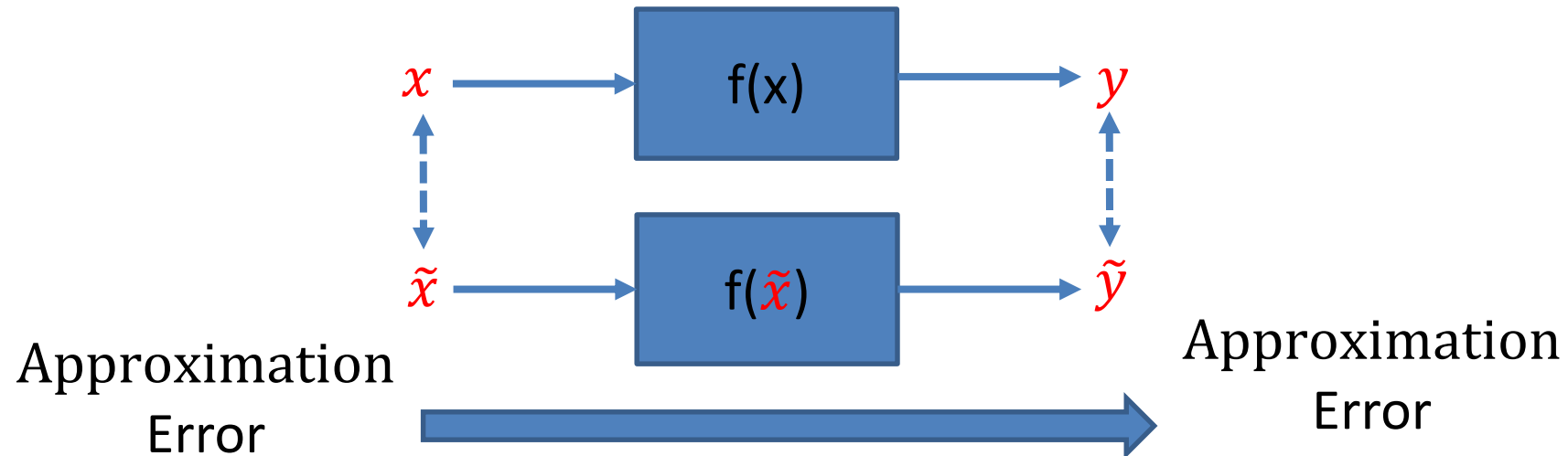
- What is the impact of the function f on the error?

Condition Number

- Condition number quantifies the impact of f on the error.

$$\text{Condition number} = \frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})}$$

- A large condition number means $f(\tilde{x})$ may contain a large error.



Example

Problem Statement. Compute the condition number for

$$f(x) = \tan x \quad \text{for } \tilde{x} = \frac{\pi}{2} + 0.1\left(\frac{\pi}{2}\right)$$

$$f(x) = \tan x \quad \text{for } \tilde{x} = \frac{\pi}{2} + 0.01\left(\frac{\pi}{2}\right)$$

Example

$$f(x) = \tan x$$

$$f'(x) = 1/\cos^2 x$$

$$\text{Condition number} = \frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})} = \frac{\tilde{x}(1/\cos^2 x)}{\tan \tilde{x}}$$

$$\text{For } \tilde{x} = \pi/2 + 0.1(\pi/2),$$

$$\text{Condition number} = \frac{1.7279(40.86)}{-6.314} = -11.2$$

Example

$$f(x) = \tan x$$

$$f'(x) = 1/\cos^2 x$$

$$\text{Condition number} = \frac{\tilde{x} f'(\tilde{x})}{f(\tilde{x})} = \frac{\tilde{x}(1/\cos^2 x)}{\tan \tilde{x}}$$

$$\text{For } \tilde{x} = \pi/2 + 0.01(\pi/2),$$

$$\text{Condition number} = \frac{1.5865(4053)}{-63.66} = -101$$

END

See you next week 😊!