

Name: _____

ID: _____

Question 1 (20 marks) Diagonalize the matrix $A = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$ by the following steps.

- (4 marks) Find the eigenvalues of A .
- (9 marks) Find the eigenvectors of A .
- (3 marks) Construct a 4x4 matrix P from the eigenvectors and a 4x4 diagonal matrix D from the eigenvalues of A .
- (2 marks) Calculate $B = AP$.
- (2 marks) Calculate $C = PD$. Can you make sure $B = C$?

Soln: (i) Upper triangular matrix. Eigenvalues are $\lambda_1=5, \lambda_2=3, \lambda_3=\lambda_4=2$.

(ii) For $\lambda_1=5$. $(A-\lambda_1 I|0) \sim \begin{pmatrix} 0 & -3 & 0 & 9 & | & 0 \\ 0 & -2 & 1 & -2 & | & 0 \\ 0 & 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & -3 & | & 0 \end{pmatrix} \Rightarrow \begin{cases} -3x_2 + 9x_4 = 0 \\ -2x_2 + x_3 - 2x_4 = 0 \\ -3x_3 = 0 \\ -3x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$

$\Rightarrow \vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} t, t \neq 0$. Set $t=1 \Rightarrow \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

* For $\lambda_2=3$. $(A-\lambda_2 I|0) \sim \begin{pmatrix} 2 & -3 & 0 & 9 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & -1 & 0 & | & 0 \\ 0 & 0 & 0 & -1 & | & 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_1 - 3x_2 + 9x_4 = 0 \\ x_3 - 2x_4 = 0 \\ -x_3 = 0 \\ -x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{3}{2}t \\ x_2 = t \\ x_3 = 0 \\ x_4 = 0 \end{cases}$

$\Rightarrow \vec{x}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \\ 0 \end{pmatrix} t, t \neq 0$. Set $t=2 \Rightarrow \vec{u}_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

* For $\lambda_3=\lambda_4=2$

$(A-2I|0) \sim \begin{pmatrix} 3 & -3 & 0 & 9 & | & 0 \\ 0 & 1 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow \begin{cases} 3x_1 - 3x_2 + 9x_4 = 0 \\ x_2 + x_3 - 2x_4 = 0 \\ x_3 = t \\ x_4 = s \end{cases} \Rightarrow \begin{cases} x_1 = -t - s \\ x_2 = -t + 2s \\ x_3 = t \\ x_4 = s \end{cases}$

$\Rightarrow \vec{x}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix} s, t+s \neq 0$. Set $t=1, s=0 \Rightarrow \vec{u}_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$

Set $t=0, s=1 \Rightarrow \vec{u}_4 = \begin{pmatrix} -1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

(iii) $P = (\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4) = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $D = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

(iv) $B = AP = \begin{pmatrix} 5 & -3 & 0 & 9 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 9 & -2 & -2 \\ 0 & 6 & -2 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

(v) $C = PD = \begin{pmatrix} 1 & 3 & -1 & -1 \\ 0 & 2 & -1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 9 & -2 & -2 \\ 0 & 6 & -2 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

$\boxed{B = C}$

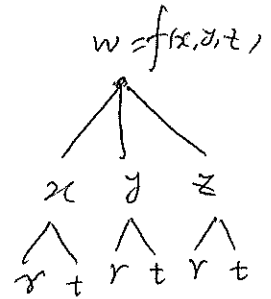
Question 2 (10 marks) It is given that $w = f(x, y, z)$, where $x = r \cos t$, $y = r \sin t$, $z = t^3$.

(i) (5 marks) Find $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial t}$.

(ii) (5 marks) If $\frac{\partial f}{\partial x} = x$, $\frac{\partial f}{\partial y} = -1$, and $\frac{\partial f}{\partial z} = z^3$, find the values of $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial t}$ at the point $(r, t) = (2, \pi)$.

Solution. (i)
$$\begin{aligned}\frac{\partial w}{\partial r} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r} \\ &= \frac{\partial f}{\partial x} \cdot \cos t + \frac{\partial f}{\partial y} \cdot \sin t + \frac{\partial f}{\partial z} \cdot 0 \\ &= \frac{\partial f}{\partial x} \cos t + \frac{\partial f}{\partial y} \sin t.\end{aligned}$$

$$\begin{aligned}\frac{\partial w}{\partial t} &= \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial t} \\ &= \frac{\partial f}{\partial x} \cdot (-r \sin t) + \frac{\partial f}{\partial y} (r \cos t) + \frac{\partial f}{\partial z} \cdot (3t^2)\end{aligned}$$



(ii) At $(r, t) = (2, \pi) \Rightarrow (x, y, z) = (-2, 0, \pi^3)$.

$$\frac{\partial f}{\partial x} = x, \quad \frac{\partial f}{\partial y} = -1, \quad \frac{\partial f}{\partial z} = z$$

$$\Rightarrow \frac{\partial w}{\partial r}(2, \pi) = (-2) \cos \pi + (-1) \cdot \sin \pi = 2.$$

$$\begin{aligned}\frac{\partial w}{\partial t}(2, \pi) &= (-2) \cdot (-2 \sin \pi) + (-1) \cdot (2 \cos \pi) + \pi^3 \cdot (3 \cdot \pi^2) \\ &= 2 + 3\pi^5\end{aligned}$$

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Question 1 (20 marks) Diagonalize the matrix $A = \begin{pmatrix} 5 & -2 & 0 & 4 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ by the following steps.

- (4 marks) Find the eigenvalues of A .
- (9 marks) Find the eigenvectors of A .
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- (2 marks) Calculate $B = AP$.
- (2 marks) Calculate $C = PD$. Can you make sure $B = C$?

Solution. (i) Upper triangular matrix. Eigenvalues are $\lambda_1=5, \lambda_2=2, \lambda_3=3, \lambda_4=3$.

(ii) For $\lambda_1=5$, $(A-\lambda_1 I|0) \sim \begin{pmatrix} 0 & -2 & 0 & 4 & | & 0 \\ 0 & -3 & 1 & -2 & | & 0 \\ 0 & 0 & -2 & 0 & | & 0 \\ 0 & 0 & 0 & -2 & | & 0 \end{pmatrix} \Rightarrow \begin{cases} -2x_2 + 4x_4 = 0 \\ -3x_2 + x_3 - 2x_4 = 0 \\ -2x_3 = 0 \\ -2x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = t \\ x_2 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{cases}$

$\Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} t, t \neq 0$. Set $t=1 \Rightarrow \vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

For $\lambda_2=2$, $(A-\lambda_2 I|0) \sim \begin{pmatrix} 3 & -2 & 0 & 4 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 0 \end{pmatrix} \Rightarrow \begin{cases} 3x_1 - 2x_2 + 4x_4 = 0 \\ x_3 - 2x_4 = 0 \\ x_3 = 0 \\ x_4 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{2}{3}t \\ x_2 = t \\ x_3 = 0 \\ x_4 = 0 \end{cases}$

$\Rightarrow \vec{x} = \begin{pmatrix} \frac{2}{3} \\ 1 \\ 0 \\ 0 \end{pmatrix} t, t \neq 0$. Set $t=3 \Rightarrow \vec{u}_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}$

For $\lambda_3=\lambda_4=3$, $(A-\lambda_3 I|0) \sim \begin{pmatrix} 2 & -2 & 0 & 4 & | & 0 \\ 0 & -1 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_1 - 2x_2 + 4x_4 = 0 \\ -x_2 + x_3 - 2x_4 = 0 \\ x_3 = t \\ x_4 = s \end{cases} \Rightarrow \begin{cases} x_1 = t - 2s \\ x_2 = t \\ x_3 = t \\ x_4 = s \end{cases}$

$\Rightarrow \vec{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} s, t^2 + s^2 \neq 0$. Set $t=0, s=1 \Rightarrow \vec{u}_3 = \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$, Set $t=1, s=0 \Rightarrow \vec{u}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

(iii) Define

$P = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $P = [\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4]$

$D = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(iv) $B = AP = \begin{pmatrix} 5 & -2 & 0 & 4 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 & -12 \\ 0 & 6 & 3 & -6 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

(v) $C = PD = \begin{pmatrix} 1 & 2 & -2 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 5 & 4 & 3 & -12 \\ 0 & 6 & 3 & -6 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

$B = C$

Question 2 (10 marks) It is given that $w = f(x, y)$, where $x = ts^2, y = \frac{s}{t}$.

(i) (5 marks) Find $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

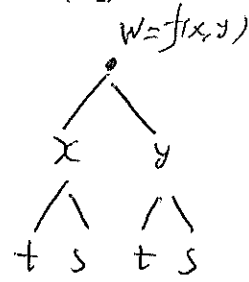
(ii) (5 marks) If $\frac{\partial f}{\partial x} = xy, \frac{\partial f}{\partial y} = \frac{x^2}{2}$, find the values of $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$ at the point $(s, t) = (2, \frac{1}{2})$.

Solution. (i)
$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= \frac{\partial f}{\partial x} \cdot (2ts) + \frac{\partial f}{\partial y} \cdot \left(\frac{1}{t}\right).$$

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= \frac{\partial f}{\partial x} \cdot (s^2) + \frac{\partial f}{\partial y} \cdot \left(\frac{-s}{t^2}\right).$$



(ii) At $(s, t) = (2, \frac{1}{2}) \Rightarrow (x, y) = (2, 4) \Rightarrow \begin{cases} \frac{\partial f}{\partial x}(2, 4) = (2)(4) = 8 \\ \frac{\partial f}{\partial y}(2, 4) = \frac{(2)^2}{2} = 2 \end{cases}$

$$\Rightarrow \frac{\partial w}{\partial s}(2, \frac{1}{2}) = 8 \cdot (2 \cdot \frac{1}{2} \cdot 2) + 2 \cdot \frac{1}{(\frac{1}{2})}$$

$$= 16 + 4$$

$$= 20$$

$$\frac{\partial w}{\partial t}(2, \frac{1}{2}) = 8 \cdot (2^2) + 2 \cdot \left(\frac{-2}{(\frac{1}{2})^2}\right)$$

$$= 32 + \frac{-4}{\frac{1}{4}}$$

$$= 32 - 16$$

$$= 16$$