

In-Class Activity 5 Solutions

Question 1: Using elementary row operations, find the inverse of the following matrix:

$$A = \begin{pmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$$

$$(A|I) = \left(\begin{array}{ccc|ccc} -2 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array} \right)$$

We now use elementary row operations to get $(I|B)$, then $B = A^{-1}$:

$$\xrightarrow{R_1+2R_3 \rightarrow R_3} \left(\begin{array}{ccc|ccc} -2 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 5 & 3 & 1 & 0 & 2 \end{array} \right)$$

$$\xrightarrow{R_3+5R_2 \rightarrow R_3} \left(\begin{array}{ccc|ccc} -2 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 8 & 1 & 5 & 2 \end{array} \right)$$

$$\xrightarrow{R_1-3R_2 \rightarrow R_1} \left(\begin{array}{ccc|ccc} -2 & 4 & 0 & 1 & -3 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 8 & 1 & 5 & 2 \end{array} \right)$$

$$\xrightarrow{R_3-8R_2 \rightarrow R_2} \left(\begin{array}{ccc|ccc} -2 & 4 & 0 & 1 & -3 & 0 \\ 0 & 8 & 0 & 1 & -3 & 2 \\ 0 & 0 & 8 & 1 & 5 & 2 \end{array} \right)$$

$$\xrightarrow{R_2-2R_1 \rightarrow R_1} \left(\begin{array}{ccc|ccc} 4 & 0 & 0 & -1 & 3 & 2 \\ 0 & 8 & 0 & 1 & -3 & 2 \\ 0 & 0 & 8 & 1 & 5 & 2 \end{array} \right)$$

$$\xrightarrow{\frac{1}{4}R_1 \rightarrow R_1 \quad \& \quad \frac{1}{8}R_2 \rightarrow R_2 \quad \& \quad \frac{1}{8}R_3 \rightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/4 & 3/4 & 1/2 \\ 0 & 1 & 0 & 1/8 & -3/8 & 1/4 \\ 0 & 0 & 1 & 1/8 & 5/8 & 1/4 \end{array} \right)$$

Therefore,

$$A^{-1} = \begin{pmatrix} -1/4 & 3/4 & 1/2 \\ 1/8 & -3/8 & 1/4 \\ 1/8 & 5/8 & 1/4 \end{pmatrix}$$

Question 2: Use Cramer's rule to solve the following system of two linear equations in two unknowns (Y and r):

$$(1 - b)Y + ar = C_0 + I_0 + G$$

$$fY - gr = M_s - M_0$$

Writing in matrix form,

$$\begin{pmatrix} 1 - b & a \\ f & -g \end{pmatrix} \begin{pmatrix} Y \\ r \end{pmatrix} = \begin{pmatrix} C_0 + I_0 + G \\ M_s - M_0 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 1 - b & a \\ f & -g \end{vmatrix} = -g(1 - b) - af$$

$$Y = \frac{\begin{vmatrix} C_0 + I_0 + G & a \\ M_s - M_0 & -g \end{vmatrix}}{|A|} = \frac{-g(C_0 + I_0 + G) - a(M_s - M_0)}{-g(1 - b) - af} = \frac{g(C_0 + I_0 + G) + a(M_s - M_0)}{g(1 - b) + af}$$

$$\begin{aligned} r &= \frac{\begin{vmatrix} 1 - b & C_0 + I_0 + G \\ f & M_s - M_0 \end{vmatrix}}{|A|} = \frac{(M_s - M_0)(1 - b) - f(C_0 + I_0 + G)}{-g(1 - b) - af} \\ &= \frac{f(C_0 + I_0 + G) - (1 - b)(M_s - M_0)}{g(1 - b) + af} \end{aligned}$$

Question 3: Find the eigenvalues and associated eigenvectors of the 2×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

We have

$$(A - \lambda I) = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1 - \lambda & 2 \\ 3 & -\lambda \end{pmatrix}$$

The eigenvalues λ are the solutions of the following:

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 2 \\ 3 & -\lambda \end{vmatrix} = 0$$

$$\implies (1 - \lambda)(-\lambda) - 6 = 0 \implies \lambda^2 - \lambda - 6 = 0 \implies \lambda^2 - 3\lambda + 2\lambda - 6 = 0 \implies (\lambda - 3)(\lambda + 2) = 0$$

The above quadratic equation has the two solutions $\lambda_1 = 3$ and $\lambda_2 = -2$, which are the eigenvalues of A .

For $\lambda = 3$, the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ becomes

$$\begin{pmatrix} 1 - \lambda & 2 \\ 3 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x_1 = x_2$$

So, the associated eigenvectors are

$$\mathbf{x} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad t \neq 0$$

For $\lambda = -2$, the system $(A - \lambda I)\mathbf{x} = \mathbf{0}$ becomes

$$\begin{pmatrix} 1 - \lambda & 2 \\ 3 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 3x_1 + 2x_2 = 0$$

Therefore, if $x_2 = t$, we have $x_1 = -\frac{2}{3}t$. So, the associated eigenvectors are

$$\mathbf{x} = t \begin{pmatrix} -2/3 \\ 1 \end{pmatrix}, \quad t \neq 0$$