

## Midterm #2

Solutions:

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Course: EE3121 *Differential Equations for Electrical Engineering*  
Date:

### Question 1

**Solution.** Find the fundamental solutions to the homogeneous equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$$

The characteristic equation is

$$r^2 + 5r + 6 = 0 \quad \Rightarrow \quad r_1 = -2, r_2 = -3.$$

The fundamental solutions are

$$y_1(t) = e^{-2t}, \quad y_2(t) = e^{-3t}.$$

(1) Calculate the Wronskian  $W(\tau)$ :

$$\begin{aligned} W(\tau) &= y_1(\tau)y_2'(\tau) - y_2(\tau)y_1'(\tau) \\ &= e^{-2\tau} \cdot (-3e^{-3\tau}) - e^{-3\tau} \cdot (-2e^{-2\tau}) = -e^{-5\tau}. \end{aligned}$$

(2) Calculate the convolution kernel  $K(t - \tau)$ :

$$\begin{aligned} K(t - \tau) &= \frac{-y_1(t)y_2'(\tau) + y_2(t)y_1'(\tau)}{W(\tau)} \\ &= \frac{-e^{-2t}e^{-3\tau} + e^{-3t}e^{-2\tau}}{-e^{-5\tau}} = e^{-2(t-\tau)} - e^{-3(t-\tau)}. \end{aligned}$$

(3) Solve the differential equation via kernel method:

$$\begin{aligned} y(t) &= \int_0^t K(t - \tau)g(\tau)d\tau = \int_0^t (e^{-2(t-\tau)} - e^{-3(t-\tau)})e^{-\tau}d\tau \\ &= e^{-2t} \int_0^t e^{\tau}d\tau - e^{-3t} \int_0^t e^{2\tau}d\tau \\ &= \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}. \end{aligned}$$

**Question 2**

**Solution.** We begin by transferring both sides of the ODEs, we have

$$L \left\{ \frac{d^2 y}{dt^2} + 7 \frac{dy}{dt} + 12y \right\} = L\{1\} = \frac{1}{s}.$$

Define  $L\{Y(t)\} = Y(s)$ . Then

$$\begin{aligned} L \left\{ \frac{dy}{dt} \right\} &= sL\{y\} - y(0) = sY(s) + 2, \\ L \left\{ \frac{d^2 y}{dt^2} \right\} &= s^2 L\{y\} - sy(0) - \dot{y}(0) = s^2 Y(s) + 2s - 1. \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad s^2 Y(s) + 2s - 1 + 7(sY(s) + 2) + 12Y(s) &= \frac{1}{s} \\ \Rightarrow \quad Y(s) = \frac{-2s^2 - 13s + 1}{s(s^2 + 7s + 12)} &= \frac{-2s^2 - 13s + 1}{s(s+3)(s+4)} = \frac{c_1}{s} + \frac{c_2}{s+3} + \frac{c_3}{s+4} \end{aligned}$$

where

$$\begin{aligned} c_1 &= \frac{-2s^2 - 13s + 1}{(s+3)(s+4)} \Big|_{s=0} = \frac{1}{12} \\ c_2 &= \frac{-2s^2 - 13s + 1}{s(s+4)} \Big|_{s=-3} = -\frac{22}{3} \\ c_3 &= \frac{-2s^2 - 13s + 1}{s(s+3)} \Big|_{s=-4} = \frac{21}{4} \end{aligned}$$

Therefore

$$Y(s) = \frac{1}{12} \frac{1}{s} - \frac{22}{3} \frac{1}{s+3} + \frac{21}{4} \frac{1}{s+4} \quad \Rightarrow \quad y(t) = \frac{1}{12} - \frac{22}{3} e^{-3t} + \frac{21}{4} e^{-4t}$$

**Question 3**

**Solution.** From the state space form we have

$$\begin{cases} \dot{x}_1(t) = 0 \\ \dot{x}_2(t) = tx_1(t) \end{cases}$$

For  $x_1(t)$ :

$$\frac{dx_1}{dt} = 0 \Rightarrow x_1(t) = c_1$$

For  $x_2(t)$ ,

$$\frac{dx_2}{dt} = tx_1(t) = tc_1 \Rightarrow x_2(t) = \frac{c_1}{2}t^2 + c_2$$

Therefore

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2c_1 \\ c_1t^2 + c_2 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ t^2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ t^2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(1) A fundamental matrix  $\Psi(t)$

$$\Psi(t) = \begin{bmatrix} 2 & 0 \\ t^2 & 1 \end{bmatrix}$$

(2) The state transition matrix  $\Phi(t, t_0)$

$$\begin{aligned} \Phi(t, t_0) &= \Psi(t)\Psi^{-1}(t_0) = \begin{bmatrix} 2 & 0 \\ t^2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ t_0^2 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 2 & 0 \\ t^2 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 0 \\ -t_0^2 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ t^2 - t_0^2 & 2 \end{bmatrix} \end{aligned}$$

(3) The solution  $x(t)$

$$x(t) = \Phi(t, t_0)x(t_0) = \Phi(t, 0) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 0 \\ t^2 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

**Question 4**

**Solution.** According to Kirchhoff circuit laws

$$\begin{aligned}i_C &= \frac{u_L}{R_2} + i_L \\e_i &= u_L + R_1 i_C + u_C \\e_o &= R_1 i_C + u_C\end{aligned}$$

Therefore,

$$\begin{aligned}C \frac{du_C}{dt} &= \frac{L}{R_2} \frac{di_L}{dt} + i_L \\e_i &= L \frac{di_L}{dt} + R_1 C \frac{du_C}{dt} + u_C \\e_o &= R_1 C \frac{du_C}{dt} + u_C\end{aligned}$$

Denote  $x_1 = i_L$ ,  $x_2 = u_C$ , we obtain

$$\begin{aligned}\frac{dx_1}{dt} &= -\frac{R_1 R_2}{(R_1 + R_2)L} x_1 - \frac{R_2}{(R_1 + R_2)L} x_2 + \frac{R_2}{(R_1 + R_2)L} e_i \\ \frac{dx_2}{dt} &= \frac{R_2}{(R_1 + R_2)C} x_1 - \frac{1}{(R_1 + R_2)C} x_2 + \frac{1}{(R_1 + R_2)C} e_i \\ e_o &= \frac{R_1 R_2}{R_1 + R_2} x_1 + \frac{R_2}{R_1 + R_2} x_2 + \frac{R_1}{R_1 + R_2} e_i\end{aligned}$$

In conclusion, the state space description is

$$\begin{aligned}\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -\frac{R_1 R_2}{(R_1 + R_2)L} & -\frac{R_2}{(R_1 + R_2)L} \\ \frac{R_2}{(R_1 + R_2)C} & -\frac{1}{(R_1 + R_2)C} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{R_2}{(R_1 + R_2)L} \\ \frac{1}{(R_1 + R_2)C} \end{bmatrix} e_i \\ e_o &= \begin{bmatrix} \frac{R_1 R_2}{R_1 + R_2} & \frac{R_2}{R_1 + R_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \frac{R_1}{R_1 + R_2} e_i\end{aligned}$$


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