

MA1201 Calculus and Basic Linear Algebra II

Solution of Problem Set 3

Application of Integration

Problem 1

(a) The required area

$$= \int_1^3 (1 + e^{3x}) dx = x + \frac{1}{3} e^{3x} \Big|_1^3 = 2 + \frac{1}{3} (e^9 - e^3).$$

(b) The required area

$$\begin{aligned} &= \int_{\frac{1}{2}}^2 |\ln x| dx = \int_{\frac{1}{2}}^1 (-\ln x) dx + \int_1^2 \ln x dx \\ &\quad \begin{array}{l} u = \ln x \\ dv = dx \\ \Rightarrow v = \int dx = x \end{array} \\ &\quad \hat{=} - \left(x \ln x \Big|_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 x d(\ln x) \right) + \left(x \ln x \Big|_1^2 - \int_1^2 x d(\ln x) \right) \\ &= \frac{1}{2} \ln \left(\frac{1}{2} \right) + \int_{\frac{1}{2}}^1 1 dx + 2 \ln 2 - \int_1^2 1 dx = \frac{1}{2} \ln \left(\frac{1}{2} \right) + x \Big|_{\frac{1}{2}}^1 + 2 \ln 2 - x \Big|_1^2 \\ &= \frac{1}{2} \ln \left(\frac{1}{2} \right) + \frac{1}{2} + 2 \ln 2 - 1 \quad \begin{array}{l} \ln \frac{1}{2} = \ln 2^{-1} = -\ln 2 \\ \hat{=} \end{array} \quad \frac{3}{2} \ln 2 - \frac{1}{2}. \end{aligned}$$

(c) The required area

$$\begin{aligned} &= \int_0^2 [(1 + x^2) - e^{-x}] dx \\ &= x + \frac{x^3}{3} + e^{-x} \Big|_0^2 = 2 + \frac{8}{3} + e^{-2} - e^{-0} \\ &= \frac{11}{3} + e^{-2}. \end{aligned}$$

(d) The intersection points can be found by solving

$$\begin{aligned} &\begin{cases} y = -x^2 + 2x + 1 \\ x + y = 1 \end{cases} \Rightarrow 1 - x = -x^2 + 2x + 1 \\ &\Rightarrow x^2 - 3x = 0 \Rightarrow x(x - 3) = 0 \\ &\Rightarrow x = 0 (y = 1), \quad x = 3 (y = -2). \end{aligned}$$

Using the graph, the required area is given by

$$\begin{aligned} &= \int_0^3 (-x^2 + 2x + 1) - (1 - x) dx = \int_0^3 (-x^2 + 3x) dx \\ &= -\frac{x^3}{3} + \frac{3x^2}{2} \Big|_0^3 = \frac{9}{2}. \end{aligned}$$

Problem 2

(a) Set $f_1(x) = f_2(x)$, we obtain the following equation

$$\begin{aligned} &e^{2x} - 3e^x - 1 = e^x - 4 \Rightarrow e^{2x} - 4e^x + 3 = 0 \\ &\quad \begin{array}{l} y = e^x \\ \hat{=} \end{array} y^2 - 4y + 3 = 0 \Rightarrow (y - 1)(y - 3) = 0 \\ &\Rightarrow y = e^x = 1 \quad \text{or} \quad y = e^x = 3 \\ &\Rightarrow x = \ln 1 = 0 \quad \text{or} \quad x = \ln 3. \end{aligned}$$

(b) Based on the critical points obtained in (a), we divide the interval $[-2, 2]$ into several parts and compare the values of $f_1(x)$ and $f_2(x)$ in each of the parts:

x	$-2 \leq x < 0$	$0 < x < \ln 3$	$\ln 3 < x < 2$
$f_1(x)$ v.s. $f_2(x)$	$f_1(x) > f_2(x)$	$f_1(x) < f_2(x)$	$f_1(x) > f_2(x)$

We conclude from the above analysis that

- $f_1(x) > f_2(x)$ for $-2 < x < 0$ or $\ln 3 < x < 2$
- $f_1(x) < f_2(x)$ for $0 < x < \ln 3$.

(c) Using the result of (b), the required area is found to be

$$\begin{aligned}
 &= \int_{-2}^0 (f_1(x) - f_2(x))dx + \int_0^{\ln 3} (f_2(x) - f_1(x))dx + \int_{\ln 3}^2 (f_1(x) - f_2(x))dx \\
 &= \int_{-2}^0 (e^{2x} - 4e^x + 3)dx + \int_0^{\ln 3} (-e^{2x} + 4e^x - 3)dx + \int_{\ln 3}^2 (e^{2x} - 4e^x + 3)dx \\
 &= \frac{1}{2}e^{2x} - 4e^x + 3 \Big|_{-2}^0 + \left(-\frac{1}{2}e^{2x} + 4e^x - 3 \right) \Big|_0^{\ln 3} + \frac{1}{2}e^{2x} - 4e^x + 3 \Big|_{\ln 3}^2 \\
 &= -\frac{1}{2}e^{-4} + 4e^{-2} + \frac{1}{2}e^4 - 4e^2 - 6\ln 3 + 14.
 \end{aligned}$$

Problem 3

We let $f(x) = (x^2 - x + 1)e^x$ and $g(x) = xe^x$.

To compare the values of $f(x)$ and $g(x)$ within $[0, 2]$, we first obtain all critical points by solving

$$f(x) = g(x) \Rightarrow (x^2 - x + 1)e^x = xe^x$$

$$\Rightarrow (x^2 - 2x + 1)e^x = 0 \Rightarrow (x - 1)^2 e^x = 0 \Rightarrow x = 1.$$

Then we divide the integral into two parts and compare the values of two functions in each part:

x	$0 \leq x < 1$	$1 < x < 2$
$f(x)$ v.s. $g(x)$	$f(x) > g(x)$	$f(x) > g(x)$

Therefore, we conclude that $f(x) \geq g(x)$ for ALL x between 0 and 2.

Using the data obtained, the required area is

$$\begin{aligned}
 \int_0^2 [f(x) - g(x)]dx &= \int_0^2 [(x^2 - x + 1)e^x - xe^x]dx = \int_0^2 (x^2 - 2x + 1)e^x dx \\
 &= \int_0^2 \underbrace{(x-1)^2}_u \underbrace{e^x}_{dv} dx \stackrel{\substack{u=(x-1)^2 \\ dv=e^x dx \\ \Rightarrow v=\int e^x dx=e^x}}{\cong} (x-1)^2 e^x \Big|_0^2 - \int_0^2 e^x d(x-1)^2 \\
 &= (e^2 - 1) - 2 \int_0^2 \underbrace{(x-1)}_u \underbrace{e^x}_{dv} dx \stackrel{\substack{u=x-1 \\ dv=e^x dx \\ \Rightarrow v=\int e^x dx=e^x}}{\cong} (e^2 - 1) - 2 \left[(x-1)e^x \Big|_0^2 - \int_0^2 e^x d(x-1) \right] \\
 &= e^2 - 1 - 2(e^2 + 1) + 2 \int_0^2 e^x dx = -e^2 - 3 + 2e^x \Big|_0^2 = e^2 - 5.
 \end{aligned}$$

Problem 4

(a) The required volume

$$\begin{aligned}
 &= \pi \int_0^{\frac{\pi}{3}} \sin^2 3x dx + \pi \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin^2 3x dx + \pi \int_{\frac{2\pi}{3}}^{\pi} \sin^2 3x dx \\
 &= \pi \int_0^{\pi} \sin^2 3x dx
 \end{aligned}$$

$$\begin{aligned}
&= \pi \int_0^{\pi} -\frac{1}{2}(\cos(3x+3x) - \cos(3x-3x))dx \\
&= \frac{\pi}{2} \int_0^{\pi} (1 - \cos 6x)dx = \frac{\pi}{2} \left(x - \frac{\sin 6x}{6} \right)_0^{\pi} = \frac{\pi^2}{2}.
\end{aligned}$$

(b) Using the graph, the required volume is

$$\begin{aligned}
&= \pi \int_0^{\frac{\pi}{2}} (1 + \cos x)^2 dx - \pi \int_0^{\frac{\pi}{2}} (1 + \cos 3x)^2 dx \\
&= \pi \int_0^{\frac{\pi}{2}} (2 \cos x + \cos^2 x - 2 \cos 3x - \cos^2 3x) dx \\
&= \pi \int_0^{\frac{\pi}{2}} \left(2 \cos x + \frac{\cos 2x + 1}{2} - 2 \cos 3x - \frac{\cos 6x + 1}{2} \right) dx \\
&= \pi \int_0^{\frac{\pi}{2}} \left(2 \cos x + \frac{\cos 2x}{2} - 2 \cos 3x - \frac{\cos 6x}{2} \right) dx \\
&= \pi \left(2 \sin x + \frac{\sin 2x}{4} - \frac{2}{3} \sin 3x - \frac{\sin 6x}{12} \right) \Big|_0^{\frac{\pi}{2}} \\
&= 2\pi + \frac{2\pi}{3} = \frac{8\pi}{3}.
\end{aligned}$$

(c) (i) The required volume

$$\int_0^{\ln 3} \pi(e^{2x})^2 dx = \pi \int_0^{\ln 3} e^{4x} dx = \frac{\pi}{4} e^{4x} \Big|_0^{\ln 3} = \frac{80\pi}{4} = 20\pi.$$

(ii) The required volume

$$\begin{aligned}
&= \int_0^9 \pi(\ln 3)^2 dy - \int_1^9 \pi \left(\frac{1}{2} \ln y \right)^2 dy \\
&= \pi(\ln 3)^2 \int_0^9 1 dy - \frac{\pi}{4} \int_1^9 \underbrace{(\ln y)^2}_u \underbrace{dy}_{dv} \\
&\quad \begin{array}{l} u = (\ln y)^2 \\ dv = dy \\ \Rightarrow v = \int dy = y \end{array} \\
&\quad \hat{=} \pi(\ln 3)^2 y \Big|_0^9 - \frac{\pi}{4} \left[y(\ln y)^2 \Big|_1^9 - \int_1^9 y d(\ln y)^2 \right] \\
&= 9(\ln 3)^2 \pi - \frac{9\pi(\ln 9)^2}{4} + \frac{\pi}{2} \int_1^9 \underbrace{\ln y}_u \underbrace{dy}_{dv} \\
&\quad \begin{array}{l} u = \ln y \\ dv = dy \\ \Rightarrow v = \int dy = y \end{array} \\
&\quad \hat{=} 9(\ln 3)^2 \pi - \frac{9\pi(2 \ln 3)^2}{4} + \frac{\pi}{2} \left(y \ln y \Big|_1^9 - \int_1^9 y d(\ln y) \right) \\
&= \frac{\pi}{2} \left(9 \ln 9 - \int_1^9 1 dy \right) = \frac{9\pi}{2} \ln 9 - \frac{\pi}{2} y \Big|_1^9 = \frac{9\pi}{2} \ln 9 - 4\pi.
\end{aligned}$$

(iii) The required volume

$$\begin{aligned}
&= \pi \int_0^{\ln 3} (e^{2x} + 1)^2 dx - \pi \int_0^{\ln 3} (1)^2 dx \\
&= \pi \int_0^{\ln 3} (e^{4x} + 2e^{2x}) dx = \pi \left(\frac{e^{4x}}{4} + e^{2x} \right) \Big|_0^{\ln 3} \\
&= \frac{81\pi}{4} - \frac{\pi}{4} + 9\pi - \pi = 28\pi.
\end{aligned}$$

$$\begin{aligned}
V &= 2\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} x \left(\sin x - \frac{1}{2} \right) dx = -2\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} x d \cos x - \frac{\pi}{2} [x^2]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \\
&= -2\pi \left([x \cos x]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \cos x dx \right) - \frac{\pi}{2} \left(\frac{25\pi^2}{36} - \frac{\pi^2}{36} \right) \\
&= -2\pi \left(\frac{5\pi}{6} \left(-\frac{\sqrt{3}}{2} \right) - \frac{\pi}{6} \left(\frac{\sqrt{3}}{2} \right) - [\sin x]_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \right) - \frac{\pi^3}{3} \\
&= -2\pi \left(-\frac{\sqrt{3}\pi}{2} - \left[\frac{1}{2} - \frac{1}{2} \right] \right) - \frac{\pi^3}{3} = \sqrt{3}\pi^2 - \frac{\pi^3}{3}.
\end{aligned}$$

(iii) Using the graph, the required volume is given by

$$\begin{aligned}
V &= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \left(\sin x - \frac{1}{2} \right)^2 dx = \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin^2 x dx - \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin x dx + \frac{\pi}{4} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} 1 dx \\
&= \pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1 - \cos 2x}{2} dx + \pi \cos x \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} + \frac{\pi}{4} x \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} = \pi \left(\frac{x}{2} - \frac{\sin 2x}{4} \right) \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}} - \pi\sqrt{3} + \frac{\pi^2}{6} = \frac{\pi^2}{2} - \frac{3\sqrt{3}}{4}\pi.
\end{aligned}$$

Problem 5

(a) The arc length

$$\begin{aligned}
&= \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{d}{dx} \ln(\sec x) \right)^2} dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \left(\frac{\sec x \tan x}{\sec x} \right)^2} dx = \int_0^{\frac{\pi}{4}} \sqrt{1 + \tan^2 x} dx \\
&= \int_0^{\frac{\pi}{4}} \sec x dx = \ln|\sec x + \tan x| \Big|_0^{\frac{\pi}{4}} = \ln|1 + \sqrt{2}|.
\end{aligned}$$

(b) The arc length

$$= \int_0^5 \sqrt{1 + \left(\frac{d}{dx} \left(\frac{1}{3} x^{\frac{3}{2}} \right) \right)^2} dx = \int_0^5 \sqrt{1 + \frac{x}{4}} dx = \frac{1}{\frac{1}{4}} \frac{\left(1 + \frac{x}{4} \right)^{\frac{3}{2}}}{\frac{3}{2}} \Big|_0^5 = \frac{8}{3} \left(\frac{27}{8} - 1 \right) = \frac{19}{3}.$$

(c) Note that

$$(y - 1)^3 = \frac{9}{4} x^2 \Rightarrow y = 1 + \sqrt[3]{\frac{9}{4} x^2}.$$

The arc length

$$\begin{aligned}
&= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{1 + \left(\frac{d}{dx} \left(1 + \sqrt[3]{\frac{9}{4} x^2} \right) \right)^2} dx = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{1 + \left(\frac{2}{3} \right)^{\frac{2}{3}} x^{-\frac{2}{3}}} dx \\
&= \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \sqrt{\frac{x^{\frac{2}{3}} + \left(\frac{2}{3} \right)^{\frac{2}{3}}}{x^{\frac{2}{3}}}} dx = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \frac{\sqrt{x^{\frac{2}{3}} + \left(\frac{2}{3} \right)^{\frac{2}{3}}}}{x^{\frac{1}{3}}} dx \\
&\stackrel{y=x^{\frac{2}{3}}+\left(\frac{2}{3}\right)^{\frac{2}{3}}}{=} \int_{\left(\frac{2}{3}\right)^{\frac{2}{3}}}^{\left(\frac{2}{3}\right)^{\frac{2}{3}}(3)} \frac{3}{2} \sqrt{y} dy = y^{\frac{3}{2}} \Big|_{\left(\frac{2}{3}\right)^{\frac{2}{3}}}^{\left(\frac{2}{3}\right)^{\frac{2}{3}}(3)} = \frac{2}{3} (3\sqrt{3} - 1).
\end{aligned}$$

(d) The arc length

$$\begin{aligned}
&= \int_0^a \sqrt{1 + \left(\frac{d}{dx} \left(\frac{a}{2} \left(e^{\frac{x}{a}} + e^{-\frac{x}{a}} \right) \right) \right)^2} dx = \int_0^a \sqrt{1 + \left(\frac{1}{2} e^{\frac{x}{a}} - \frac{1}{2} e^{-\frac{x}{a}} \right)^2} dx \\
&= \int_0^a \sqrt{1 + \left(\frac{1}{2} e^{\frac{x}{a}} - \frac{1}{2} e^{-\frac{x}{a}} \right)^2} dx = \int_0^a \sqrt{\frac{1}{4} e^{\frac{2x}{a}} + \frac{1}{2} + \frac{1}{4} e^{-\frac{2x}{a}}} dx \\
&= \int_0^a \sqrt{\left(\frac{1}{2} e^{\frac{x}{a}} + \frac{1}{2} e^{-\frac{x}{a}} \right)^2} dx = \int_0^a \left(\frac{1}{2} e^{\frac{x}{a}} + \frac{1}{2} e^{-\frac{x}{a}} \right) dx \\
&= \frac{a}{2} e^{\frac{x}{a}} - \frac{a}{2} e^{-\frac{x}{a}} \Big|_0^a = \frac{a}{2} \left(e - \frac{1}{e} \right).
\end{aligned}$$

(e) The arc length

$$= \int_0^{\frac{\pi}{2}} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt = \int_0^{\frac{\pi}{2}} \sqrt{(t \cos t)^2 + (t \sin t)^2} dt = \int_0^{\frac{\pi}{2}} \sqrt{t^2} dt = \int_0^{\frac{\pi}{2}} t dt = \frac{t^2}{2} \Big|_0^{\frac{\pi}{2}} = \frac{\pi^2}{8}.$$

(f) The arc length

$$\begin{aligned}
&= \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt = \int_0^{2\pi} \sqrt{(1 - \cos t)^2 + (\sin t)^2} dt \\
&= \int_0^{2\pi} \sqrt{1 - 2 \cos t + \cos^2 t + \sin^2 t} dt = \int_0^{2\pi} \sqrt{2 - 2 \cos t} dt \\
&= \sqrt{2} \int_0^{2\pi} \sqrt{1 - \cos t} dt = \sqrt{2} \int_0^{2\pi} \sqrt{\cos 0 - \cos t} dt \\
&= \sqrt{2} \int_0^{2\pi} \sqrt{-2 \sin \left(\frac{0+t}{2} \right) \sin \left(\frac{0-t}{2} \right)} dt = \sqrt{2} \int_0^{2\pi} \sqrt{-2 \sin \frac{t}{2} \sin \left(-\frac{t}{2} \right)} dt \\
&= 2 \int_0^{2\pi} \sqrt{\sin^2 \frac{t}{2}} dt = 2 \int_0^{2\pi} \sin \frac{t}{2} dt = -4 \cos \frac{t}{2} \Big|_0^{2\pi} = 8.
\end{aligned}$$

(g) The arc length

$$\begin{aligned}
&= \int_0^a \sqrt{\left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2} dt = \int_0^a \sqrt{(2at)^2 + (2a)^2} dt \\
&= 2a \int_0^a \sqrt{1 + t^2} dt \stackrel{t=\tan \theta}{=} 2a \int_0^{\tan^{-1} a} \sqrt{1 + (\tan \theta)^2} \sec^2 \theta d\theta \\
&= 2a \int_0^{\tan^{-1} a} \sec^3 \theta d\theta = 2a \left[\frac{1}{2} (\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|) \right] \Big|_0^{\tan^{-1} a} \\
&= a^2 \sqrt{1 + a^2} + a \ln |a + \sqrt{1 + a^2}|.
\end{aligned}$$

Problem 6

(a) The surface area

$$= 2\pi \int_0^2 x^3 \sqrt{1 + \left(\frac{d}{dx} x^3 \right)^2} dx$$

$$\begin{aligned}
&= 2\pi \int_0^2 x^3 \sqrt{1+9x^4} dx \\
&\quad y=1+9x^4 \\
&\quad \frac{dy}{dx}=36x^3 \\
&\quad \cong \frac{2\pi}{36} \int_1^{145} \sqrt{y} dy = \frac{\pi}{18} \frac{y^{\frac{3}{2}}}{\frac{3}{2}} \Big|_1^{145} \\
&= \frac{\pi}{27} \left(145^{\frac{3}{2}} - 1 \right).
\end{aligned}$$

(b) The surface area

$$\begin{aligned}
&= 2\pi \int_0^4 \sqrt{4-x} \sqrt{1 + \left(\frac{d}{dx} \sqrt{4-x} \right)^2} dx \\
&= 2\pi \int_0^4 \sqrt{4-x} \sqrt{1 + \frac{1}{4(4-x)}} dx \\
&= \pi \int_0^4 \sqrt{17-4x} dx \\
&= \pi \left(-\frac{1}{4} \frac{(17-4x)^{\frac{3}{2}}}{\frac{3}{2}} \right) \Big|_0^4 = \frac{\pi}{6} (17^{\frac{3}{2}} - 1).
\end{aligned}$$

(c) The surface area

$$\begin{aligned}
I &= 2\pi \int_0^1 e^x \sqrt{1 + \left(\frac{d}{dx} e^x \right)^2} dx \\
&= 2\pi \int_0^1 e^x \sqrt{1 + e^{2x}} dx
\end{aligned}$$

Let $e^x = \tan \theta$ (so that $\sqrt{1 + e^{2x}} = \sqrt{1 + \tan^2 \theta} = \sec \theta$),

$$\Rightarrow \frac{d}{d\theta} e^x = \frac{d}{d\theta} \tan \theta \Rightarrow e^x \frac{dx}{d\theta} = \sec^2 \theta \Rightarrow dx = \frac{\sec^2 \theta}{e^x} d\theta.$$

When $x = 1$, $\theta = \tan^{-1} e$. When $x = 0$, $\theta = \tan^{-1} 1 = \frac{\pi}{4}$.

The integral then becomes

$$\begin{aligned}
I &= 2\pi \int_{\frac{\pi}{4}}^{\tan^{-1} e} e^x \sqrt{1 + e^{2x}} \left(\frac{\sec^2 \theta}{e^x} d\theta \right) = 2\pi \int_{\frac{\pi}{4}}^{\tan^{-1} e} \sqrt{1 + \tan^2 \theta} \sec^2 \theta d\theta \\
&= 2\pi \int_{\frac{\pi}{4}}^{\tan^{-1} e} \sec^3 \theta d\theta = 2\pi \left[\frac{1}{2} (\sec x \tan x + \ln |\sec x + \tan x|) \right]_{\frac{\pi}{4}}^{\tan^{-1} e} \\
&= \pi \left(e\sqrt{1 + e^2} + \ln |e + \sqrt{1 + e^2}| \right) - \pi (\sqrt{2} + \ln |\sqrt{2} + 1|).
\end{aligned}$$

(d) The surface area

$$\begin{aligned}
&= \left[2\pi \int_1^2 x \sqrt{1 + \left(\frac{d}{dx} x \right)^2} dx + 2\pi \int_1^2 (2-x) \sqrt{1 + \left(\frac{d}{dx} (2-x) \right)^2} dx \right] \\
&\quad + \left[2\pi \int_2^3 (4-x) \sqrt{1 + \left(\frac{d}{dx} (4-x) \right)^2} dx + 2\pi \int_2^3 (x-2) \sqrt{1 + \left(\frac{d}{dx} (x-2) \right)^2} dx \right] \\
&= \left[2\sqrt{2}\pi \int_1^2 x + 2\sqrt{2}\pi \int_1^2 (2-x) dx \right] + \left[2\sqrt{2}\pi \int_2^3 (4-x) dx + 2\sqrt{2}\pi \int_2^3 (x-2) dx \right]
\end{aligned}$$

$$\begin{aligned}
&= 2\sqrt{2}\pi \left(\frac{x^2}{2}\right) \Big|_1^2 + 2\sqrt{2}\pi \left(2x - \frac{x^2}{2}\right) \Big|_1^2 + 2\sqrt{2}\pi \left(4x - \frac{x^2}{2}\right) \Big|_2^3 + 2\sqrt{2}\pi \left(\frac{x^2}{2} - 2x\right) \Big|_2^3 \\
&= 2\sqrt{2}\pi \left(\frac{3}{2} + \frac{1}{2} + \frac{3}{2} + \frac{1}{2}\right) = 8\sqrt{2}\pi.
\end{aligned}$$

Problem 7

(a) The required area

$$= \int_0^2 (e^x - e^{-x}) dx = e^x + e^{-x} \Big|_0^2 = e^2 + e^{-2} - 2.$$

(b) (i) The required volume

$$\begin{aligned}
&= \pi \int_0^2 (e^x)^2 dx - \pi \int_0^2 (e^{-x})^2 dx = \pi \int_0^2 (e^{2x} - e^{-2x}) dx \\
&= \pi \left(\frac{e^{2x}}{2} + \frac{e^{-2x}}{2} \right) \Big|_0^2 = \pi \left(\frac{e^4}{2} + \frac{e^{-4}}{2} - 1 \right).
\end{aligned}$$

(ii) The required volume

$$\begin{aligned}
&= \left(\pi \int_{e^{-2}}^1 2^2 dy - \pi \int_{e^{-2}}^1 (-\ln y)^2 dy \right) + \left(\pi \int_1^{e^2} 2^2 dy - \pi \int_1^{e^2} (\ln y)^2 dy \right) \\
&= \pi \int_{e^{-2}}^{e^2} 2^2 dy - \pi \int_{e^{-2}}^{e^2} \underbrace{(\ln y)^2}_u \underbrace{dy}_{dv} \\
&= 4\pi y \Big|_{e^{-2}}^{e^2} - \pi \left[y(\ln y)^2 \Big|_{e^{-2}}^{e^2} - \int_{e^{-2}}^{e^2} y d(\ln y)^2 \right] \\
&= 4\pi(e^2 - e^{-2}) - 4\pi(e^2 - e^{-2}) + 2\pi \int_{e^{-2}}^{e^2} y \ln y dy \\
&= 2\pi \left(y \ln y \Big|_{e^{-2}}^{e^2} - \int_{e^{-2}}^{e^2} y d(\ln y) \right) \\
&= 4\pi(e^2 + e^{-2}) - 2\pi \int_{e^{-2}}^{e^2} 1 dy \\
&= 4\pi(e^2 + e^{-2}) - 2\pi y \Big|_{e^{-2}}^{e^2} = 2\pi e^2 + 6\pi e^{-2}.
\end{aligned}$$

(iii) The required volume

$$\begin{aligned}
&= \pi \int_0^2 (e^x + 1)^2 dx - \pi \int_0^2 (e^{-x} + 1)^2 dx \\
&= \pi \int_0^2 (e^{2x} + 2e^x - e^{-2x} - 2e^{-x}) dx \\
&= \pi \left(\frac{e^{2x}}{2} + 2e^x + \frac{e^{-2x}}{2} + 2e^{-x} \right) \Big|_0^2 \\
&= \pi \left(\frac{e^4}{2} + 2e^2 + \frac{e^{-4}}{2} + 2e^{-2} - 5 \right).
\end{aligned}$$

Problem 8

(a) The intersection points can be obtained by solving

$$\begin{cases} y = x^2 \\ x = y^2 \end{cases} \Rightarrow x^2 = \sqrt{x} \Rightarrow \sqrt{x} \left(x^{\frac{3}{2}} - 1 \right) = 0$$

$$\Rightarrow x = 0 \ (y = 0) \text{ or } x = 1 \ (y = 1).$$

The required area

$$= \int_0^1 (\sqrt{x} - x^2) dx = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^3}{3} \Big|_0^1 = \frac{1}{3}.$$

(b) (i) The required volume

$$\begin{aligned} &= \pi \int_0^1 (\sqrt{x})^2 dx - \pi \int_0^1 (x^2)^2 dx = \pi \int_0^1 (x - x^4) dx \\ &= \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right) \Big|_0^1 = \frac{3\pi}{10}. \end{aligned}$$

(ii) The required volume

$$\begin{aligned} &= \pi \int_0^1 (\sqrt{y})^2 dy - \pi \int_0^1 (y^2)^2 dy = \pi \int_0^1 (y - y^4) dy \\ &= \pi \left(\frac{y^2}{2} - \frac{y^5}{5} \right) \Big|_0^1 = \frac{3\pi}{10}. \end{aligned}$$

(iii) The required volume

$$\begin{aligned} &= \pi \int_0^1 (\sqrt{x} + 1)^2 dx - \pi \int_0^1 (x^2 + 1)^2 dx \\ &= \pi \int_0^1 (x + 2\sqrt{x} - x^4 - 2x^2) dx \\ &= \pi \left(\frac{x^2}{2} + \frac{4}{3}x^{\frac{3}{2}} - \frac{x^5}{5} - \frac{2}{3}x^3 \right) \Big|_0^1 = \frac{29\pi}{30}. \end{aligned}$$

(c) The required arc length

$$\begin{aligned} &= \int_0^1 \sqrt{1 + \left(\frac{d}{dx} \sqrt{x} \right)^2} dx + \int_0^1 \sqrt{1 + \left(\frac{d}{dx} x^2 \right)^2} dx \\ &= \int_0^1 \sqrt{1 + \frac{1}{4x}} dx + \int_0^1 \sqrt{1 + 4x^2} dx \\ &= \int_0^1 \frac{\sqrt{4x+1}}{2\sqrt{x}} dx + \int_0^1 \sqrt{1+4x^2} dx \\ &\stackrel{y=\sqrt{x}}{\cong} \int_0^1 \sqrt{1+4y^2} dy + \int_0^1 \sqrt{1+4x^2} dx \\ &= 2 \int_0^1 \sqrt{1+4x^2} dx \\ &\stackrel{x=\frac{1}{2}\tan\theta}{\cong} 2 \int_0^{\tan^{-1}2} \sqrt{1+\tan^2\theta} \left(\frac{1}{2} \sec^2\theta d\theta \right) \\ &= \int_0^{\tan^{-1}2} \sec^3\theta d\theta \\ &= \frac{1}{2} (\sec\theta \tan\theta + \ln|\sec\theta + \tan\theta|) \Big|_0^{\tan^{-1}2} \\ &= \frac{1}{2} (2\sqrt{5} + \ln|2 + \sqrt{5}|). \end{aligned}$$

Problem 9

(a) The required area

$$\begin{aligned}
 A &= \int_0^1 \sqrt{x} dx + \int_1^{\sqrt{2}} \sqrt{2-x^2} dx \stackrel{\substack{x=\sqrt{2} \sin \theta \\ \frac{dx}{d\theta}=\sqrt{2} \cos \theta}}{\cong} \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}} \right]_0^1 + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{2 \cos^2 \theta} (\sqrt{2} \cos \theta d\theta) \\
 &= \frac{2}{3} + 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 \theta d\theta = \frac{2}{3} + 2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta \\
 &= \frac{2}{3} + \left[\theta + \frac{\sin 2\theta}{2} \right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{4} + \frac{1}{6}.
 \end{aligned}$$

(b) (i) The required volume is

$$\begin{aligned}
 &\pi \int_0^1 (\sqrt{x})^2 dx + \pi \int_1^{\sqrt{2}} (\sqrt{2-x^2})^2 dx = \pi \int_0^1 x dx + \pi \int_1^{\sqrt{2}} (2-x^2) dx \\
 &= \pi \left[\frac{x^2}{2} \right]_0^1 + \pi \left[2x - \frac{x^3}{3} \right]_1^{\sqrt{2}} = \frac{4\sqrt{2}-7}{6} \pi.
 \end{aligned}$$

(ii) The required volume is

$$\begin{aligned}
 &\pi \int_0^1 (\sqrt{2-y^2})^2 dy - \pi \int_0^1 (y^2)^2 dy \\
 &= \pi \int_0^1 (2-y^2-y^4) dy = \pi \left[2y - \frac{y^3}{3} - \frac{y^5}{5} \right]_0^1 = \frac{22\pi}{15}.
 \end{aligned}$$

(c) The surface area is

$$\begin{aligned}
 &= 2\pi \left(\int_0^1 \sqrt{x} \sqrt{1 + \left(\frac{d}{dx} \sqrt{x} \right)^2} dx + \int_1^{\sqrt{2}} \sqrt{2-x^2} \sqrt{1 + \left(\frac{d}{dx} \sqrt{2-x^2} \right)^2} dx \right) \\
 &= 2\pi \left(\int_0^1 \sqrt{x} \sqrt{1 + \frac{1}{4x}} dx + \int_1^{\sqrt{2}} \sqrt{2-x^2} \sqrt{1 + \frac{x^2}{2-x^2}} dx \right) \\
 &= 2\pi \left(\int_0^1 \frac{1}{2} \sqrt{4x+1} dx + \int_1^{\sqrt{2}} \sqrt{2} dx \right) \\
 &= 2\pi \left(\left[\frac{1}{12} (4x+1)^{\frac{3}{2}} \right]_0^1 + [\sqrt{2}x]_1^{\sqrt{2}} \right) = \frac{\pi}{6} \left(5^{\frac{3}{2}} - 1 \right) + 2\pi(2 - \sqrt{2}).
 \end{aligned}$$