

EE3122: Analogue Circuit Fundamentals

Frequency Response

Department of Electrical Engineering

City University of Hong Kong

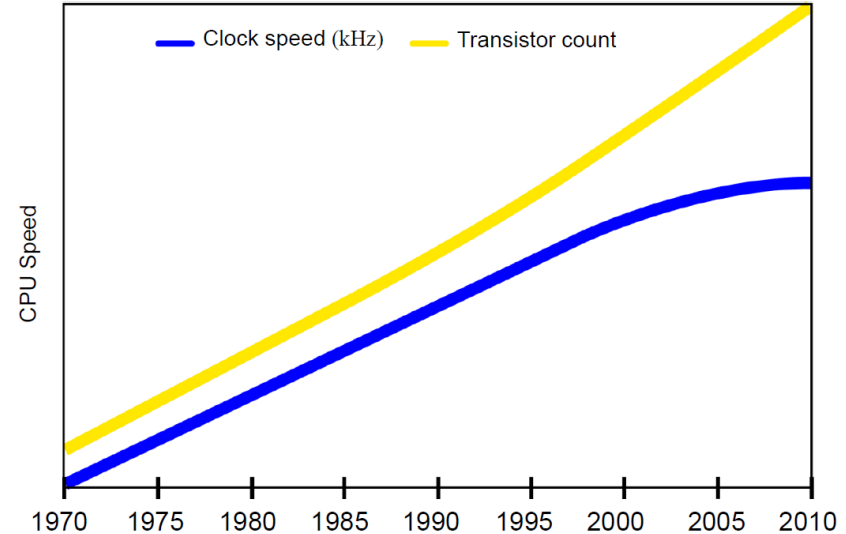
Prof. Arindam BASU

Introduction

Audio



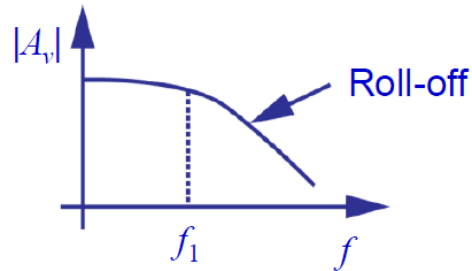
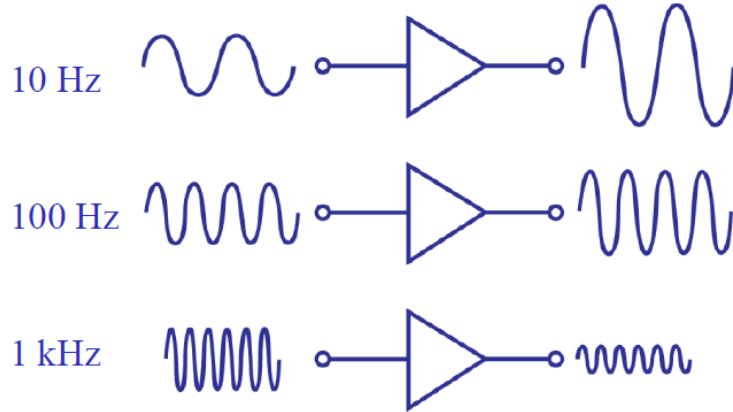
Real life signals have different frequencies



Why don't we have a 100 GHz CPU for cellphone?

For electronic circuits, what is the limitation for signal frequency?

Motivation: Why study frequency response



- An ideal amplifier would have an infinite bandwidth.
- The gain would remain constant at all frequencies.
- However, **the bandwidth is finite in practice, and the gain changes with frequency.**
- This is due to the fact that capacitive/ inductive elements in the circuit becomes prominent when the signal frequency changes.
- This effect occurs in all circuits including amplifier design.

Gain of amplifier reduces at high frequency! –WHY?

(Gain is a graph –not 1 number!)

Recap

Three terms fully describe a sine wave:

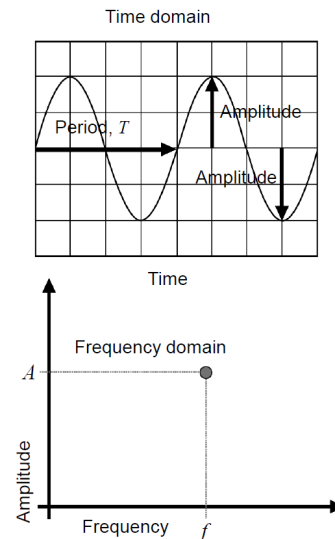
- Frequency
- Amplitude
- Phase

$$\omega = 2\pi \times f = \frac{2\pi}{T}$$

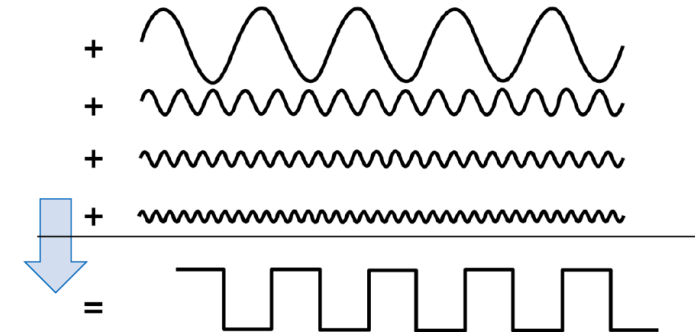
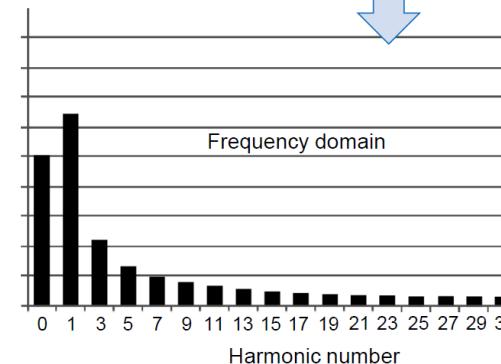
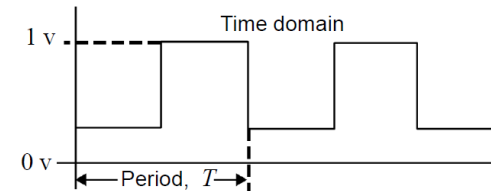
Angular frequency,
in radians/sec

Sine-wave
frequency, in Hz

Non-periodic signals analyzed in frequency domain by
Fourier Transform



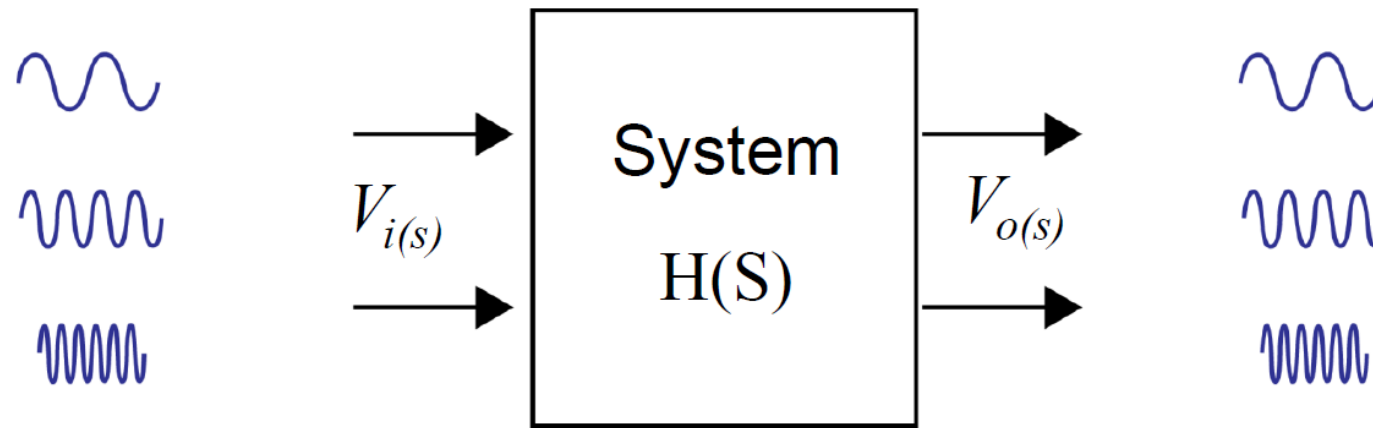
Periodic signals described in terms of sinusoids by Fourier series



Real world signals can be
decomposed into many single
frequency primitives, namely
0th harmonic, 1st harmonic, etc.

Recap: Transfer Function

Transfer function, also known as the **system function** or **network function** is a mathematical representation, in terms of spatial or temporal frequency, of the relation between the input and output of a linear time-invariant system.



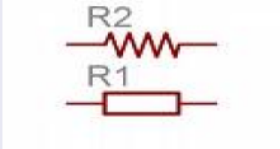
Transfer function is defined by,

$$H(S) = \frac{V_o(S)}{V_i(S)} .$$

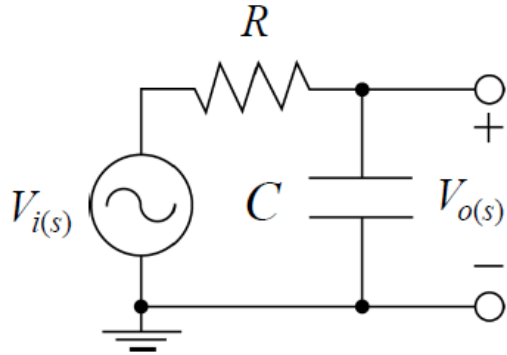
(Laplace transform being used to capture frequency dependence)

Impedances & frequency dependence

Translation from the circuit domain to the Laplace ($s = j\omega$) domain is made by making the following substitutions of circuit elements:

Element	Definition	Impedance
 	$V = I \times R$	$Z = R$
 	$I = \frac{dQ}{dt} = C \frac{dV}{dt}$	$Z = \frac{1}{j\omega C} = \frac{1}{sC}$
 	$I = L \frac{dI}{dt}$	$Z = j\omega L = sL$

Example of RC circuit



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC}$$

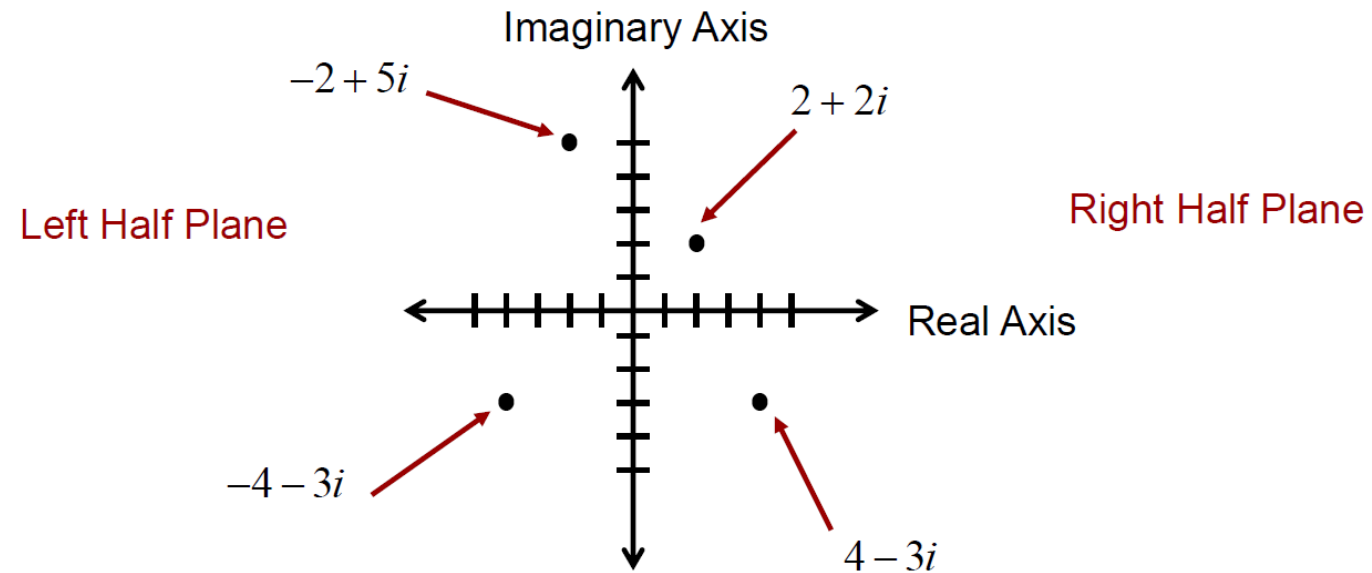
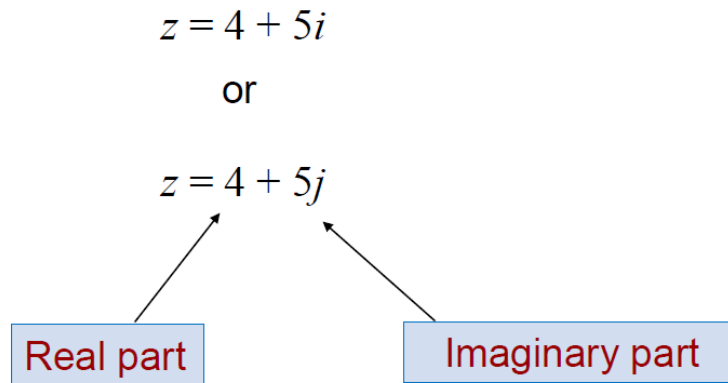
$$|H(s)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

- $H(s)$ describes the relationship between the output to the input: **gain**.
- $H(s)$ varies with frequency, due to the impedance of the capacitor changes with frequency (like a variable resistor).
- This circuit is a low pass filter: $|H(s)|$ drops when ω increases, so only low frequency signals can pass.

To get intuition, always try to find the gain for small and large ω

Recap: Complex Numbers

- A complex number has a real part and an imaginary part.
- The standard format is:



Complex Numbers Arithmetic

Addition:

$$\begin{aligned} &(-1 + 2i) + (3 + 3i) \\ &= (-1 + 3) + (2 + 3)i \\ &= 2 + 5i \end{aligned}$$

Subtraction:

$$\begin{aligned} &(-1 + 2i) - (3 + 3i) \\ &= (-1 - 3) + (2 - 3)i \\ &= -4 - i \end{aligned}$$

Multiplication:

$$\begin{aligned} i(3 + i) &= i \times 3 + i \times i \\ &= 3i + (-1) \\ &= 3i - 1 \end{aligned}$$

$$\begin{aligned} (2 + 3i)(5 + 6i) &= 2 \times (5 + 6i) + 3i \times (5 + 6i) \\ &= 10 + 12i + 15i + 18i^2 \\ &= 10 + 27i + (-18) \\ &= -8 + 27i \end{aligned}$$

Absolute Value:

$$\begin{aligned} |-2 + 5i| &= \sqrt{(-2)^2 + 5^2} \\ &= \sqrt{29} \end{aligned}$$

$$|a + bi| = \sqrt{a^2 + b^2}$$

$$|a + b\omega i| = \sqrt{a^2 + (b\omega)^2}$$

Division:

$$\begin{aligned} \frac{3 + 11i}{-1 - 2i} &= \frac{3 + 11i}{-1 - 2i} \times \frac{-1 + 2i}{-1 + 2i} \\ &= \frac{(3 + 11i) \times (-1 + 2i)}{(-1 - 2i) \times (-1 + 2i)} \\ &= \frac{-3 + 6i - 11i + 22i^2}{(-1)^2 - (2i)^2} \\ &= \frac{-25 - 5i}{1 - (-4)} \\ &= -5 - i \end{aligned}$$

Complex Numbers: Roots of polynomials

Solve: $x^2 - 6x + 13 = 0$

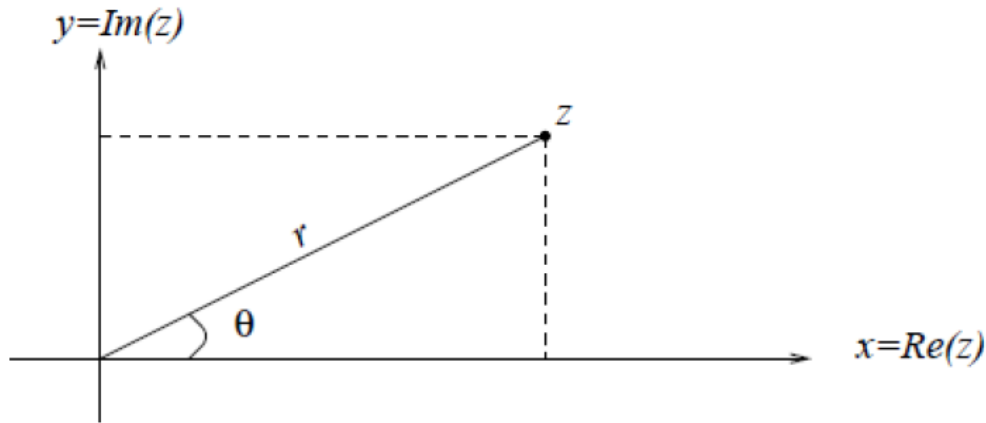
$$x = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$x = \frac{6 \pm \sqrt{-16}}{2}$$

$$x = \frac{6 \pm \sqrt{16}\sqrt{-1}}{2}$$

$$x = 3 \pm 2i \quad (\text{Two roots, a pair of complex conjugates})$$

Complex Numbers: Polar Form



$$z = x + iy = r \cos(\theta) + ir \sin(\theta) = re^{i\theta}$$

$$|z| = r$$

$$\angle(z) = \theta$$

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

Given: $Z_1 = a + bi$, $Z_2 = c + di$, $Z_3 = e + fi$

$$|Z_1 \times Z_2 \times Z_3| = |Z_1| \times |Z_2| \times |Z_3|$$

$$\log(|Z_1 \times Z_2 \times Z_3|) = \log|Z_1| + \log|Z_2| + \log|Z_3|$$

$$\log\left(\left|\frac{Z_1}{Z_2}\right|\right) = \log|Z_1| - \log|Z_2|$$

$$\begin{aligned} \angle(Z_1 \times Z_2 \times Z_3) \\ = \angle(Z_1) + \angle(Z_2) + \angle(Z_3) \end{aligned}$$

$$\angle\left(\frac{Z_1}{Z_2}\right) = \angle(Z_1) - \angle(Z_2)$$

General Transfer Function

$$H(s) = \frac{b_0 + b_1s + b_2s^2 + \dots + b_{m-1}s^{m-1} + b_ms^m}{a_0 + a_1s + a_2s^2 + \dots + a_{n-1}s^{n-1} + a_ns^n} = \frac{H_o \left(1 + \frac{s}{z_1}\right) \left(1 + \frac{s}{z_2}\right) \dots \left(1 + \frac{s}{z_m}\right)}{\left(1 + \frac{s}{p_1}\right) \left(1 + \frac{s}{p_2}\right) \dots \left(1 + \frac{s}{p_n}\right)}$$

H_o : Low-frequency gain

Zeros: Roots of numerator, $-z_1, -z_2, \dots, -z_m$

Poles: Roots of denominator, $-p_1, -p_2, \dots, -p_n$

$$H(S) = \frac{S^2 - 5S + 6}{aS^2 + bS + c}$$

zeros: $z_1 = -2, z_2 = -3$

$$\text{poles: } p_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$p_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

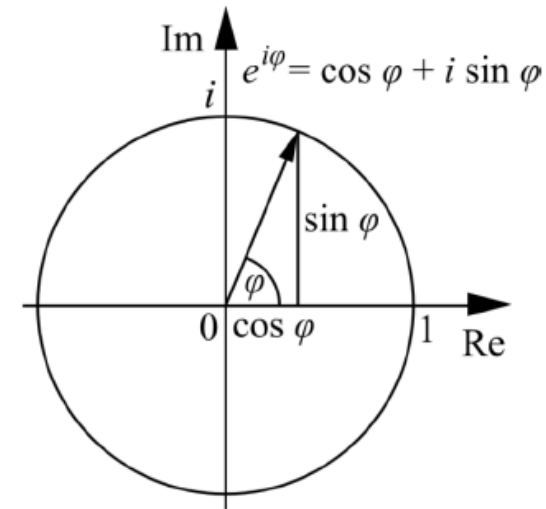
Poles and Zeros

- The values of poles and zeros can be either real or imaginary.
- For a **physical system**, all the coefficients are real, hence, the poles and zeros can only have two forms:
 - 1.The roots are real.
 - 2.If one root is complex, its complex conjugate has to be another root, i.e., if $p = a + jb$ is a root, $p^* = a - jb$ must be another root.
- Both poles and zeros can be located in either left half-plane(LHP) or the right half-plane(RHP).

Bode Plots

To study the frequency response of a system, the transfer function $H(s)$ is re-written in the polar form.

$$H(s) = |H(s)| \cdot \angle H(s)$$



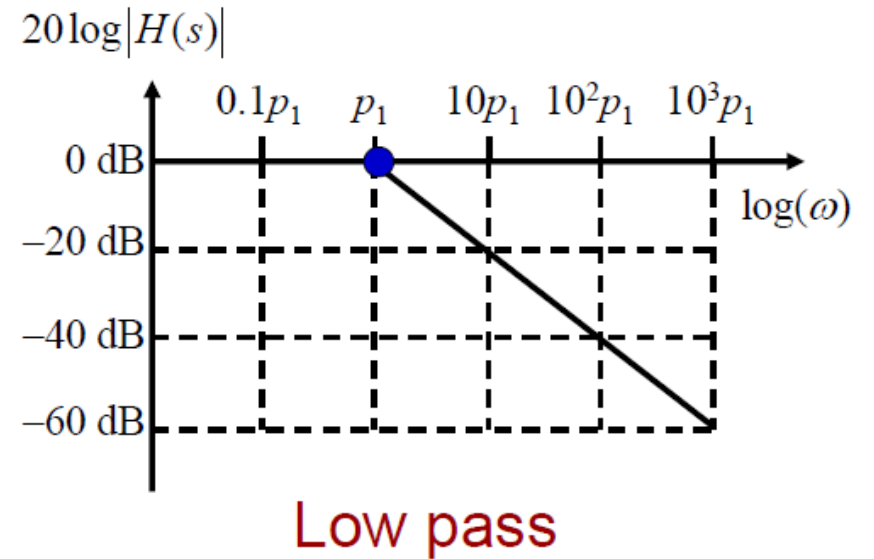
Bode plots are a very useful way to represent the gain and phase of a system as a function of frequency.

Bode Plots: Magnitude of Single pole system

$$H(s) = \frac{1}{1 + \frac{s}{p_1}} \Rightarrow |H(s)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{p_1}\right)^2}}$$

$$dB(H) = 20 \log \left(\frac{1}{\sqrt{1 + \left(\frac{\omega}{p_1}\right)^2}} \right) = -10 \log \left[1 + \left(\frac{\omega}{p_1}\right)^2 \right]$$

$$dB(H) = \begin{cases} 0 & \text{when } \omega \ll p_1 \\ -20 \log \omega + 20 \log p_1 & \text{when } \omega \gg p_1 \end{cases}$$



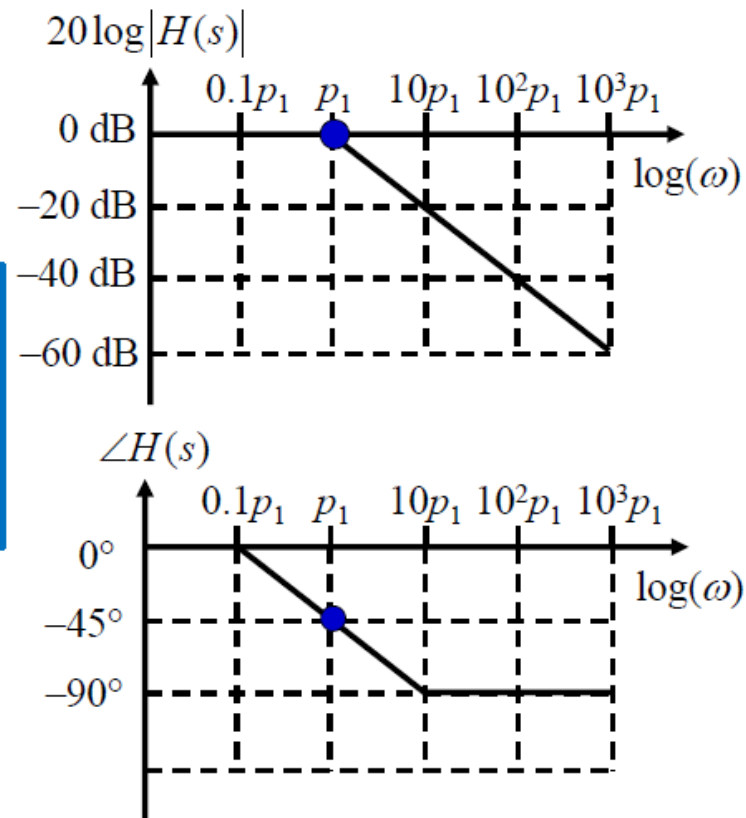
Try some frequency $\omega_1 \gg p_1$, $dB[H(\omega_1)]$,
 $= -20 \log \omega_1 + 20 \log p_1$

Try another frequency $\omega_2 = 10 * \omega_1$, $dB[H(\omega_2)]$,
 $= -20 \log \omega_1 - 20 + 20 \log p_1$

When we hit a pole, p_1 , the Bode magnitude falls with a slope of -20 dB/dec .

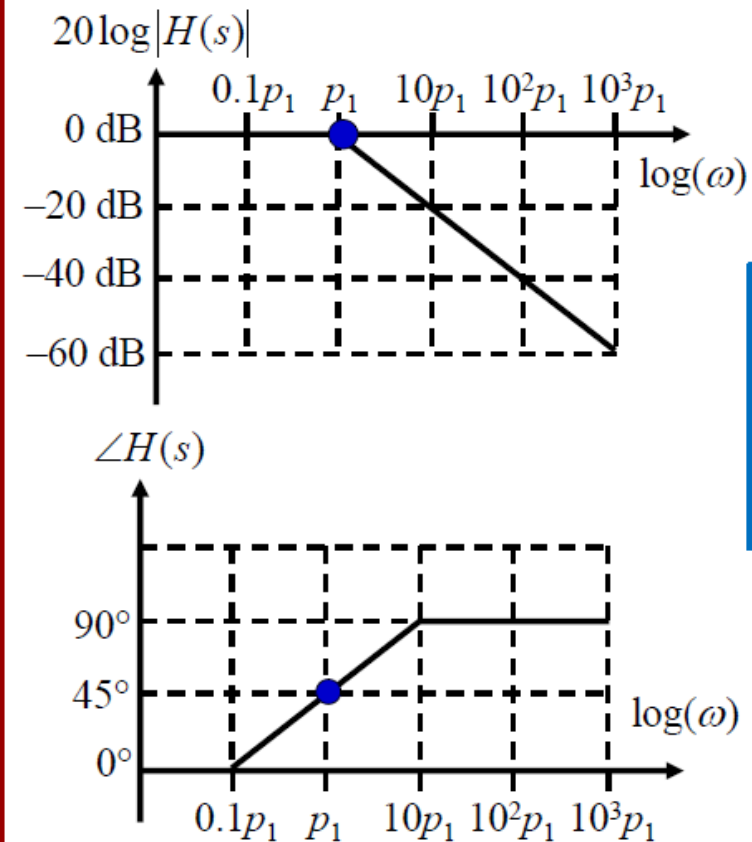
Bode Plots: Summary of Single pole system

Low pass, LHP pole



$$H(s) = \frac{1}{1 + \frac{s}{p_1}}$$

Low pass, RHP pole



$$H(s) = \frac{1}{1 - \frac{s}{p_1}}$$

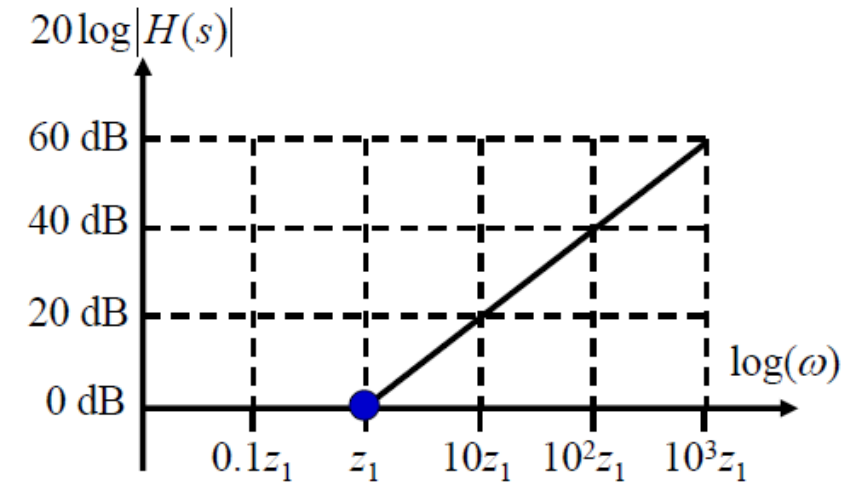
When we hit a pole, p_1 , the Bode magnitude falls with a slope of -20 dB/dec.

Bode Plots: Magnitude of Single zero system

$$H(s) = 1 + \frac{s}{z_1} \Rightarrow |H(s)| = |H(j\omega)| = \sqrt{1 + \left(\frac{\omega}{z_1}\right)^2}$$

$$dB(H) = 20 \log \left(\sqrt{1 + \left(\frac{\omega}{z_1}\right)^2} \right) = 10 \log \left[1 + \left(\frac{\omega}{z_1}\right)^2 \right]$$

$$dB(H) = \begin{cases} 0 & \text{when } \omega \ll z_1 \\ 20 \log \omega - 20 \log z_1 & \text{when } \omega \gg z_1 \end{cases}$$



High pass

Try some frequency $\omega_1 \gg z_1$, $dB[H(\omega_1)] = 20 \log \omega_1 - 20 \log z_1$

Try another frequency $\omega_2 = 10 * \omega_1$, $dB[H(\omega_2)] = -20 \log \omega_1 + 20 + 20 \log z_1$

When we hit a zero, z_1 , the Bode magnitude rises with a slope of +20 dB/dec.

Bode Plots: Many pole-zero system

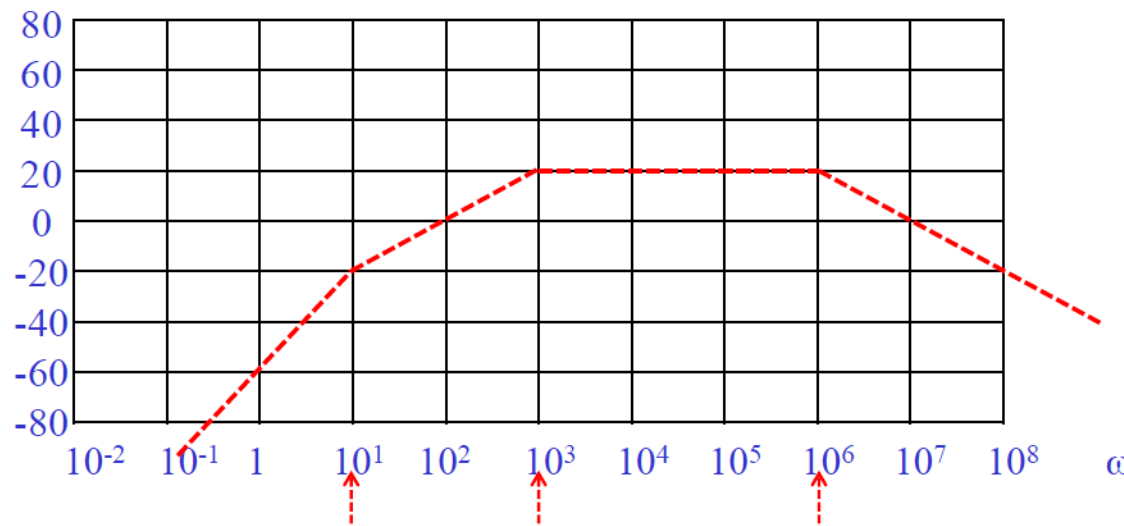
1. Replace “ s ” by “ $j\omega$ ” to get $H(j\omega)$. Then take magnitude of the resulting complex number $|H(j\omega)|$.
2. Label the frequency of poles (p_1, p_2 etc.) and zeros (z_1, z_2 etc.) on the “ ω ” axis.
3. Find the slope of each segment between each pole/ zero frequency: suppose, there are n_p poles and n_z zeros at frequencies **lower** than ω_0 . Then, the slope at ω_0 is $n_z \times (20 \text{ dB/ dec}) + n_p \times (-20 \text{ dB/ dec})$, i.e. it is the sum of contributions of all poles and zeros at frequencies lower than ω_0 .
4. Pick a frequency other than any pole/ zero frequency: usually at least one decade away from p_i or z_i . Using this frequency value to find out the exact complex number of the $H(j\omega)$. Calculate its amplitude and convert to dB.
5. Now you should have the complete Bode plot drawing.

Bode Plots: Many pole-zero system

Given: $H(s) = \frac{10^8 s^2}{(s-10)(s-10^3)(s-10^6)}$

2 Zeros: $z_1 = z_2 = 0$

3 Poles: $p_1 = 10, p_2 = 10^3, p_3 = 10^6$

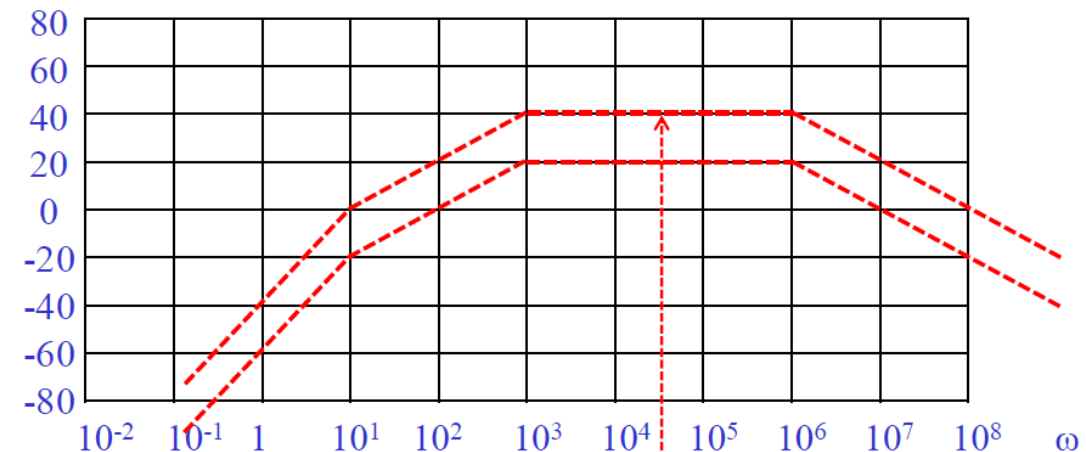


Step 1:
Shape

Try any specific frequency you prefer, for example, 10^4 :

$$20\log|H(s = j10^4)| = 20\log \left| \frac{10^8 (j10^4)^2}{(j10^4 - 10)(j10^4 - 10^3)(j10^4 - 10^6)} \right| = 40$$

Step 2:
Height



-3 dB frequency

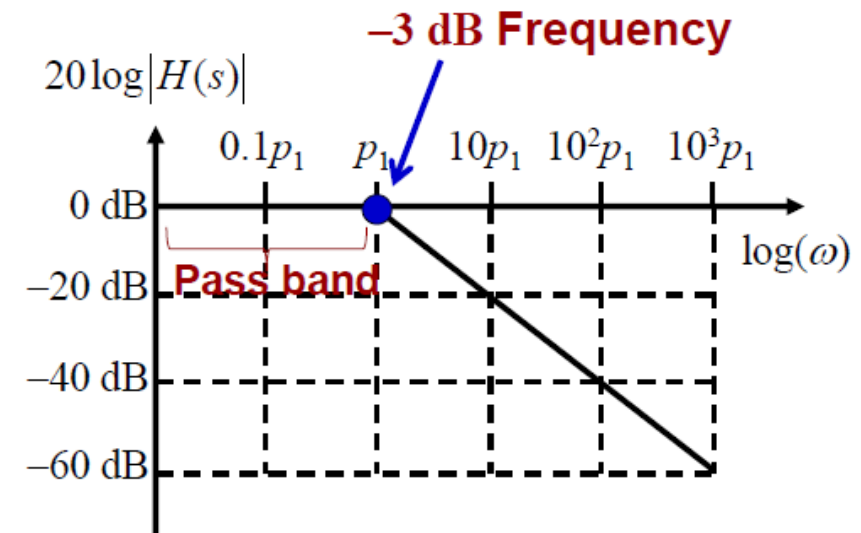
$$H(s) = 1 + \frac{s}{p_1} \Rightarrow |H(s)| = \frac{1}{\sqrt{1 + \left(\frac{\omega}{p_1}\right)^2}}$$

$$dB(H) = 20 \log \left(\sqrt{1 + \left(\frac{\omega}{p_1}\right)^2} \right) = -10 \log \left[1 + \left(\frac{\omega}{p_1}\right)^2 \right]$$

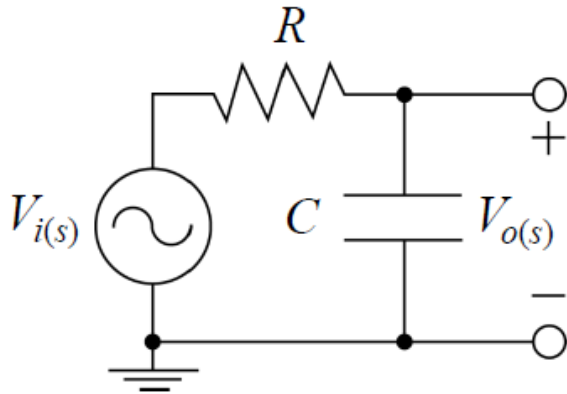
$$dB(H) = \begin{cases} 0 & \text{when } \omega \ll p_1 \\ -20 \log \omega + 20 \log p_1 & \text{when } \omega \gg p_1 \end{cases}$$

How about exactly at frequency p_1 ?

$$dB[H(p_1)] = -10 \log 2 \simeq -3 \text{ when } \omega = p_1$$

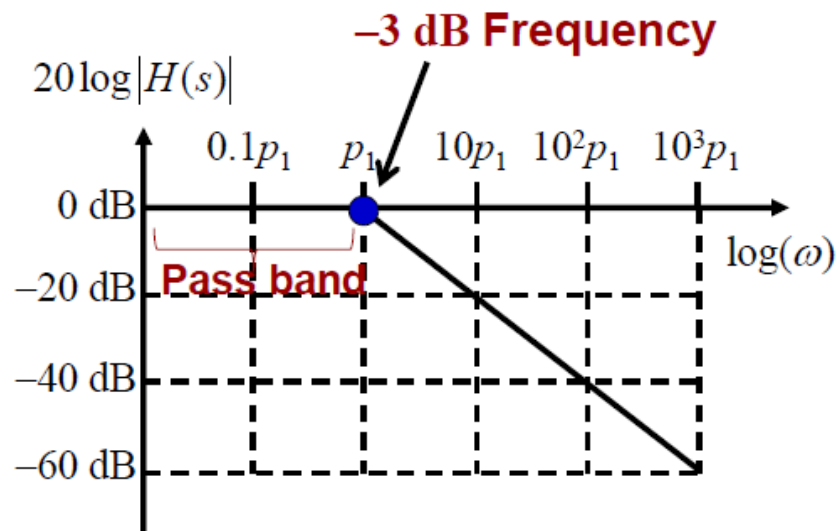


Real circuit examples



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{\frac{1}{sC}}{R + \frac{1}{sC}} = \frac{1}{1 + sRC}, \quad p_1 = \frac{1}{RC}$$

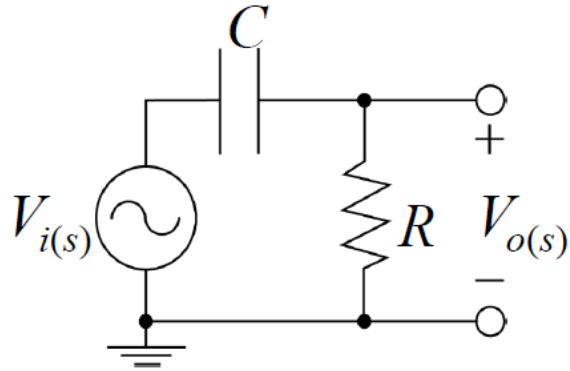
$$20 \log |H(\omega \rightarrow 0)| = 0, \quad 20 \log |H(\omega = p_1)| = -3$$



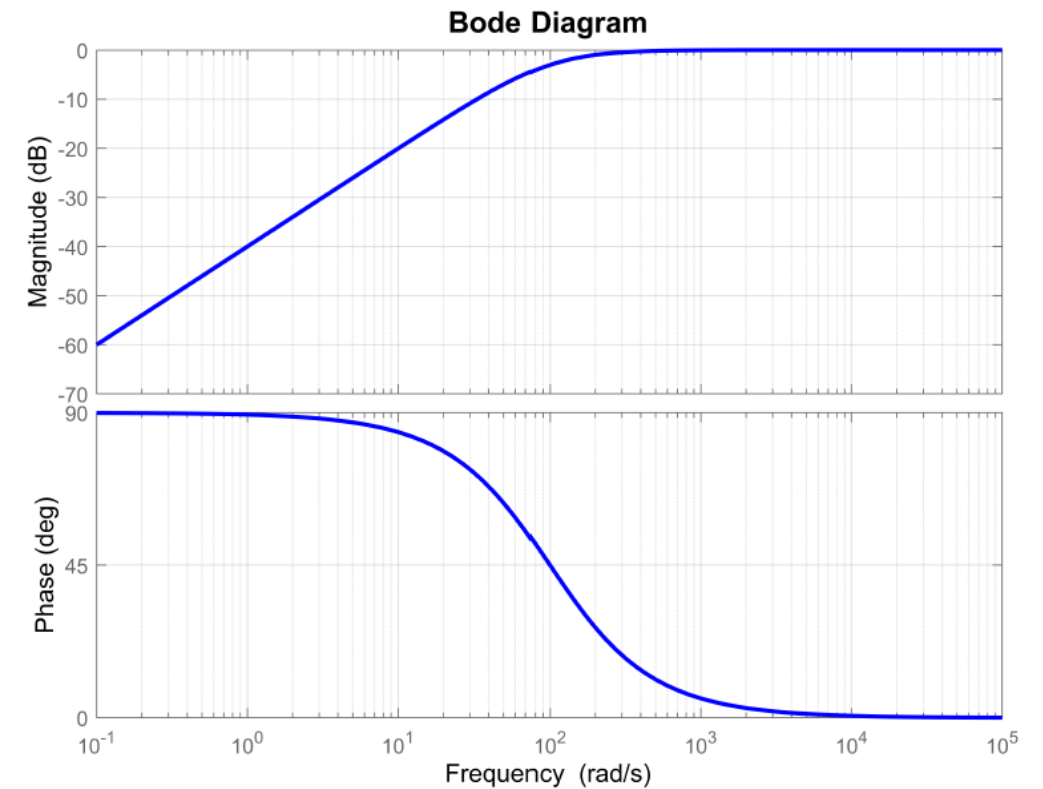
- At low frequency, the capacitor is effectively open and the gain is flat. As frequency increases, the capacitor tends to a short and the gain starts to decrease.
- At a special frequency, the gain drops by 3 dB; This gives **idea** of frequency beyond which $|H|$ starts rolling off quickly → **pass band**

Real circuit examples

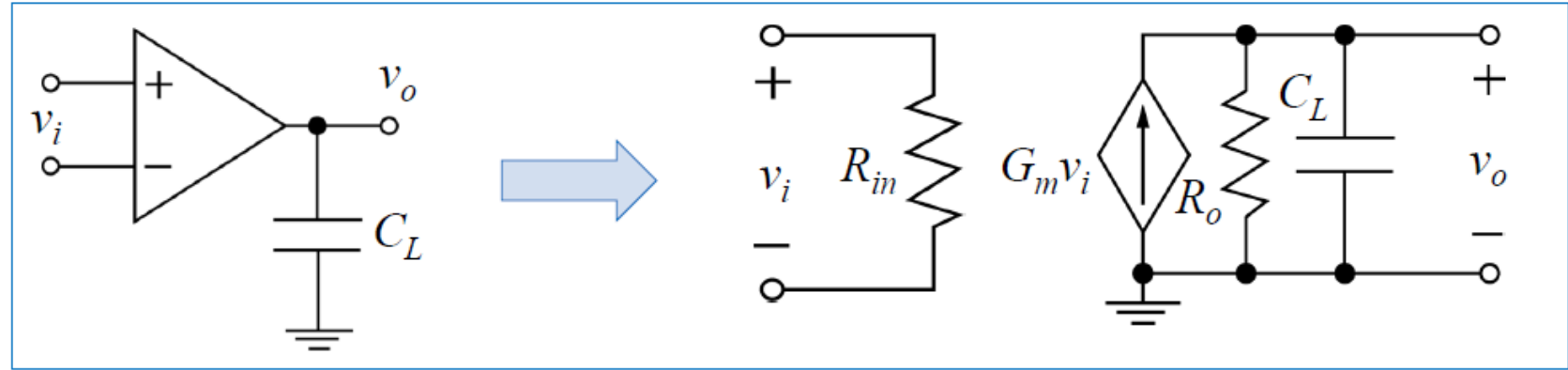
Swap the capacitor with the resistor:



$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{R}{R + \frac{1}{sC}} = \frac{sRC}{1 + sRC}$$



Real circuit examples

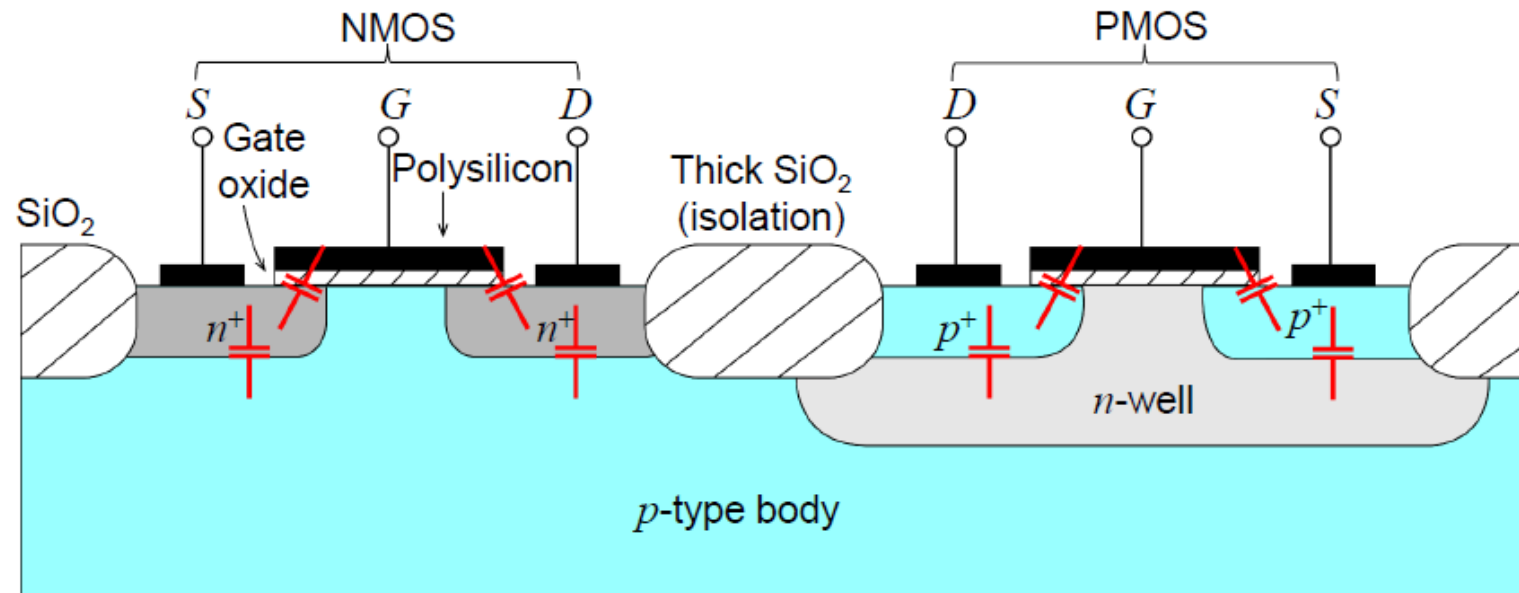


$$V_o(s) = [V_i(s) \times G_m] \times \left[R_o // \frac{1}{sC_L} \right]$$

$$H(s) = \frac{V_o(s)}{V_i(s)} = G_m \times \left(R_o // \frac{1}{sC_L} \right) = \frac{G_m R_o}{1 + sR_o C_L} = \frac{G_m R_o}{1 + \frac{s}{p_1}} \quad \text{where } p_1 = \frac{1}{R_o C}$$

A single pole low pass system, its low frequency gain: $|H(\omega \rightarrow 0)| = \left| \frac{G_m R_o}{1 + j\omega R_o C_L} \right| \approx G_m R_o$

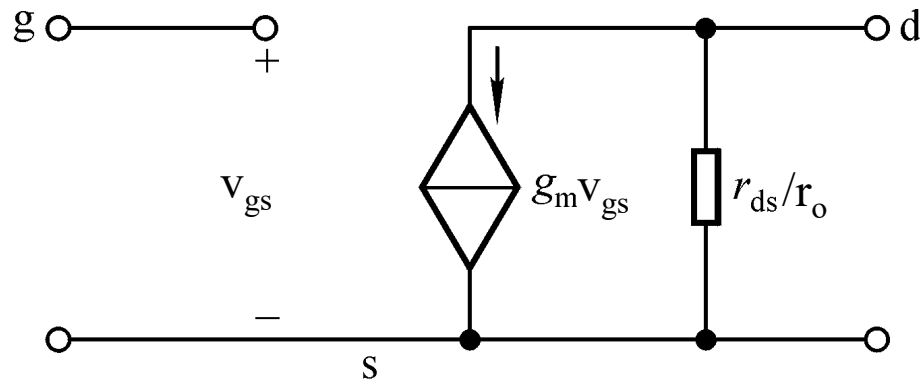
Parasitic Capacitors in Transistors



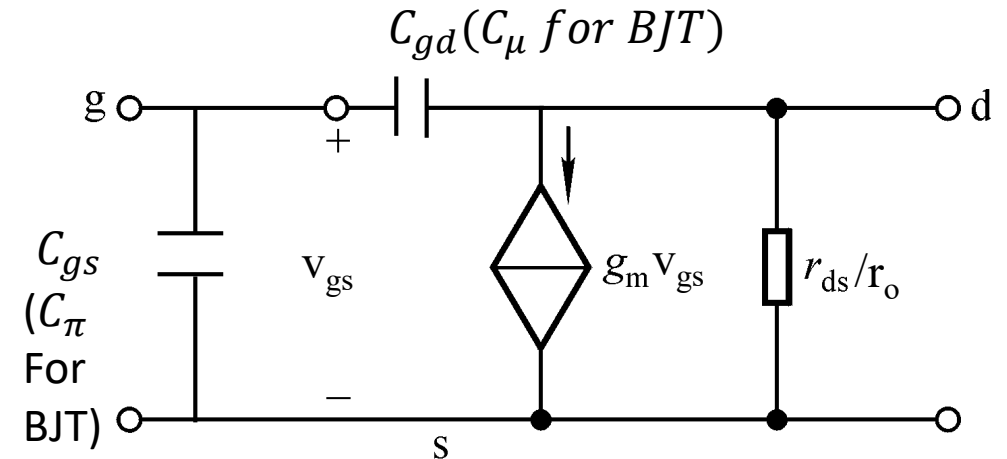
1. Capacitance between **Gate/Source** and **Gate/Drain** due to the overlap of gate electrode (Parallel-plate capacitor)
2. Junction capacitance between **Source/Body** and **Drain/Body** (Reverse-bias junction)

Typically $C_{gs} \gg C_{gd}$ (though effect of C_{gd} may be amplified by Miller Effect)

New small signal model



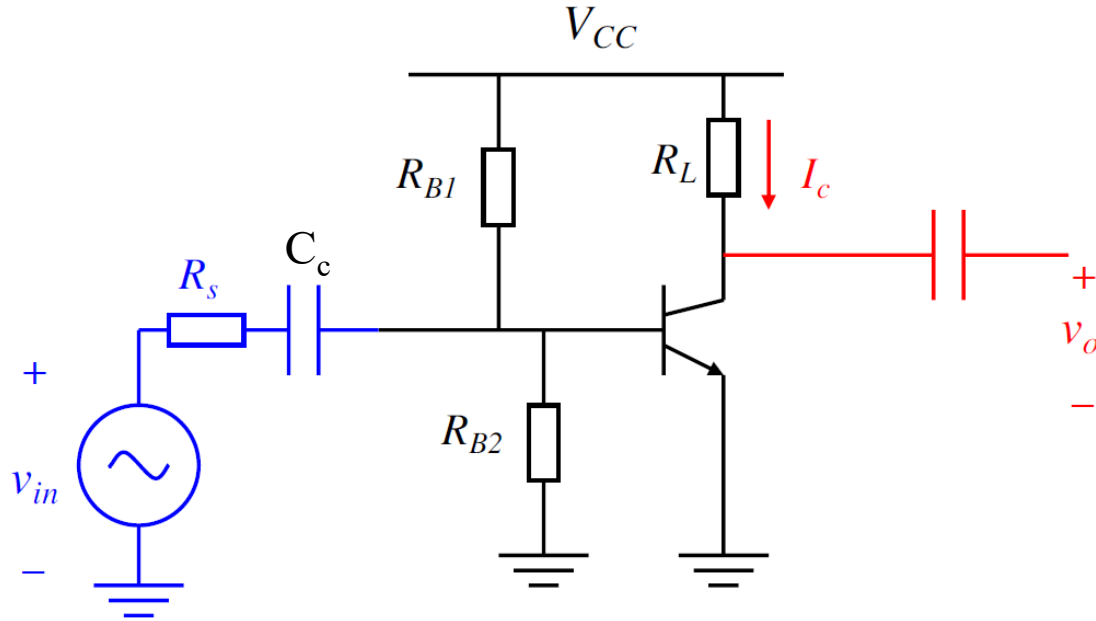
Earlier model (valid for low frequency)



New model (including capacitors)

Typically $C_{gs} \gg C_{gd}$ (though effect of C_{gd} may be amplified by Miller Effect)

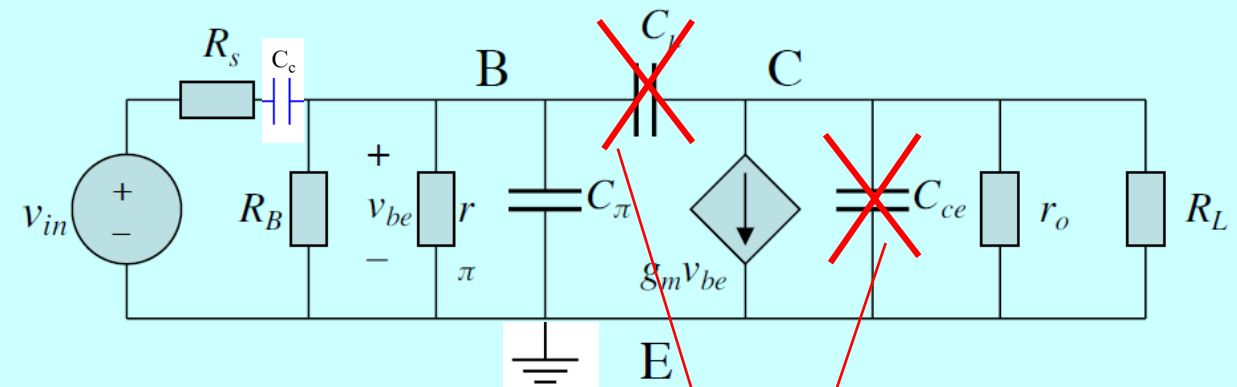
Example: CE amplifier



Recall that the voltage gain at low frequencies is

$$\frac{v_o}{v_{in}} = \left[\frac{R_{B1} \parallel R_{B2}}{(R_{B1} \parallel R_{B2}) + R_s} \right] \left[g_m (R_L \parallel r_o) \right] \approx g_m R_L \quad \text{if } R_s \approx 0 \text{ and } r_o \rightarrow \infty$$

Complete small-signal model:

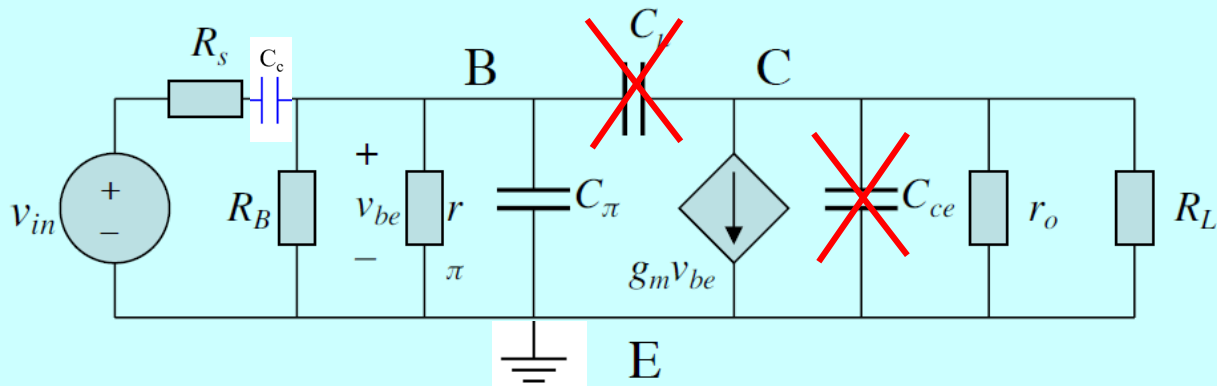


Ignored since very small

Also, $C_c \gg C_{\pi}$

Example: CE amplifier

Complete small-signal model:



$$V_{out} = -g_m v_{be} * (R_L || r_o)$$

Denote $R_B || r_{\pi} = R_B'$

Denote $Z_{\pi} = R_B' || (1/sC_{\pi}) = R_B'/(1+s R_B' C_{\pi})$

Denote $Z_s = R_s + 1/sC_c = (1+s R_s C_c)/sC_c$

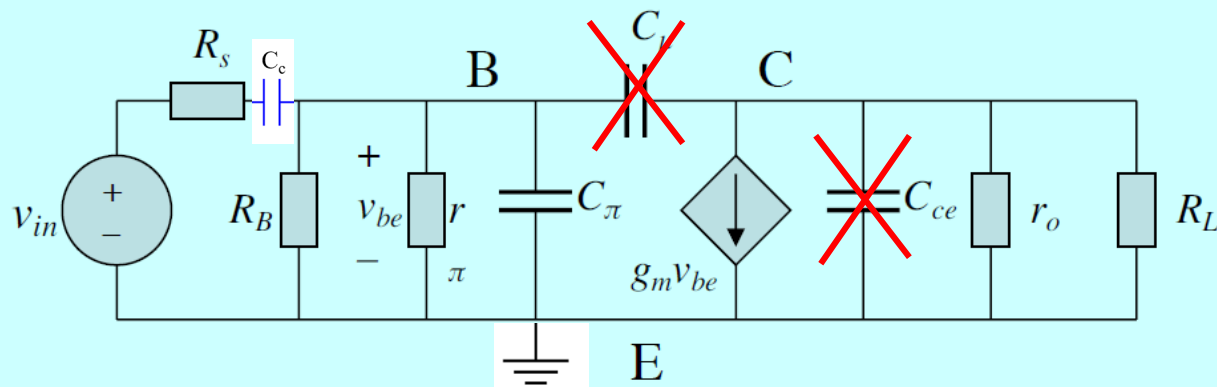
v_{be} is formed due to potential divider between Z_s and Z_{π} :

$$v_{be} = v_{in} * (Z_{\pi}) / (Z_{\pi} + Z_s) = \frac{v_{in} \times s R_B' C_c}{1 + s(C_C R_B' + C_{\pi} R_B' + C_C R_S) + s^2 C_{\pi} C_C R_B' R_S} = \frac{v_{in} \times \frac{R_B'}{R_B' + R_S} \times s(R_B' + R_S) C_c}{1 + s(C_C R_B' + C_{\pi} R_B' + C_C R_S) + s^2 C_{\pi} C_C R_B' R_S}$$

$$\therefore \frac{v_{out}}{v_{in}} = \frac{-g_m (R_L || r_o) \times \frac{R_B'}{R_B' + R_S} \times s(R_B' + R_S) C_c}{1 + s(C_C R_B' + C_{\pi} R_B' + C_C R_S) + s^2 C_{\pi} C_C R_B' R_S}$$

Example: CE amplifier

Complete small-signal model:



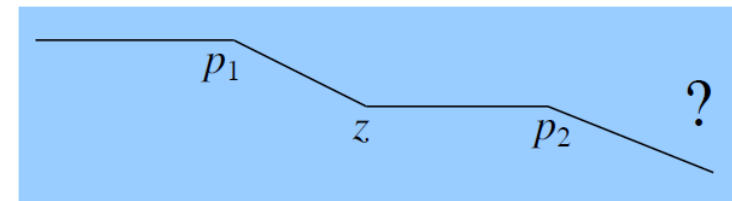
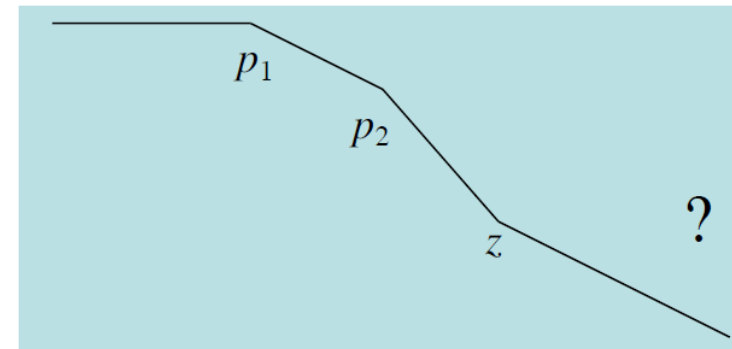
$$\therefore \frac{v_{out}}{v_{in}} = \frac{-g_m(R_L || r_o) \times \frac{R'_B}{R'_B + R_S} \times s(R'_B + R_S)C_c}{1 + s(C_C R'_B + \cancel{C_\pi R'_B} + C_C R_S) + s^2 C_\pi C_C R'_B R_S}$$

$C_c \gg C_\pi$

Highest power of s in numerator is 1 $\rightarrow$ 1 zero

Highest power of s in denominator is 2 $\rightarrow$ 2 poles

Tough to say anything else!!!



Most important is p_1 which dominates the bandwidth $\rightarrow$ Can approximate based on that!!

Dominant pole approximation

For Transfer function with 2 poles p_1, p_2 :

$$H(s) = \frac{1}{(1+s/p_1)(1+s/p_2)} = \frac{1}{1+s\left(\frac{1}{p_1}+\frac{1}{p_2}\right)+s^2/p_1p_2} = \frac{1}{1+as+bs^2}$$

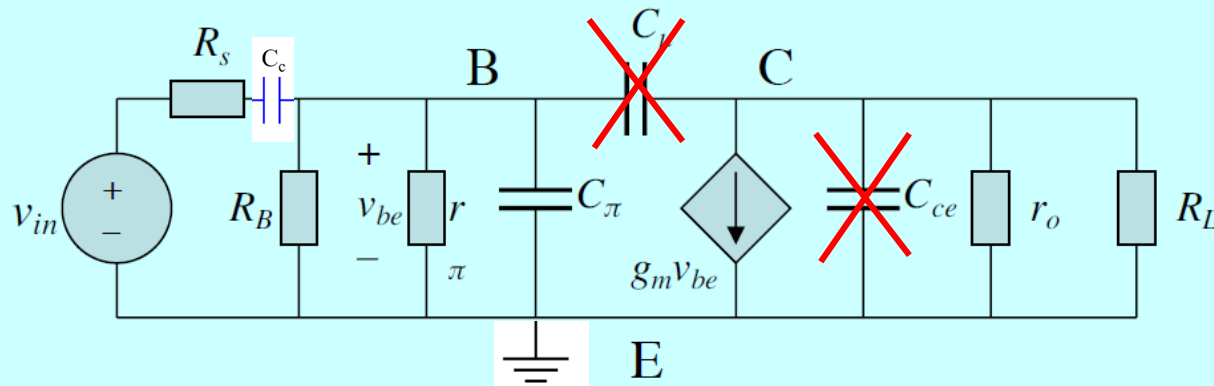
Pole p_1 is dominant means $p_1 \ll p_2$

Hence, $a \approx 1/p_1 \rightarrow p_1 = 1/a$

Also, $b = 1/p_1p_2 \approx a/p_2 \rightarrow p_2 = a/b$

Example: CE amplifier

Complete small-signal model:



Applying dominant pole approximation,

$$p1 \sim 1/(C_C R'_B + C_C R_S)$$

$$p2 \sim C_C (R'_B + R_S) / C_\pi C_C R'_B R_S = 1 / (C_\pi (R'_B || R_S))$$

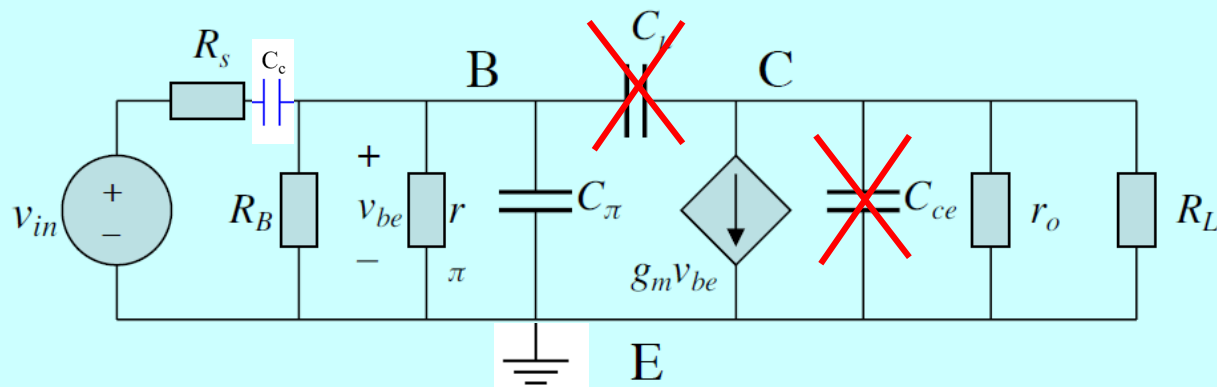
$$\therefore \frac{v_{out}}{v_{in}} = \frac{-g_m (R_L || r_o) \times \frac{R'_B}{R'_B + R_S} \times s(R'_B + R_S) C_c}{1 + s(C_C R'_B + \cancel{C_\pi R'_B} + C_C R_S) + s^2 C_\pi C_C R'_B R_S}$$

$C_c \gg C_\pi$

$$\therefore \frac{v_{out}}{v_{in}} = \frac{-g_m (R_L || r_o) \times \frac{R'_B}{R'_B + R_S} \times s(R'_B + R_S) C_c}{(1 + s C_C (R'_B + R_S))(1 + s C_\pi (R'_B || R_S))}$$

Example: CE amplifier

Complete small-signal model:

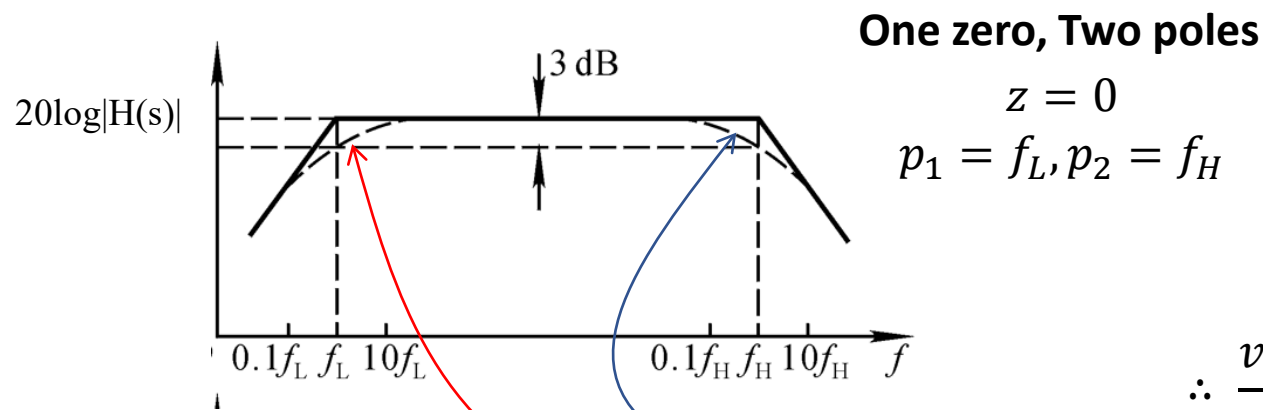


Applying dominant pole approximation,

$$p1 \sim 1/(C_C R_B' + C_C R_S)$$

$$p2 \sim C_C (R_B' + R_S) / C_\pi C_C R_B' R_S = 1 / (C_\pi (R_B' || R_S))$$

(Open-circuit/Short-circuit time constant method can allow you to directly get p1,p2)



$$\therefore \frac{v_{out}}{v_{in}} = \frac{-g_m (R_L || r_o) \times \frac{R_B'}{R_B' + R_S} \times s(R_B' + R_S) C_c}{(1 + s C_C (R_B' + R_S)) (1 + s C_\pi (R_B' || R_S))}$$

Appendix

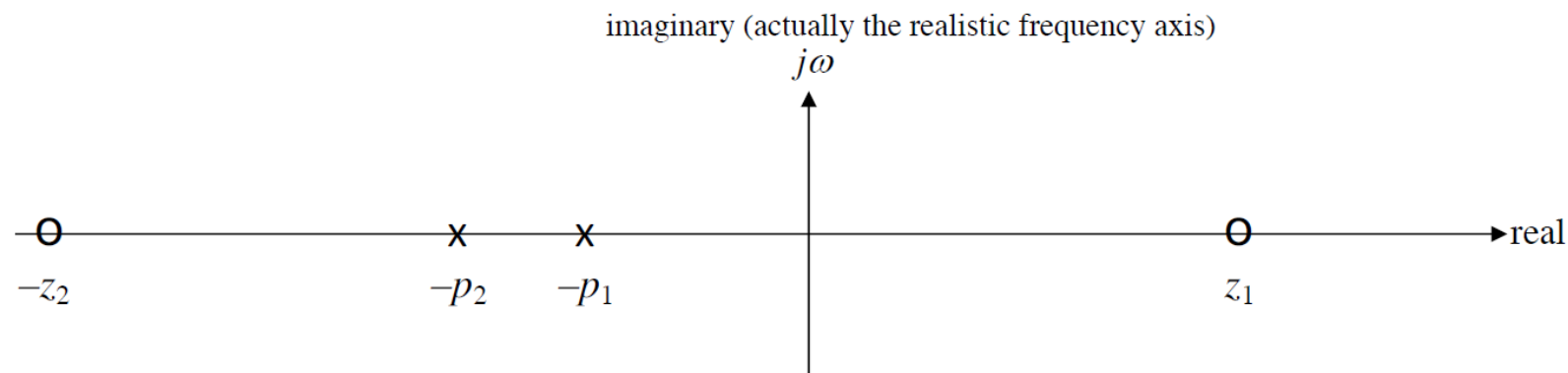
Poles & Zeros (Year 1 Linear Systems)

Theory (if you still remember!)

On the complex number plane, a pole is the point where the output response is ∞ even when the input is finite. Also, a zero is the point where the output response is 0 even when the input is finite.

All poles must be on LHS of the complex plane. Otherwise, the circuit is unstable. But zeros can be anywhere.

For circuits with ONLY ONE type of reactive elements (capacitors only), all poles and zeros are real numbers. Thus, all poles are LHS real numbers. [Some textbooks use the term *negative real poles* instead of LHS real poles!]



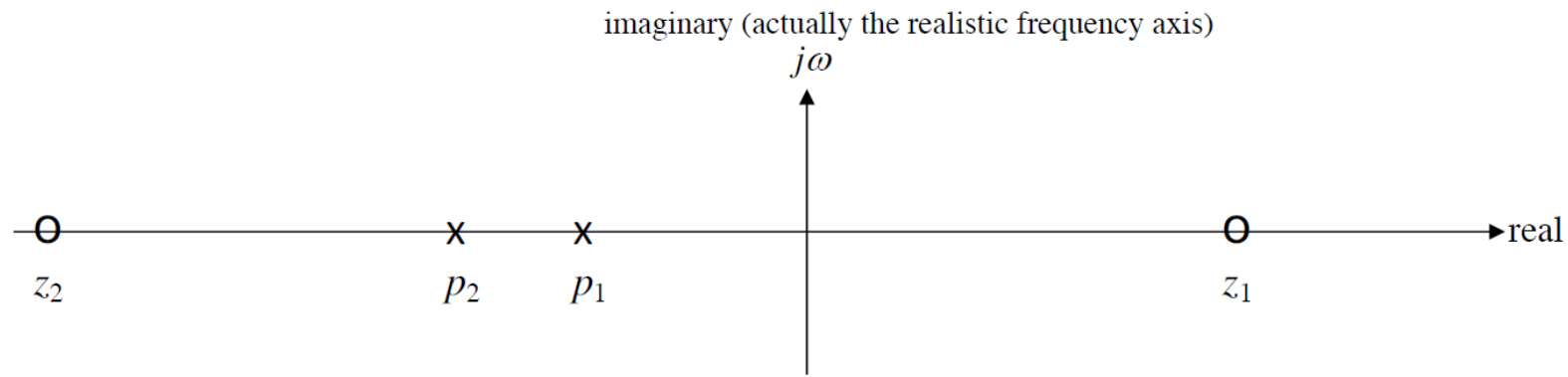
Poles & Zeros (Year 1 Linear Systems)

In transfer function forms, we write

$$F(j\omega) = A_o \frac{\left(1 + \frac{j\omega}{z_2}\right) \left(1 - \frac{j\omega}{z_1}\right)}{\left(1 + \frac{j\omega}{p_1}\right) \left(1 + \frac{j\omega}{p_2}\right)}$$

where A_o is the low-frequency gain.

Careful! Watch the sign for the RHP zero!

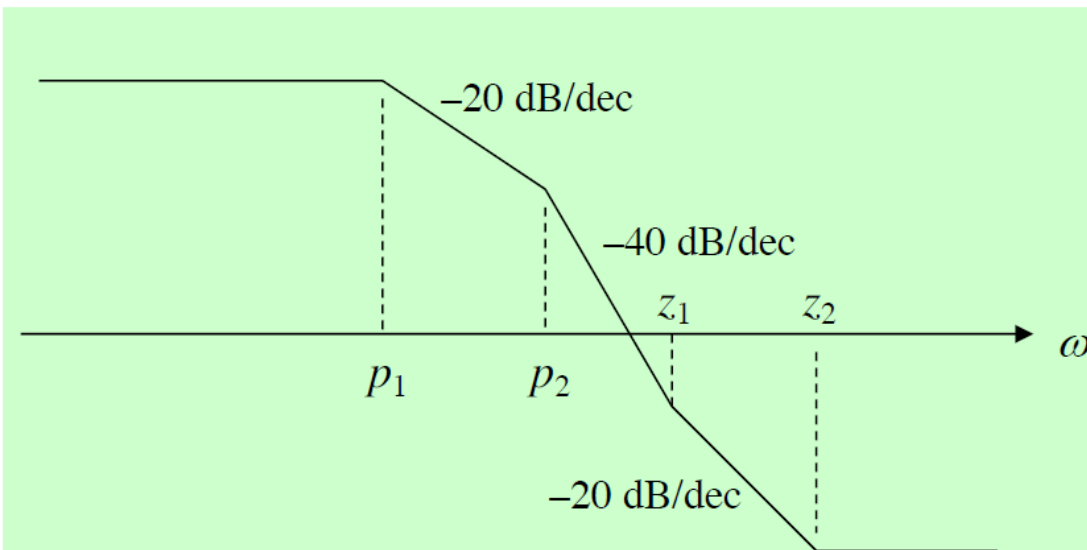


Poles & Zeros (Year 1 Linear Systems)

In practice

We look at the imaginary axis, which is actually the realistic frequency axis.

A pole [or zero] becomes just a corner point on the realistic frequency axis where the response starts to fall [rise] at a rate of 20 dB/dec.



We know all poles are LHS poles, but zeros can be LHS or RHS.

What is the difference between a RHS zero and a LHS zero? To see the difference, we have to look at the phase.

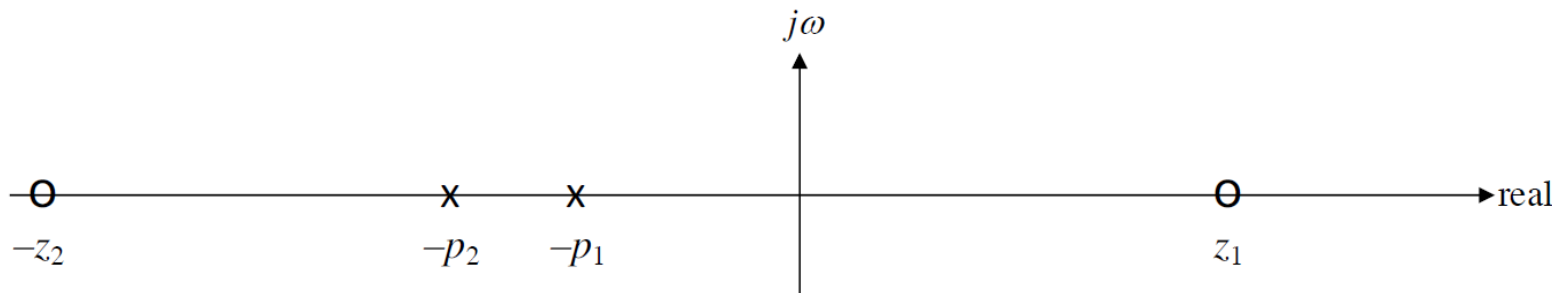
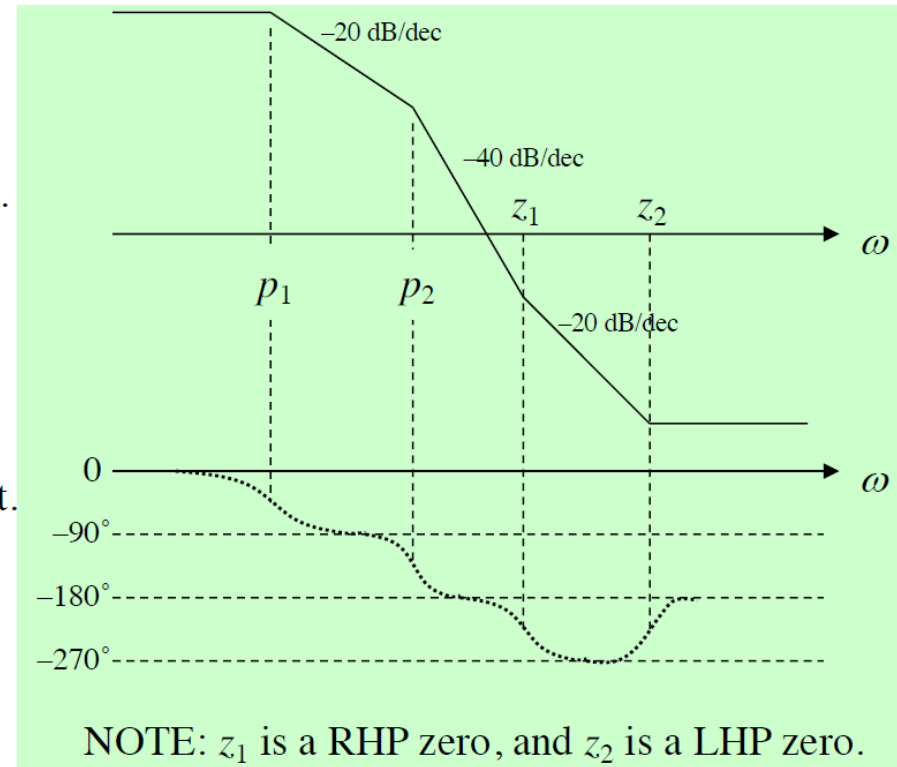
Poles & Zeros (Year 1 Linear Systems)

Phase shift

A LHS pole comes with negative phase shift.
One LHS pole gives -90° .

A LHS zero comes with positive phase shift.
One LHS zero gives $+90^\circ$.

A RHS zero comes with **negative** phase shift.
One RHS zero gives -90° .



Bode Plots: Phase of Single pole system

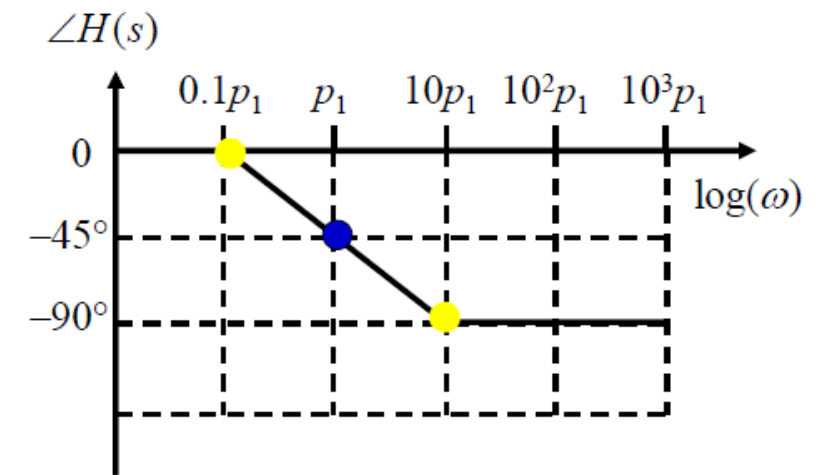
$$H(s) = \frac{1}{1 + \frac{s}{p_1}}$$

$$\begin{aligned}\angle H(j\omega) &= \angle(\text{Numerator}) - \angle(\text{Denominator}) \\ &= \angle(1) - \angle\left(1 + \frac{j\omega}{p_1}\right) = 0 - \arctan\left(\frac{\omega}{p_1}\right)\end{aligned}$$

$$\text{At } \omega_1 = p_1, \angle H(j\omega_1) = -\arctan\left(\frac{p_1}{p_1}\right) = -45^\circ$$

$$\text{At } \omega_2 = 10 p_1, \angle H(j\omega_2) = -\arctan\left(\frac{10 p_1}{p_1}\right) = -84.3^\circ \approx -90^\circ$$

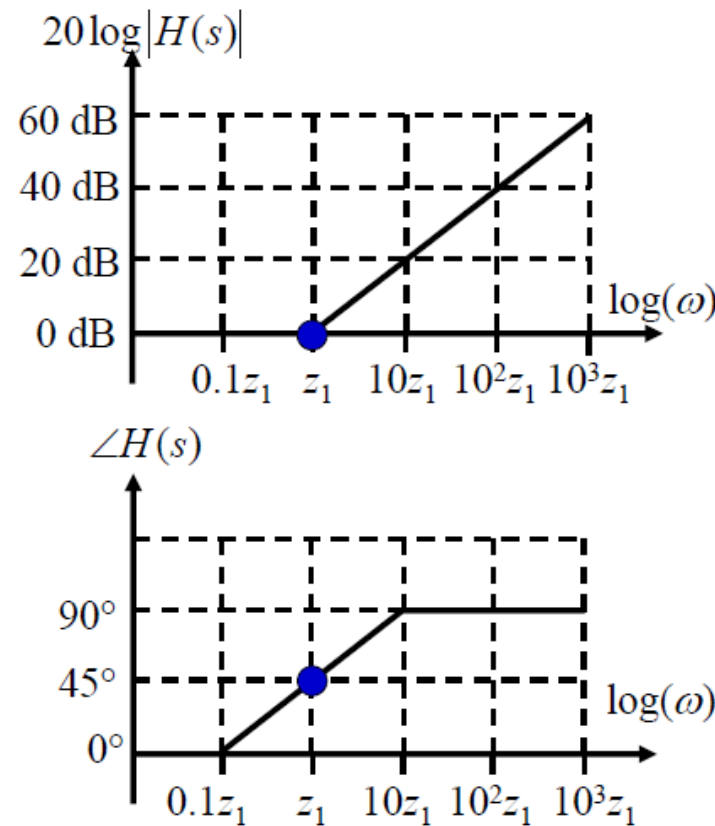
$$\text{At } \omega_3 = 100 p_1, \angle H(j\omega_3) = -\arctan\left(\frac{100 p_1}{p_1}\right) = -89.4^\circ \approx -90^\circ$$



At p_1 , phase shift is -45° .
Phase shift is -90° when frequency is very high.

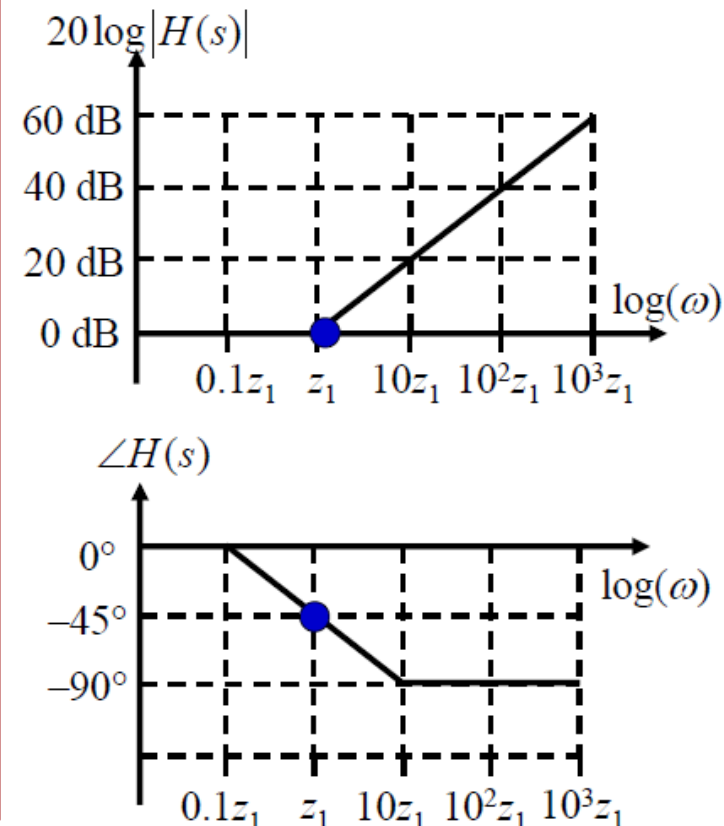
Bode Plots: Summary of Single zero system

High pass, LHP zero



$$H(s) = 1 + \frac{s}{z_1}$$

High pass, RHP zero



$$H(s) = 1 - \frac{s}{z_1}$$

When we hit a zero, z_1 , the Bode magnitude rises with a slope of +20 dB/dec.