

MA1300 Assignment # 3, Solutions

1. (P91, #12, 14) Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a .

$$(1) \quad f(x) = x^2 + \sqrt{7-x}, \quad a = 4.$$

$$(2) \quad f(t) = \frac{2t-3t^2}{1+t^3}, \quad a = 1.$$

Proof. (1), since $\lim_{x \rightarrow 4} 7-x = 3 > 0$, $\lim_{x \rightarrow 4} f(x) = 16 + \sqrt{3} = f(4)$. So f is continuous at a . (2), since f is a rational function and $a = 1$ is in the domain of f , so $\lim_{x \rightarrow 1} f(t) = f(1)$, and f is continuous at $a = 1$.

2. (P91, #15) Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval:

$$f(x) = \frac{2x+3}{x-2}, \quad (2, \infty).$$

Proof. Since f is a rational function and $(2, \infty)$ is a subset of the domain of f , we have for any $a \in (2, \infty)$, $\lim_{x \rightarrow a} f(x) = f(a)$. So f is continuous on the interval $(2, \infty)$.

For Questions 3 ~ 4, explain why the function is discontinuous at the given number 1. Sketch the graph of the function.

$$3. \text{ (P91, \#19) } f(x) = \begin{cases} 1-x^2, & \text{if } x < 1 \\ \frac{1}{x}, & \text{if } x \geq 1 \end{cases} \quad a = 1.$$

Solution. Since $\lim_{x \rightarrow 1^-} f(x) = 0 \neq 1 = f(1)$, f is not continuous at $a = 1$. The function is sketched in Figure 1.

$$4. \text{ (P91, \#20) } f(x) = \begin{cases} \frac{x^2-x}{x^2-1}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases} \quad a = 1.$$

Solution. Since $\lim_{x \rightarrow 1} \frac{x^2-x}{x^2-1} = \lim_{x \rightarrow 1} \frac{x(x-1)}{(x+1)(x-1)} = \frac{1}{2} \neq 1 = f(1)$, f is not continuous at $a = 1$. The function is sketched in Figure 1.

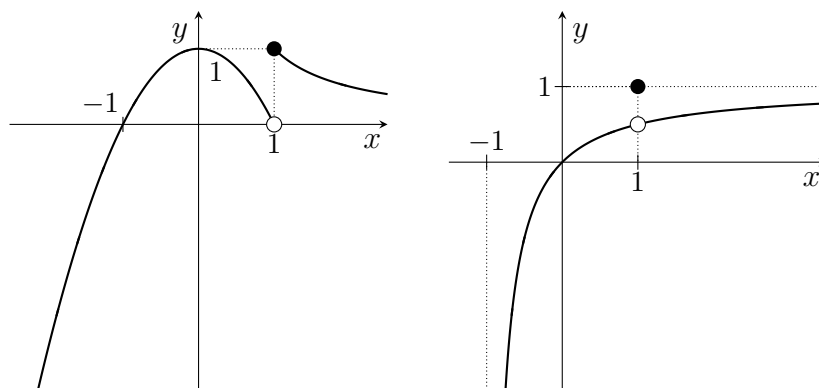


Figure 1: Sketch of the functions in Question 3 and 4. Left: for Question 3, Right, for Question 4.

5. (P91, #36, 38) Use continuity to evaluate the limit.

$$(1) \quad \lim_{x \rightarrow \pi} \sin(x + \sin x)$$

$$(2) \quad \lim_{x \rightarrow 2} (x^3 - 3x + 1)^{-3}$$

Solution. (1) Since the domain of sine function is $\mathbb{R}$ and $\lim_{x \rightarrow \pi} (x + \sin x) = \pi \in \mathbb{R}$, we have

$$\lim_{x \rightarrow \pi} \sin(x + \sin x) = \sin(\lim_{x \rightarrow \pi} (x + \sin x)) = \sin \pi = 0.$$

(2) Since $\lim_{x \rightarrow 2} (x^3 - 3x + 1) = 3$ is within the domain of the function $f(x) = \frac{1}{x^3}$, we have

$$\lim_{x \rightarrow 2} (x^3 - 3x + 1)^{-3} = (\lim_{x \rightarrow 2} (x^3 - 3x + 1))^{-3} = \frac{1}{27}.$$

6. (P92, #39) Show that f is continuous on $(-\infty, \infty)$:

$$f(x) = \begin{cases} x^2, & \text{if } x < 1, \\ \sqrt{x}, & \text{if } x \geq 1. \end{cases}$$

Proof. Since $\lim_{x \rightarrow 1^-} x = 1 > 0$, we have $\lim_{x \rightarrow 1^-} x^2 = 1$. Similarly, $\lim_{x \rightarrow 1^+} \sqrt{x} = 1$. Therefore $\lim_{x \rightarrow 1} f(x) = 1 = f(1)$, and thus f is continuous at $x = 1$. Since on $(-\infty, 1)$, f is a polynomial, and on $(1, \infty)$, f is a root function, so f is continuous on the whole real line $(-\infty, \infty)$.

7. (P92, #44) The gravitational force exerted by the earth on a unit mass at a distance r from the center of the planet is

$$F(r) = \begin{cases} \frac{GM}{R^3}r, & \text{if } r < R \\ \frac{GM}{r^2}, & \text{if } r \geq R \end{cases}$$

where M is the mass of the earth, R is its radius, and G is the gravitational constant. Is F a continuous function of r ? Explain why.

Solution. For $r \geq R > 0$, F is a rational function with domain containing $[R, \infty)$, so F is continuous. For $0 < r < R$, F is a polynomial and thus continuous. at $r = R$, we have

$$\lim_{r \rightarrow R^-} F(r) = \frac{GM}{R^2} = \lim_{r \rightarrow R^+} F(r).$$

Therefore F is continuous on $(0, \infty)$.

8. (P92, #45) For what values of the constant c is the function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x, & \text{if } x < 2, \\ x^3 - cx, & \text{if } x \geq 2. \end{cases}$$

Solution. Since $\lim_{x \rightarrow 2^-} f(x) = 4c + 4$ and $\lim_{x \rightarrow 2^+} f(x) = 8 - 2c$, let $8 - 2c = 4c + 4$ to give $c = 2/3$.

9. (P92, #49) If $f(x) = x^2 + 10 \sin x$, show that there is a number c such that $f(c) = 1000$.

Proof. Since $f(0) = 0 < 1000$, $f(100) = 10,000 - 10 \sin(100) \geq 9990 > 1000$, and that f is continuous on $(-\infty, \infty)$, by the intermediate value theorem, there exists a constant $c \in (0, 100)$ such that $f(c) = 1000$.

10. (P92, #50) Suppose f is continuous on $[1, 5]$ and the only solutions to the equation $f(x) = 6$ are $x = 1$ and $x = 4$. If $f(2) = 8$, explain why $f(3) > 6$.

Solution. If $f(3) = 6$, there is one more solution $x = 3$ to the equation $f(x) = 6$, a contradiction. If $f(3) < 6$, since f is continuous on $[1, 5]$, and $f(2) = 8 > 6$, by the intermediate value theorem, there exists a constant $c \in (2, 3)$ such that $f(c) = 6$, so $x = c$ is a solution to $f(x) = 6$ other than 1 and 4, another contradiction. Therefore $f(3) > 6$.