

MA3160: Probability and Stochastic Processes
Midterm exam (March 15th, 2019, 12–2:50pm)

Only a non-programmable calculator and a two-page (A4 size) formula sheet are allowed.

This exam consists of 5 independent exercises (120 points at total), and all answers need to be justified carefully (unless explicitly specified).

1. (20 points) We study a medical test, used to detect a disease which occurs in 1% of the population. If a given person has the disease, then the test detects it correctly with probability 0.98, while if this person is healthy, the test wrongly detects the disease with probability 0.03.
 - (i) (10 points) If the test says that a person has the disease, find the probability that this person is indeed sick.
 - (ii) (10 points) If the test says that a person is healthy, find the probability that this person does not have the disease.
2. (24 points) Let $p \in (0, 1)$. Consider two independent random variables X and Y , each equal to 1 with probability p , and to -1 with probability $1 - p$. Let $Z = X \cdot Y$.
 - (i) (6 points) What is the range E_Z of Z ? Determine $\mathbb{P}(Z = z)$ for each $z \in E_Z$.
 - (ii) (8 points) Compute $\mathbb{P}((X, Z) = (x, z))$ for all possible values (x, z) that the pair (X, Z) can take.
 - (iii) (10 points) For which values of $p \in (0, 1)$ are X and Z independent (explain)?
3. (20 points) Let X be a continuous random variable, with density $f_X(x)$. We denote by F_X its cumulative distribution function.
 - (i) (10 points) Let $Y = X^2$. Find its cumulative distribution function and its density, in terms of F_X and f_X .
 - (ii) (10 points) Assume now that X has exponential distribution with parameter 2, i.e. it is continuous with density $f_X(x) = 2e^{-2x} \mathbb{1}_{[0, +\infty)}(x)$. Determine the cumulative distribution function and the density of X^2 .

4. (28 points) The arrival times of customers in a bakery can be modeled by a Poisson process $(N_t)_{t \geq 0}$ with rate $\lambda = 4$.
- (i) (6 points) Let $\mu > 0$, and let X be a random variable having exponential distribution with parameter μ (i.e. it is continuous with density $f_X(x) = \mu e^{-\mu x} \mathbb{1}_{[0, +\infty)}(x)$). Prove that it has expectation $\mathbb{E}[X] = \frac{1}{\mu}$.
 - (ii) (4 points) Denote by $(S_n)_{n \geq 1}$ the successive arrival times of customers, and $S_0 = 0$. For each $n \geq 1$, recall what the distribution of $(S_n - S_{n-1})$ is.
 - (iii) (4 points) For $n \geq 1$, determine the expectation of $(S_n - S_{n-1})$, and then of S_n .
 - (iv) (10 points) Compute the probability of the following event: {the second customer arrives before time 2, two customers arrive during $[4, 10]$, and at least one arrives in $[8, 16]$ }.
 - (v) (4 points) We now assume that upon entering the bakery, each customer decides to buy either one bread with probability $\frac{2}{3}$, or one croissant with probability $\frac{1}{3}$ (i.e. the customer chooses exactly one of these two options, it never chooses both), independently of the other customers. What can you say about the corresponding counting processes, denoted (respectively) by $(N_t^B)_{t \geq 0}$ and $(N_t^C)_{t \geq 0}$?
5. (28 points) The arrival times of customers in two neighboring restaurants (called A and B) can be modeled by two independent Poisson processes $(N_t)_{t \geq 0}$ and $(N'_t)_{t \geq 0}$, with respective rates λ and λ' .
- (i) (10 points) For $t \geq 0$, compute $\text{Cov}(N_{4t} - N_t, N_{5t} - N_{2t})$.
 - (ii) (4 points) In each of the two restaurants, we consider the arrival time of the first customer: it is denoted by T in restaurant A, and T' in restaurant B. Recall what the distributions of T and T' are.
 - (iii) (10 points) Compute the probability that the first customer to arrive enters the first restaurant, i.e. $\mathbb{P}(T < T')$.
 - (iv) (4 points) What can you say about the counting process $(N_t + N'_t)_{t \geq 0}$ of customers entering one of the two restaurants?