

MA1200 CALCULUS AND BASIC LINEAR ALGEBRA I

LECTURE: CG1

Chapter 8 **Applications of Differentiation**

Applications of Differentiation

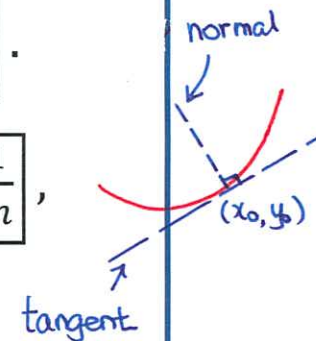
In this chapter, we will study the following applications of differentiation:

- 1. Equation of tangent / normal to the curve**
- 2. Rate of change**
 - rate of change of a variable with respect to time
- 3. Local extrema of functions**
 - find the local maximum or local minimum values of the graph of a function
- 4. Optimization problem**
 - maximize or minimize a variable
- 5. L'Hôpital's rule**
 - find the limits of functions of indeterminate forms
- 6. Taylor's series / Maclaurin series**
 - approximate a given function by using higher order polynomials

1. Equation of tangent / normal to the curve

Given the curve of a function $F(x, y) = 0$. Suppose that the slope of the function y , i.e. $\frac{dy}{dx}$, at the point (x_0, y_0) exists, and let $m = \left. \frac{dy}{dx} \right|_{(x_0, y_0)}$ be the value of $\frac{dy}{dx}$ of the curve at (x_0, y_0) . Then, m is the **slope of the tangent** to the curve at (x_0, y_0) .

- The **equation of the tangent to the curve** at the point (x_0, y_0) is $\boxed{\frac{y-y_0}{x-x_0} = m}$.
- The **equation of the normal to the curve** at the point (x_0, y_0) is $\boxed{\frac{y-y_0}{x-x_0} = -\frac{1}{m}}$, provided $m \neq 0$.



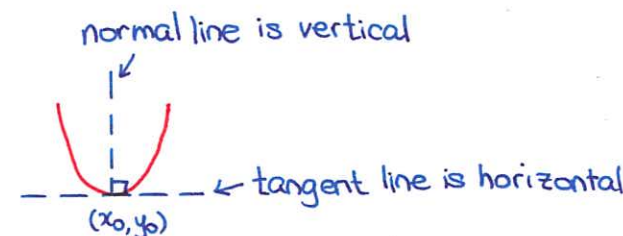
Recall:

$$\boxed{\text{slope of normal} = \frac{-1}{\text{slope of tangent}}}$$

(from Chapter 1)

Note that if $m = 0$, then

- the equation of the tangent to the curve at (x_0, y_0) is the horizontal line $y = y_0$; and
- the equation of the normal to the curve at (x_0, y_0) is the vertical line $x = x_0$.



Example 1

Given the parametric equations $\begin{cases} x = 3 \cos t \\ y = 2 \sin t \end{cases}$, where $0 \leq t \leq 2\pi$.

- (i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
- (ii) Find the equation of the tangent to the curve at $t = \frac{\pi}{4}$.
- (iii) Find the equation of the normal to the curve at $t = \frac{\pi}{4}$.

Solution

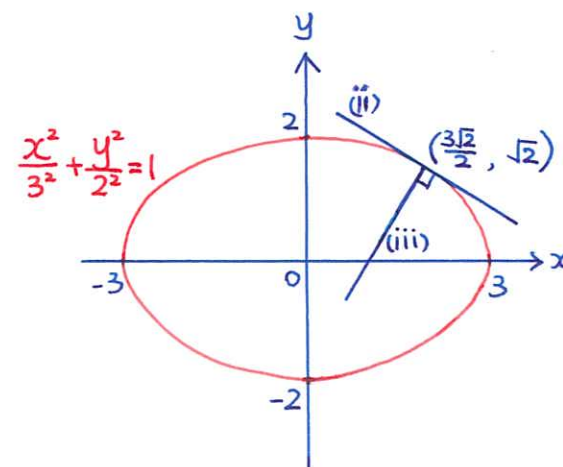
$$(i) \quad x = 3 \cos t \Rightarrow \frac{dx}{dt} = 3(-\sin t) = -3 \sin t$$

$$y = 2 \sin t \Rightarrow \frac{dy}{dt} = 2 \cos t$$

$$\text{Then } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 \cos t}{-3 \sin t} = -\frac{2}{3} \cot t \quad (t \neq 0, \pi, 2\pi)$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{\frac{d}{dt} \left(\frac{dy}{dx} \right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt} \left(-\frac{2}{3} \cot t \right)}{-3 \sin t} = \frac{-\frac{2}{3} (-\operatorname{cosec}^2 t)}{-3 \sin t} = -\frac{2}{9} \operatorname{cosec}^3 t \quad (t \neq 0, \pi, 2\pi)$$

$\frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx}$



$$(ii) \quad \text{At } t = \frac{\pi}{4}, \quad y = 2 \sin\left(\frac{\pi}{4}\right) = 2 \cdot \frac{\sqrt{2}}{2} = \sqrt{2}$$

$$x = 3 \cos\left(\frac{\pi}{4}\right) = 3 \cdot \frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2}$$

The slope of the tangent to the curve at $t = \frac{\pi}{4}$ is

$$\frac{dy}{dx} = -\frac{2}{3} \cot\left(\frac{\pi}{4}\right) = -\frac{2}{3} \cdot \frac{1}{\tan\left(\frac{\pi}{4}\right)} = -\frac{2}{3} \cdot \frac{1}{1} = -\frac{2}{3}$$

$\therefore$ The equation of the tangent to the curve at $t = \frac{\pi}{4}$ is given by

$$\frac{y - \sqrt{2}}{x - \frac{3\sqrt{2}}{2}} = -\frac{2}{3}$$

$$\Rightarrow y - \sqrt{2} = -\frac{2}{3} \left(x - \frac{3\sqrt{2}}{2} \right)$$

$$\Rightarrow y = -\frac{2}{3}x + 2\sqrt{2}$$

(iii) The slope of the normal to the curve at $t = \frac{\pi}{4}$ is

$$\frac{-1}{\left(-\frac{2}{3}\right)} = \frac{3}{2}.$$

$\therefore$ The equation of the normal to the curve at $t = \frac{\pi}{4}$ is given by

$$\frac{y - \sqrt{2}}{x - \frac{3\sqrt{2}}{2}} = \frac{3}{2}$$

$$\Rightarrow y - \sqrt{2} = \frac{3}{2} \left(x - \frac{3\sqrt{2}}{2} \right)$$

$$\Rightarrow y = \frac{3}{2}x - \frac{5\sqrt{2}}{4}$$

Example 2 (Explicit function)

Find the equation of the tangent line to the curve $y = \frac{\sqrt{x}}{x^2+1}$ at the point $\left(1, \frac{1}{2}\right)$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{(x^2 + 1) \cdot \frac{d}{dx} \left(x^{\frac{1}{2}} \right) - x^{\frac{1}{2}} \cdot \frac{d}{dx} (x^2 + 1)}{(x^2 + 1)^2} = \frac{(x^2 + 1) \cdot \frac{1}{2} x^{-\frac{1}{2}} - x^{\frac{1}{2}} \cdot (2x)}{(x^2 + 1)^2} \\ &= \frac{-3x^2 + 1}{2\sqrt{x} (x^2 + 1)^2}\end{aligned}$$

The slope of the tangent line to the curve at the point $\left(1, \frac{1}{2}\right)$ is

$$\left. \frac{dy}{dx} \right|_{x=1} = \frac{-3(1)^2 + 1}{2\sqrt{1} (1^2 + 1)^2} = -\frac{1}{4}.$$

$\therefore$ The equation of the tangent line to the curve at $\left(1, \frac{1}{2}\right)$ is

$$\frac{y - \frac{1}{2}}{x - 1} = -\frac{1}{4} \Rightarrow y = -\frac{1}{4}x + \frac{3}{4} \quad (\text{or written as } 4y + x - 3 = 0).$$

Example 3 (Implicit function)

Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$.

Solution

Note that $x^2 + y^2 = 25$ is an **implicit** function.

Differentiate both sides of $x^2 + y^2 = 25$ w.r.t. x :

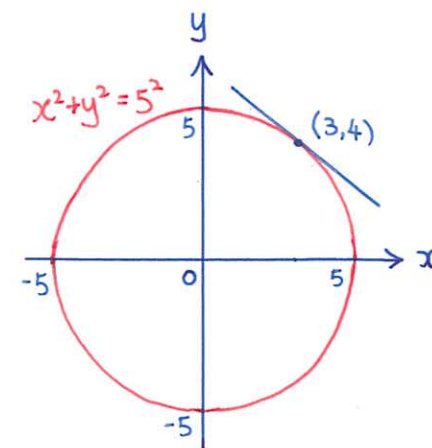
$$\begin{aligned} 2x + 2y \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} &= \frac{-2x}{2y} = -\frac{x}{y} \end{aligned}$$

$\therefore$ The slope of the tangent line to $x^2 + y^2 = 25$ at the point $(3, 4)$ is

$$\left. \frac{dy}{dx} \right|_{(x,y)=(3,4)} = -\frac{3}{4}.$$

$\therefore$ The equation of the tangent line to the curve at the point $(3, 4)$ is

$$\frac{y - 4}{x - 3} = -\frac{3}{4} \quad \Rightarrow \quad y = -\frac{3}{4}x + \frac{25}{4}$$



Class Exercise

Given that $x^2 + 2xy^2 - y^3 - 1 = 0$. ← implicit function

Find the equations of the tangent and normal to the curve at the point $P(1, 2)$.

Solution

Differentiate both sides of $x^2 + 2xy^2 - y^3 - 1 = 0$ w.r.t. x :

$$\frac{d}{dx}(x^2) + 2 \frac{d}{dx}(xy^2) - \frac{d}{dx}(y^3) - \frac{d}{dx}(1) = \frac{d}{dx}(0)$$

$$\Rightarrow 2x + 2 \left[x \cdot \frac{d}{dx}(y^2) + y^2 \cdot \frac{d}{dx}(x) \right] - 3y^2 \cdot \frac{dy}{dx} - 0 = 0$$

$$\Rightarrow 2x + 2 \left[x \cdot 2y \frac{dy}{dx} + y^2 \cdot 1 \right] - 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow (4xy - 3y^2) \frac{dy}{dx} = -2x - 2y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x - 2y^2}{4xy - 3y^2}$$

At the point $P(1,2)$,

$$\frac{dy}{dx} = \frac{-2(1) - 2(2)^2}{4(1)(2) - 3(2)^2} = \frac{-10}{-4} = \frac{5}{2}$$

$\therefore$ The equation of the tangent to the curve at $P(1,2)$ is

$$\frac{y-2}{x-1} = \frac{5}{2} \Rightarrow 2(y-2) = 5(x-1)$$

$$\Rightarrow 2y - 5x + 1 = 0$$

$\therefore$ The equation of the normal to the curve at $P(1,2)$ is

$$\frac{y-2}{x-1} = \frac{-1}{(\frac{5}{2})} \Rightarrow 5(y-2) = -2(x-1)$$

$$\Rightarrow 5y + 2x - 12 = 0$$

2. Rate of change

We usually use the notation

$$\frac{d(?)}{dt}$$

to denote “the rate of change of (?) with respect to time t ”.

E.g. If $h(t)$ denotes the height of an object at time t , then

$$\frac{dh}{dt} = \text{the rate of change of height with respect to time.}$$

E.g. Suppose an object is moving along a straight line with the **position function**

$$x = x(t) \quad (\text{i.e. the position of the object at time } t).$$

$$\text{Then } v(t) = \frac{dx}{dt} = \text{velocity of the object at time } t$$

(i.e. $v(t)$ is the rate of change of its position x w.r.t. time t),

$$\text{and } a(t) = \frac{dv}{dt} = \frac{d}{dt} \left(\frac{dx}{dt} \right) = \frac{d^2x}{dt^2} = \text{acceleration of the object at time } t$$

(i.e. $a(t)$ is the rate of change of velocity v w.r.t. time t).

Example 4

If the velocity $v(t)$ of a body varies inversely as the square root of the distance $s(t)$, prove that the acceleration $a(t)$ varies as the fourth power of the velocity.

Solution

We have $v(t) = \frac{k}{\sqrt{s(t)}}$, where k is a constant. Then

$$\begin{aligned} a(t) &= \frac{d}{dt}[v(t)] = \frac{d}{dt}\left[\frac{k}{\sqrt{s(t)}}\right] = \frac{d}{dt}\left[k(s(t))^{-\frac{1}{2}}\right] \\ &= k\left(-\frac{1}{2}\right)(s(t))^{-\frac{3}{2}} \cdot \frac{d}{dt}[s(t)] \\ &= -\frac{1}{2}k(s(t))^{-\frac{3}{2}} \cdot v(t) = -\frac{1}{2}k\left\{\left[\frac{k}{v(t)}\right]^2\right\}^{-\frac{3}{2}} \cdot v(t) \\ &= -\frac{1}{2}k\left[\frac{k}{v(t)}\right]^{-3} \cdot v(t) \\ &= -\frac{[v(t)]^4}{2k^2} \end{aligned}$$

Procedure for solving rate of change problems

Step 1: Write an equation which relates relevant variables.

Step 2: Use chain rule to differentiate (implicitly) both sides of the equation w.r.t. time t .

Step 3: Substitute in the values for all known variables and derivatives.

Step 4: Solve for the remaining rate.

Example 5

A ladder which is 5m long is resting against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1m/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 3m from the wall?

$$\left. \frac{dy}{dt} \right|_{x=3} = ?$$

Solution

Let $x(t)$ be the perpendicular distance from the wall to the bottom of the ladder at time t , and $y(t)$ be the distance from the top of the ladder to the ground at time t .

By Pythagoras' Theorem,

$$x^2 + y^2 = 5^2.$$

Differentiate both sides w.r.t. t :

$$\frac{d}{dx}(x^2) \cdot \frac{dx}{dt} \rightarrow 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{x}{y} \cdot \frac{dx}{dt}$$

(Handwritten red notes: $\frac{d}{dy}(y^2) \cdot \frac{dy}{dt}$ with arrows pointing to the $2y \cdot \frac{dy}{dt}$ term)

Given that $\frac{dx}{dt} = 1$ m/s.

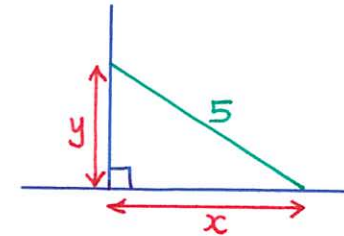
When $x = 3$ m,

$$y = \sqrt{5^2 - x^2} = \sqrt{5^2 - 3^2} = \sqrt{16} = 4 \text{ m}$$

and

$$\left. \frac{dy}{dt} \right|_{x=3, y=4} = \left(-\frac{x}{y} \cdot \frac{dx}{dt} \right) \Big|_{x=3, y=4} = -\frac{3}{4} \cdot 1 = -\frac{3}{4} \text{ m/s.}$$

$\therefore$ The top of the ladder is sliding down the wall at a rate of $\frac{3}{4}$ m/s.



Example 6

A paper cup has the shape of a right circular cone with height 10 cm and diameter 7 cm (at the top). If drinking water is poured into the cup at a rate of $\underline{6 \text{ cm}^3/\text{sec}}$, how fast is the water level rising when the water is 6 cm deep?

↑
 $\frac{dV}{dt}$

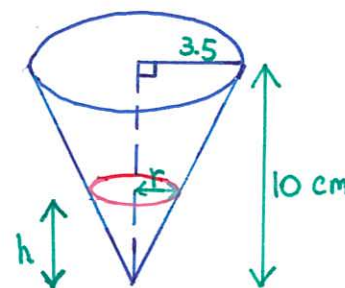
(Hint: Volume of right circular cone is $V = \frac{1}{3}\pi r^2 h$.)

Solution

Let V , r and h be the volume of the water, the radius of the surface of water, and the height of water at time t seconds, respectively.

Objective: Find $\left. \frac{dh}{dt} \right|_{h=6 \text{ cm}}$

Given information: $\frac{dV}{dt} = 6 \text{ cm}^3/\text{sec}$



By similar triangles, $\frac{r}{h} = \frac{3.5}{10} \Rightarrow r = 0.35 h$

$$\therefore V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(0.35 h)^2 h = \frac{1}{3}\pi(0.35)^2 h^3 \text{ (cm}^3\text{)}$$

Differentiate both sides w.r.t. t :

$$\frac{dV}{dt} = \frac{1}{3}\pi(0.35)^2 \cdot \frac{d}{dt}(h^3) = \frac{1}{3}\pi(0.35)^2 \cdot \boxed{3h^2 \frac{dh}{dt}} = \pi(0.35)^2 h^2 \frac{dh}{dt}$$

$\nwarrow \frac{d}{dh}(h^3) \cdot \frac{dh}{dt}$

$$\Rightarrow \frac{dh}{dt} = \frac{\frac{dV}{dt}}{\pi(0.35)^2 h^2} = \frac{6}{\pi(0.35)^2 h^2}$$

When the height of water is $h = 6$ cm,

$$\left. \frac{dh}{dt} \right|_{h=6 \text{ cm}} = \frac{6}{\pi(0.35)^2 6^2} \approx 0.433 \text{ cm/sec}$$

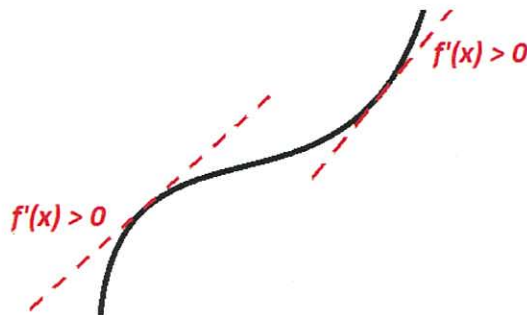
3. Local extrema (local minima / local maxima) of functions

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b) .

The sign of $f'(x)$ tells us whether $f(x)$ is an increasing or decreasing function.

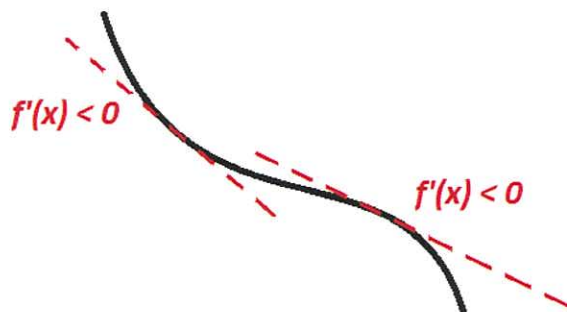
➤ Increasing function:

If $f'(x) \geq 0$ ($f'(x) > 0$) for every $x \in (a, b)$, then $f(x)$ is **increasing (strictly increasing)** on $[a, b]$.



➤ Decreasing function:

If $f'(x) \leq 0$ ($f'(x) < 0$) for every $x \in (a, b)$, then $f(x)$ is **decreasing (strictly decreasing)** on $[a, b]$.



Example 7

Identify the increasing and decreasing intervals for $f(x) = x^3 - 3x^2 + 1$.

Solution

$$f'(x) = 3x^2 - 6x$$

Set $f'(x) = 0 \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x - 2) = 0 \Rightarrow x = 0$ or 2 .

	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
Sign of x	—	0	+	+	+
Sign of $x - 2$	—	—	—	0	+
Sign of $f'(x)$	+	0	—	0	+

Observe that $f'(x) > 0$ when $x < 0$ or $x > 2$;

and $f'(x) < 0$ when $0 < x < 2$.

Therefore, f is strictly increasing on $(-\infty, 0]$ and $[2, \infty)$, and strictly decreasing on $[0, 2]$.

Example 8

Show that $\ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$ for $x > 0$.

Solution:

$$\text{Let } f(x) = \ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3}.$$

Then

$$\underline{f'(x)} = \frac{1}{1+x} - 1 + x - x^2 = \frac{1 - (1+x) + x(1+x) - x^2(1+x)}{1+x} = \frac{-x^3}{1+x} < 0 \text{ for } x > 0.$$

$\therefore f$ is decreasing on $(0, \infty)$.

$\therefore f(x) < f(0)$ for any $x > 0$,

i.e. $\ln(1+x) - x + \frac{x^2}{2} - \frac{x^3}{3} < \ln(1+0) - 0 + 0 - 0 = 0$ for any $x > 0$.

$$\Rightarrow \ln(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \text{ for } x > 0.$$

Extreme values

Maximum and **minimum values** of a function are referred to as **extreme values**.

Absolute extreme values

- A function f has an **absolute** (or **global**) **maximum** value $f(x_0)$ at the point $x = x_0$ in $\text{Dom}(f)$ if $f(x) \leq f(x_0)$ for every $x \in \text{Dom}(f)$.
- A function f has an **absolute** (or **global**) **minimum** value $f(x_0)$ at the point $x = x_0$ in $\text{Dom}(f)$ if $f(x) \geq f(x_0)$ for every $x \in \text{Dom}(f)$.

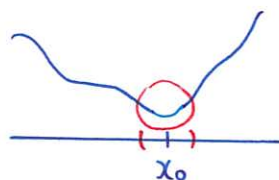
Local extreme values

(or relative maximum)

- A function f has a **local maximum** value $f(x_0)$ at the point $x = x_0$ in $\text{Dom}(f)$ if there is a number $h > 0$ such that $f(x) \leq f(x_0)$ for every x in the open interval $(x_0 - h, x_0 + h)$.



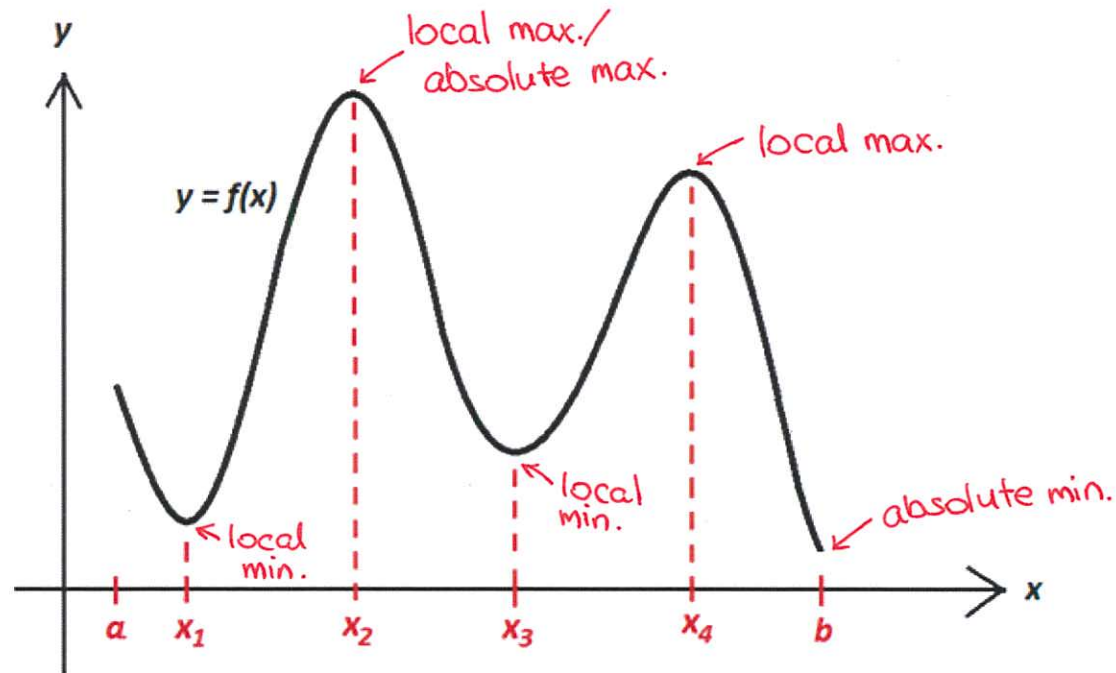
- A function f has a **local minimum** value $f(x_0)$ at the point $x = x_0$ in $\text{Dom}(f)$ if there is a number $h > 0$ such that $f(x) \geq f(x_0)$ for every x in the open interval $(x_0 - h, x_0 + h)$.

**Remarks:**

1. In other words, local maximum and minimum values correspond to the points on the graph that are higher or lower than neighboring points.
2. The absolute maximum is the highest of the local maxima (or occurs at boundary point); the absolute minimum is the lowest of the local minima (or occurs at boundary point).

Illustration:

Consider the graph of the function $y = f(x)$ whose domain is $Dom(f) = [a, b]$.



Observe that

- $f(x)$ has local maximum values (or **local maxima**) at x_2 and x_4 , and local minimum values (or **local minima**) at x_1 and x_3 .
- $f(x)$ has an **absolute maximum** at x_2 , and an **absolute minimum** at b .

Local maximum:**Definition:**

Let f be continuous at $x = c$.

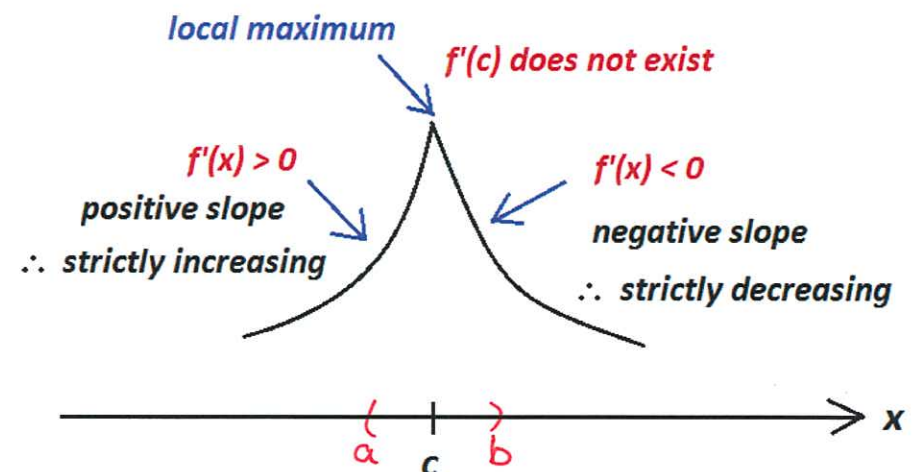
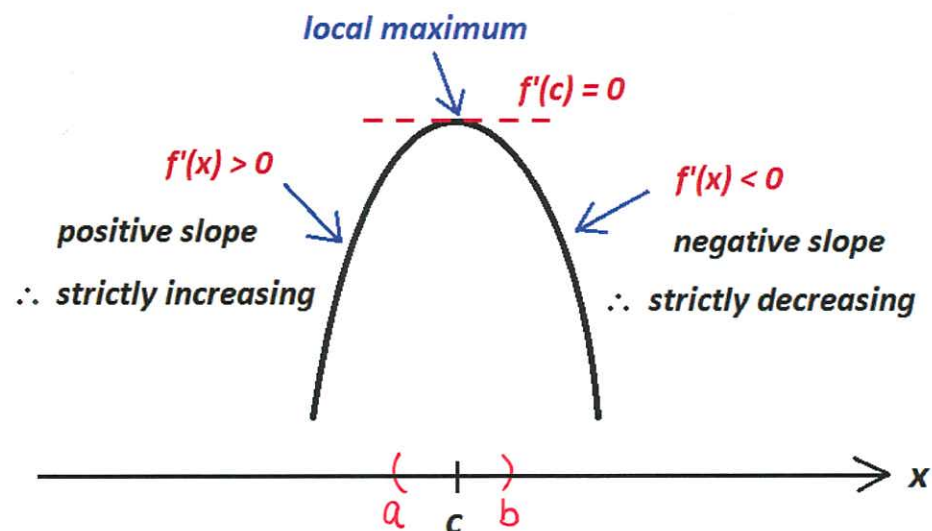
If there exists an open interval (a, b) containing c such that

$$f'(x) > 0 \text{ when } a < x < c$$

and $f'(x) < 0$ when $c < x < b$,

then f has a **local maximum** value at $x = c$.

First Derivative Test



Local minimum:**Definition:**

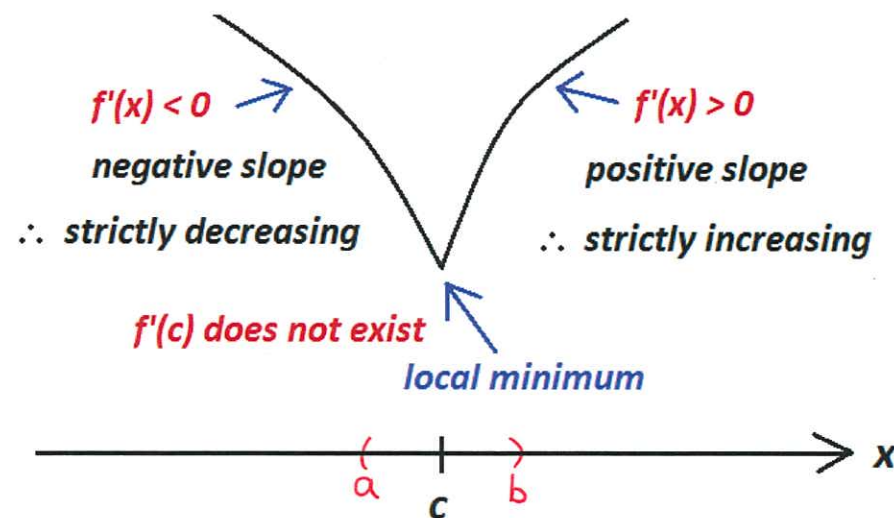
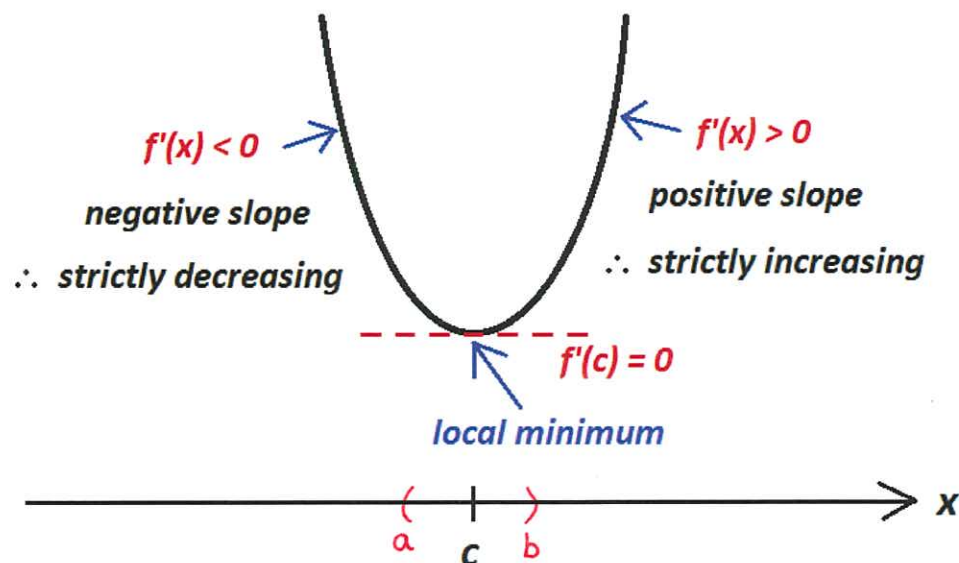
Let f be continuous at $x = c$.

If there exists an open interval (a, b) containing c such that

$$f'(x) < 0 \text{ when } a < x < c$$

and $f'(x) > 0$ when $c < x < b$,

then f has a **local minimum** value at $x = c$.



Theorem

Suppose f has a local maximum or local minimum point at $x = c$. If $f'(c)$ exists, then

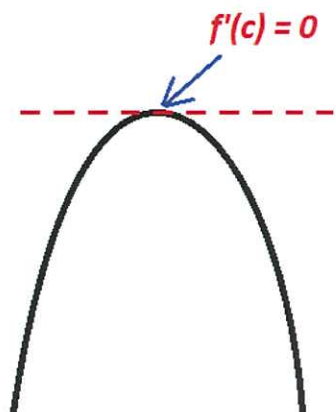
$$\boxed{f'(c) = 0}.$$

Definition

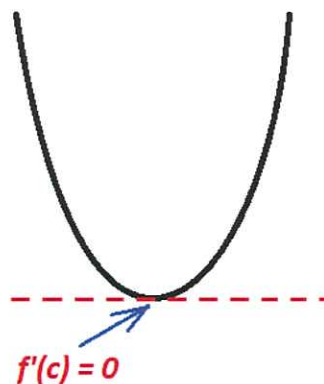
A function f has a **stationary point** ^(or critical point) at $x = c$ if

$$\boxed{f'(c) = 0}.$$

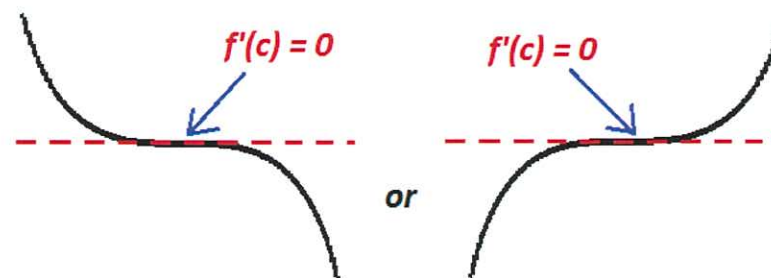
A stationary point can be a **local maximum**, **local minimum**, or **point of inflection**.



local maximum



local minimum



point of inflection

Example 9

Find and classify all stationary points of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x + 1)(x - 2).$$

$$\text{Set } f'(x) = 0 \Rightarrow 12x(x + 1)(x - 2) = 0 \Rightarrow x = -1, 0, 2.$$

$\therefore f$ has stationary points at $x = -1, 0, 2$.

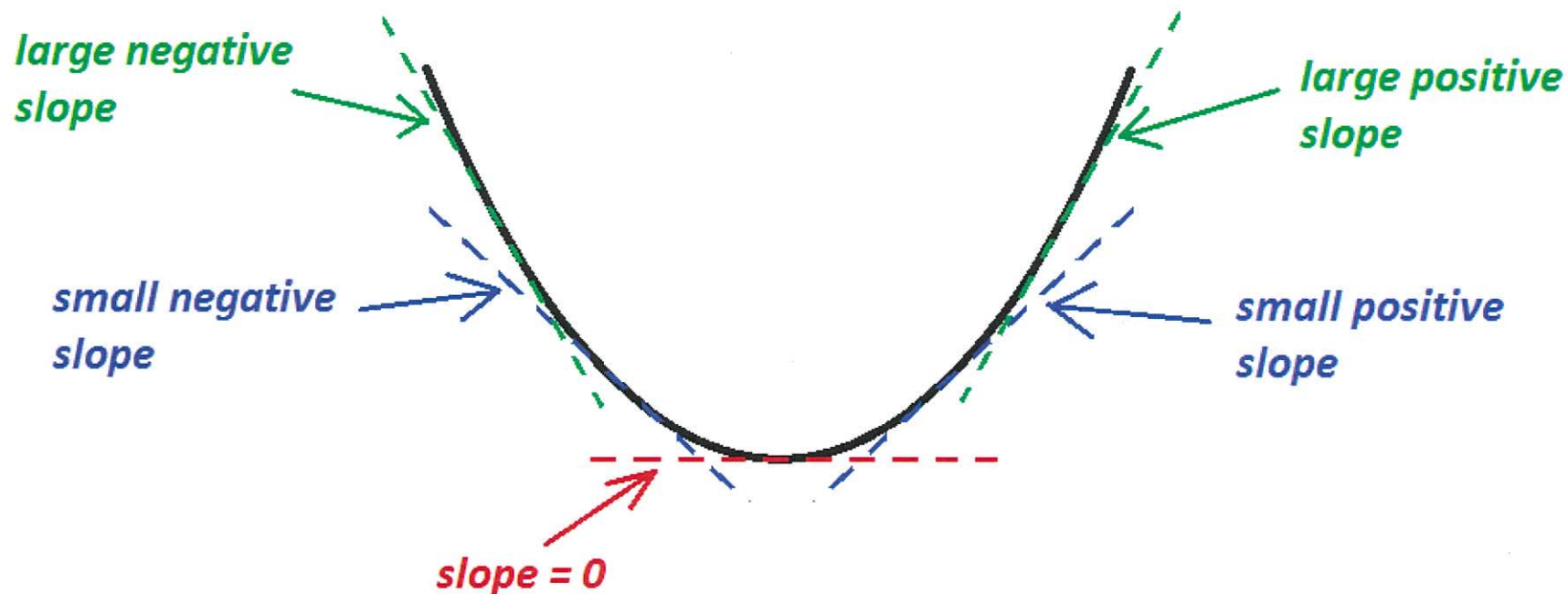
	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
Sign of $x + 1$	—	0	+	+	+	+	+
Sign of x	—	—	—	0	+	+	+
Sign of $x - 2$	—	—	—	—	—	0	+
Sign of $f'(x)$	—	0	+	0	—	0	+

$\therefore f$ has local maximum at $x = 0$, and the corresponding maximum value is $f(0) = 5$.

f has local minima at $x = -1$ and $x = 2$, and the corresponding local minimum values are $f(-1) = 0$ and $f(2) = -27$, respectively.

Concavity (p.27-31: For your reference)

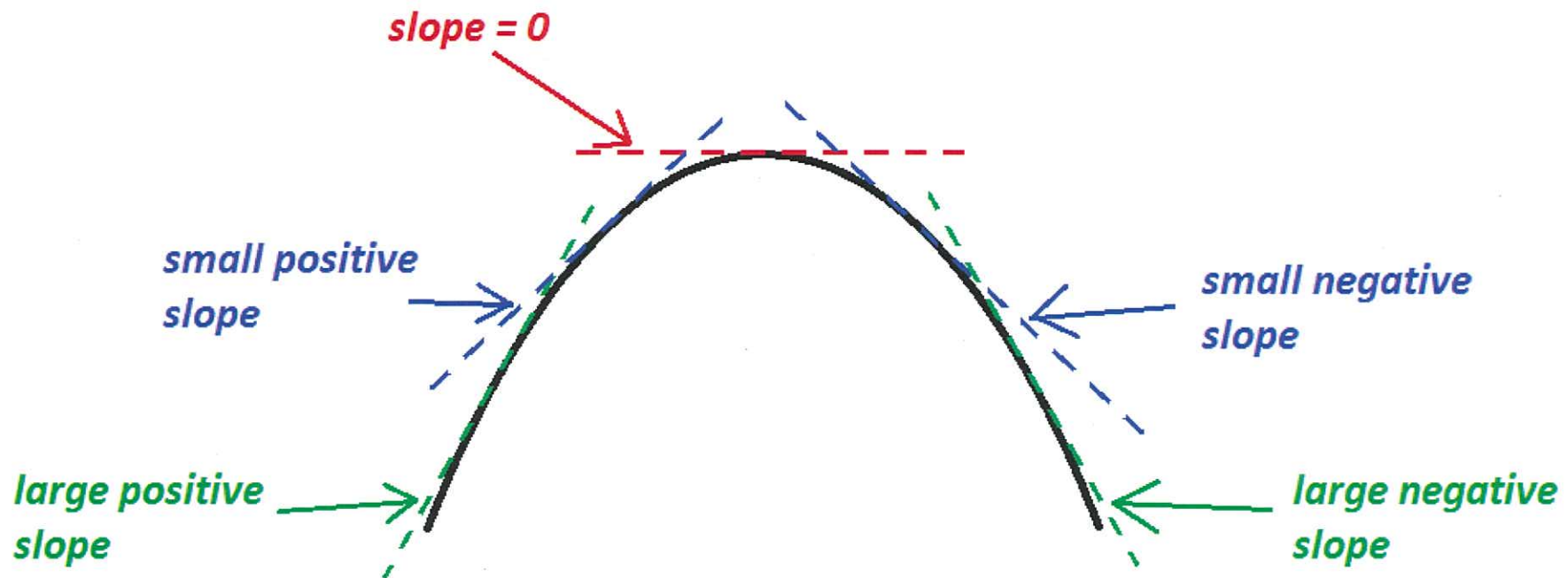
- A function f is **concave up** on an interval I if it is differentiable on I and $f'(x)$ is increasing on I , i.e. $f''(x) > 0$ for every $x \in I$.



Concave up: $f'(x)$ increases as x increases.

$$\therefore f''(x) > 0$$

- A function f is **concave down** on an interval I if it is differentiable on I and $f'(x)$ is decreasing on I , i.e. $f''(x) < 0$ for every $x \in I$.



Concave down:

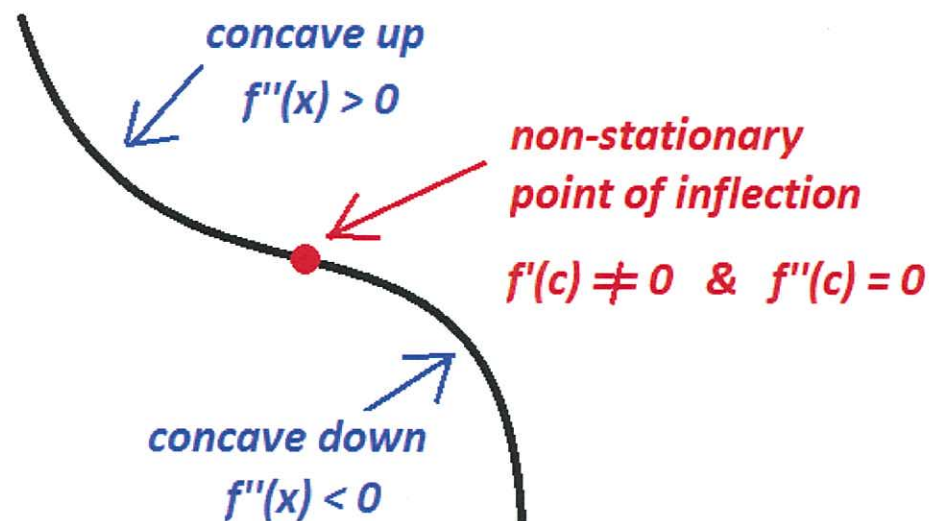
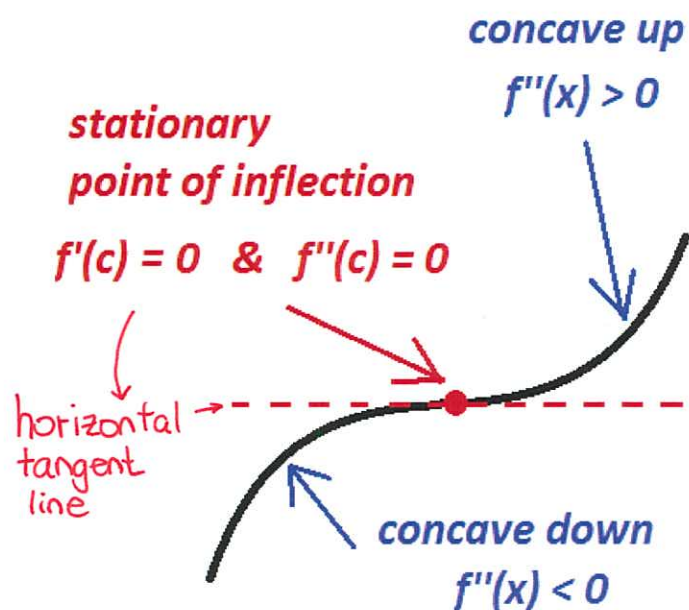
$f'(x)$ decreases as x increases.

$$\therefore f''(x) < 0$$

- Let f be a twice differentiable function. Then f has a **point of inflection** (or **point of inflexion**) at $x = c$ if $\boxed{f''(c) = 0}$ and f is concave up on one side of c and concave down on the other side of c (i.e. the sign of $f''(x)$ changes at $x = c$).

In addition, if $f'(c) = 0$, we say that $(c, f(c))$ is a **stationary point of inflection**.

If $f'(c) \neq 0$, then $(c, f(c))$ is a **non-stationary point of inflection**.



Summary: Determination of points of inflection of a function

Given a twice differentiable function $y = f(x)$.

To determine the points of inflection of the curve:

Step 1: Find the second derivative $f''(x)$.

Step 2: Set $f''(x) = 0$, and find all the values of c in $\text{Dom}(f)$ such that $f''(c) = 0$.

Step 3: If the sign of f'' changes at $x = c$, then f has a point of inflection at $x = c$, i.e. at the point $(c, f(c))$.

Example 10

Determine all (if any) points of inflection for the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

Solution

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$f''(x) = 36x^2 - 24x - 24 = 12(3x^2 - 2x - 2)$$

$$\text{Set } f''(x) = 0 \Rightarrow 12(3x^2 - 2x - 2) = 0 \Rightarrow 3x^2 - 2x - 2 = 0$$

$$\Rightarrow x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-2)}}{2(3)} = \frac{2 \pm \sqrt{28}}{6} = \frac{2 \pm 2\sqrt{7}}{6} = \frac{1 \pm \sqrt{7}}{3}$$

	$x < \frac{1 - \sqrt{7}}{3}$	$x = \frac{1 - \sqrt{7}}{3}$	$\frac{1 - \sqrt{7}}{3} < x < \frac{1 + \sqrt{7}}{3}$	$x = \frac{1 + \sqrt{7}}{3}$	$x > \frac{1 + \sqrt{7}}{3}$
Sign of $f''(x)$	+	0	−	0	+
	C.U.		C.D.		C.U.

Since the sign of f'' changes at $x = \frac{1 - \sqrt{7}}{3}$ and $x = \frac{1 + \sqrt{7}}{3}$, f has points of inflection at

$x = \frac{1 - \sqrt{7}}{3}$ and $x = \frac{1 + \sqrt{7}}{3}$, i.e. at $(-0.5486, 2.3207)$ and $(1.2153, -13.3578)$.

Classification of the stationary point from the second derivative (Second Derivative Test)

Suppose that the function f has a **stationary point** at $x = c$, i.e. $f'(c) = 0$.

- If $f''(c) > 0$, then f has a **local minimum** at $x = c$. $\cup$
- If $f''(c) < 0$, then f has a **local maximum** at $x = c$. $\cap$
- If $f''(c) = 0$, then no conclusion can be drawn. The function f may have a local maximum at $x = c$, a local minimum at $x = c$, or a (stationary) point of inflection at $x = c$.
 $\uparrow$ Second derivative test fails

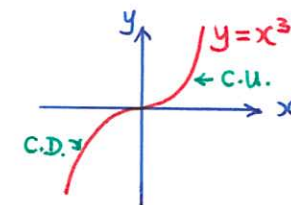
E.g. 1: Consider $f(x) = x^3$. Then $f'(x) = 3x^2$. Setting $f'(x) = 0$ gives $x = 0$.

$\therefore f$ has a stationary point at $x = 0$, $y = 0$.

$f''(x) = 6x$. Then $f''(0) = 6 \times 0 = 0$, so the Second Derivative Test fails.

Now, $f''(0) = 0$ and let's consider the sign of f'' in the neighborhood of $x = 0$:

	$x < 0$	$x = 0$	$x > 0$
Sign of $f''(x)$	$-$ C.D.	0	$+$ C.U.



Since the sign of $f''(x)$ changes at $x = 0$, the function f has a (stationary) point of inflection at $(0, 0)$.
 $\uparrow$
 $\because f'(0) = 0$

E.g. 2: Consider $f(x) = x^4$. Then $f'(x) = 4x^3$.

Setting $f'(x) = 0$ gives $4x^3 = 0 \Rightarrow x = 0$

$\therefore f$ has a stationary point at $x = 0, y = 0$.

$f''(x) = 12x^2$. Then $f''(0) = 12 \times 0^2 = 0$, so the Second Derivative Test fails.

Now, $f''(0) = 0$ and let's consider the sign of f'' in the neighborhood of $x = 0$:

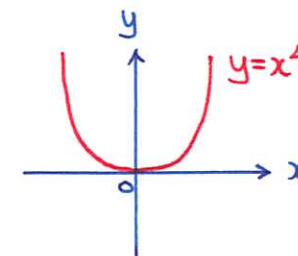
	$x < 0$	$x = 0$	$x > 0$
Sign of $f''(x)$	+	0	+

Since the sign of $f''(x)$ does not change at $x = 0$, the function f does not have a point of inflection at $(0, 0)$.

To classify the stationary point at $x = 0$, we use the First Derivative Test:

	$x < 0$	$x = 0$	$x > 0$
Sign of $f'(x)$	- $\downarrow$	0 $\cup$	+ $\uparrow$

$\therefore$ The function has a local minimum at $(0, 0)$.



Summary: Determination of local extrema (local minima / local maxima) of a function

Given a twice differentiable function $y = f(x)$.

To determine the local extrema of the curve:

Step 1: Find the first derivative $f'(x)$.

Step 2: Set $f'(x) = 0$, and find all the values of x_0 in $Dom(f)$ such that $f'(x_0) = 0$.

Step 3: Classify whether x_0 is a local maximum or local minimum by using one of the following tests:

➤ **The First Derivative Test:**

(a) If there exists an open interval (a, b) containing x_0 such that

$$f'(x) > 0 \text{ when } a < x < x_0$$

and $f'(x) < 0$ when $x_0 < x < b$,



then f has a **local maximum** value at $x = x_0$, i.e. at the point $(x_0, f(x_0))$.

(b) If there exists an open interval (a, b) containing x_0 such that

$$f'(x) < 0 \text{ when } a < x < x_0$$

$$\text{and } f'(x) > 0 \text{ when } x_0 < x < b,$$

then f has a **local minimum** value at $x = x_0$, i.e. at the point $(x_0, f(x_0))$.



(c) If the sign of $f'(x)$ does not change at $x = x_0$, then $x = x_0$ is neither a local maximum nor a local minimum.

➤ **The Second Derivative Test:**

(a) If $f'(x_0) = 0$ and $f''(x_0) \leq 0$, then f has a **local maximum** value at $x = x_0$, i.e. at the point $(x_0, f(x_0))$.

c.d.

(b) If $f'(x_0) = 0$ and $f''(x_0) \geq 0$, then f has a **local minimum** value at $x = x_0$, i.e. at the point $(x_0, f(x_0))$.

c.u.

(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$, no conclusion can be drawn.

(→ Use the First Derivative Test.)

Example 11

Determine and classify all local extrema of the function $f(x) = x^2 e^x$.

Solution product rule

$$f'(x) = \overset{\text{product rule}}{x^2 \cdot \frac{d}{dx}(e^x)} + e^x \cdot \frac{d}{dx}(x^2) = x^2 \cdot e^x + e^x \cdot 2x = e^x x(x + 2).$$

$$\text{Set } f'(x) = 0 \Rightarrow e^x x(x + 2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = -2 \quad (\text{Note: } e^x > 0 \text{ for all } x \in \mathbb{R}.)$$

Use either the **First Derivative Test**:

	$x < -2$	$x = -2$	$-2 < x < 0$	$x = 0$	$x > 0$
Sign of x	−	−	−	0	+
Sign of $(x + 2)$	−	0	+	+	+
Sign of $f'(x)$	+	0	−	0	+
		(local max.)		(local min.)	

$\therefore f$ has a **local maximum** at $x = -2$, $y = 4e^{-2}$; and a **local minimum** at $x = 0$, $y = 0$.

Or the **Second Derivative Test**:

Product rule for differentiating $f \cdot g \cdot h$ w.r.t. x :

$$(f \cdot g \cdot h)'(x) = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$$

$$f''(x) = \frac{d}{dx} [e^x x(x+2)]$$

$$= e^x x \cdot \frac{d(x+2)}{dx} + e^x (x+2) \cdot \frac{d(x)}{dx} + x(x+2) \cdot \frac{d(e^x)}{dx}$$

$$= e^x x \cdot 1 + e^x (x+2) \cdot 1 + x(x+2) \cdot e^x$$

$$= e^x (x^2 + 4x + 2)$$

$$f''(-2) = e^{-2} [(-2)^2 + 4(-2) + 2] = -2e^{-2} < 0$$

$\Rightarrow f$ has a **local maximum** at $x = -2$, $y = 4e^{-2}$.

$$f''(0) = e^0 [(0)^2 + 4(0) + 2] = 2 > 0$$

$\Rightarrow f$ a **local minimum** at $x = 0$, $y = 0$.