

EF3441: Intermediate Macroeconomics

Prepared by Dr. Tam
Lecture 6

The Main Goal of this Lecture

- ▶ Understand Solow growth model by
 - ▶ knowing how to derive the steady-state conditions,
 - ▶ using the model to do policy analysis and find transitional paths,
 - ▶ applying the Solow model to explain the data,
 - ▶ and finding facts the Solow model can/can't explain.
- ▶ Understand the endogenous growth model by
 - ▶ knowing the assumptions and facts that motivate this model

Recall the Solow Model

- ▶ Household

- ▶ Exogenous population growth:

$$N' = (1 + n)N,$$

- ▶ “Rule of thumb” for saving

$$C = (1 - s)Y$$

- ▶ Firm

- ▶ Capital accumulation function:

$$K' = (1 - \delta)K + I$$

The Firm

Take the dynamic firm's problem. *i.e.*, the firm chooses labor and investment to maximize its present-value profits.

$$\pi(K) = \max_{I, N} \left\{ Y - wN - I + \frac{\pi(K')}{1+r} \right\}$$

Subject to:

$$Y = zF(K, N)$$

and

$$K' = (1 - \delta)K + I$$

Steady State

- ▶ In Solow population always grows, use the notion of per capita **steady state**. Let lower case letters denote **per-capita quantities**

$$\frac{K'}{N'} = \frac{K}{N} = k^* \quad \rightarrow \quad k' = k = k^*$$

$$\frac{Y'}{N'} = \frac{Y}{N} = y^* \quad \rightarrow \quad y' = y = y^*$$

$$\frac{F(K, N)}{N} = F\left(\frac{K}{N}, \frac{N}{N}\right) = F\left(\frac{K}{N}, 1\right) = f(k)$$

Solving For the Steady State

- ▶ Our goal is to derive a condition for the equilibrium value of k .
This is important to understand questions like:
 - ▶ How to foster growth?
 - ▶ Why is the saving rate important?
 - ▶ Is a capital tax good or bad?
- ▶ Algebra ahead: take a deep breath!!

Solving For the Steady State

Start:

$$K' = (1 - \delta)K + I \rightarrow I = K' - (1 - \delta)K$$

Substitute I from market clearing for goods and assets:

$$Y = C + K' - (1 - \delta)K$$

Substitute rule of thumb for saving and production function:

$$Y = (1 - s)Y + K' - (1 - \delta)K \rightarrow K' = szF(K, N) + (1 - \delta)K$$

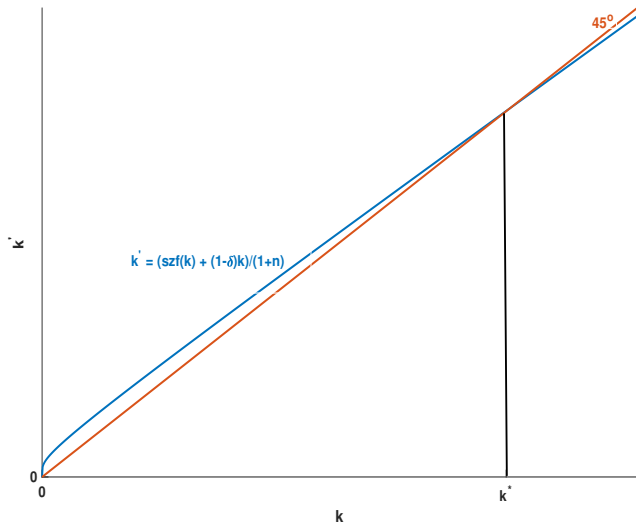
Divide by N ,

$$\frac{K'}{N} \frac{N'}{N'} = \frac{szF(K, N)}{N} + \frac{(1 - \delta)K}{N}$$

Recall $N' = (1 + n)N$,

$$\begin{aligned} k'(1 + n) &= szf(k) + (1 - \delta)k \\ \Rightarrow k' &= \frac{szf(k) + (1 - \delta)k}{1 + n} \end{aligned}$$

Capital Accumulation



Solving For the Steady State

Start:

$$K' = (1 - \delta)K + I \rightarrow I = K' - (1 - \delta)K$$

Substitute I from market clearing for goods and assets:

$$Y = C + K' - (1 - \delta)K$$

Substitute rule of thumb for saving and production function:

$$Y = (1 - s)Y + K' - (1 - \delta)K \rightarrow K' = szF(K, N) + (1 - \delta)K$$

Divide by N

$$\frac{K'}{N} \frac{N'}{N'} = \frac{szF(K, N)}{N} + \frac{(1 - \delta)K}{N}$$

Recall $N' = (1 + n)N$

$$k'(1 + n) = szf(k) + (1 - \delta)k$$

$$\Rightarrow k' = \frac{szf(k) + (1 - \delta)k}{1 + n}$$

Since in steady state $k = k'$

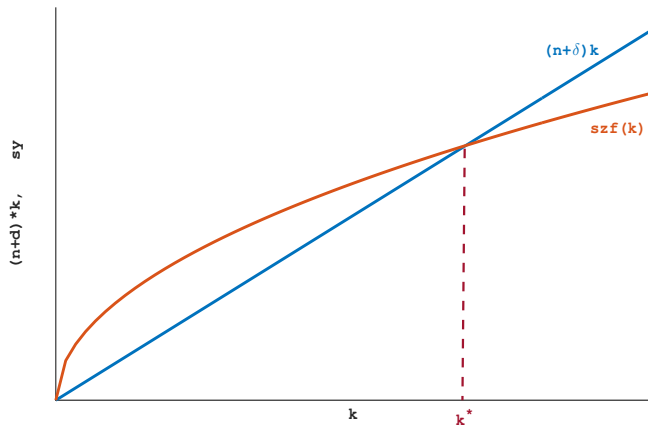
$$szf(k) = (n + \delta)k$$

Analysis of the Steady State

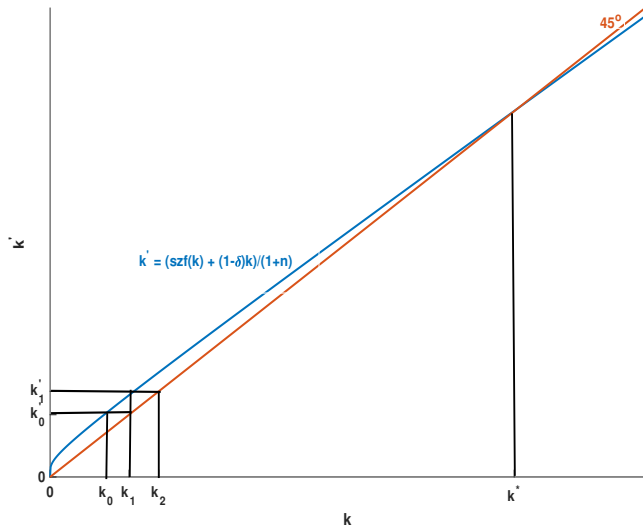
Key equilibrium condition:

$$szf(k^*) = (n + \delta)k^*$$

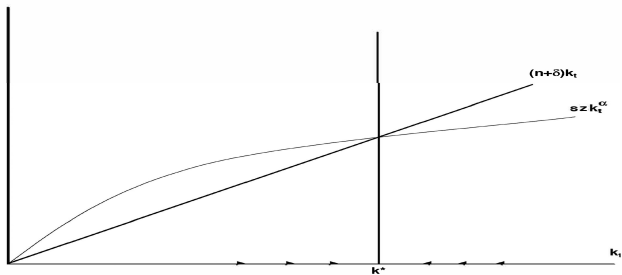
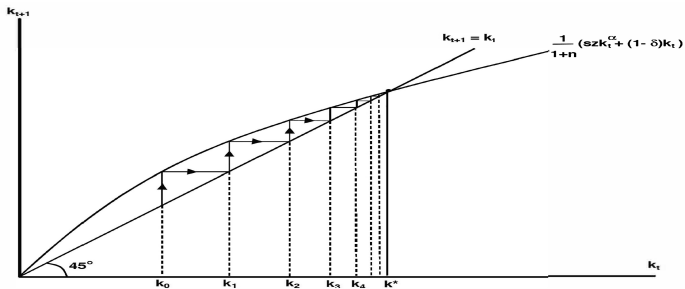
To understand it, graph both sides of the equation:



Analysis of the Steady State: Convergence



Analysis of the Steady State: Convergence

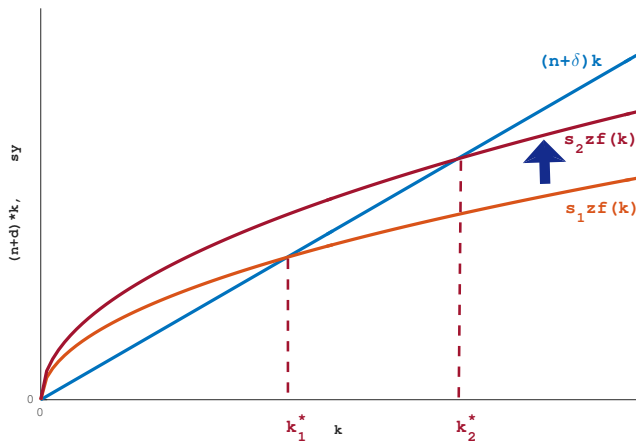


Analysis of the Steady State

What happens if s , n , d , z changes?

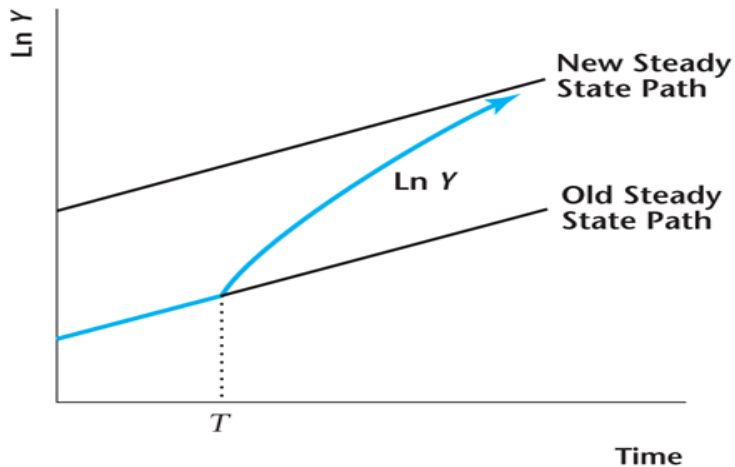
Changing the Saving Rate

Suppose s changes: s_1 to s_2 and $s_2 > s_1$.



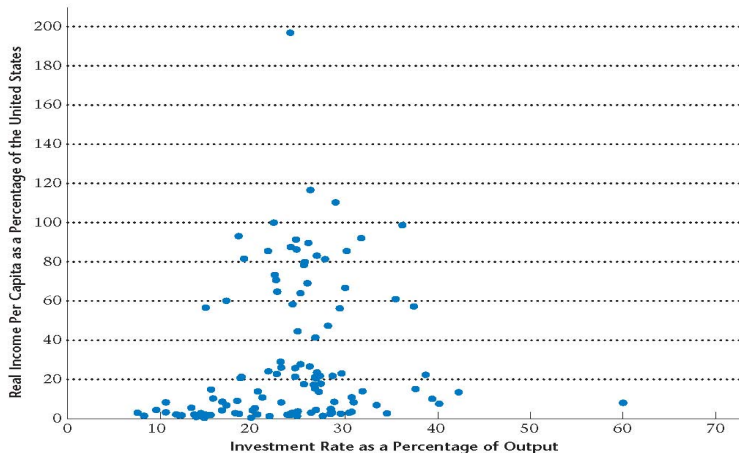
Steady state: k_1 increases to k_2 . Question: what happens to y ?

Changing the Saving Rate



Changing the Saving Rate

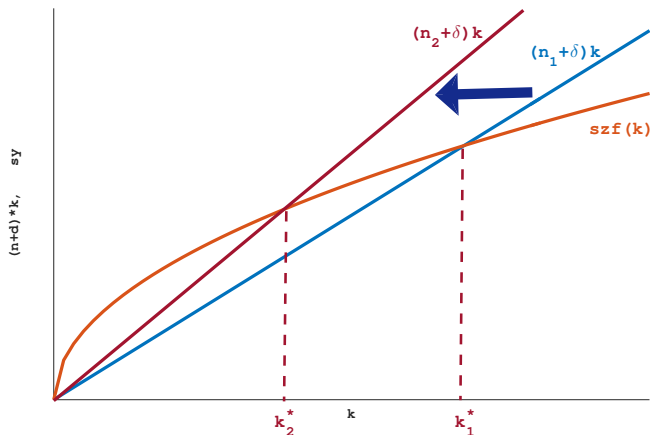
Recall: investment and GDP level



Investment is *positively* correlative with GDP levels.

Changing the Population Growth Rate

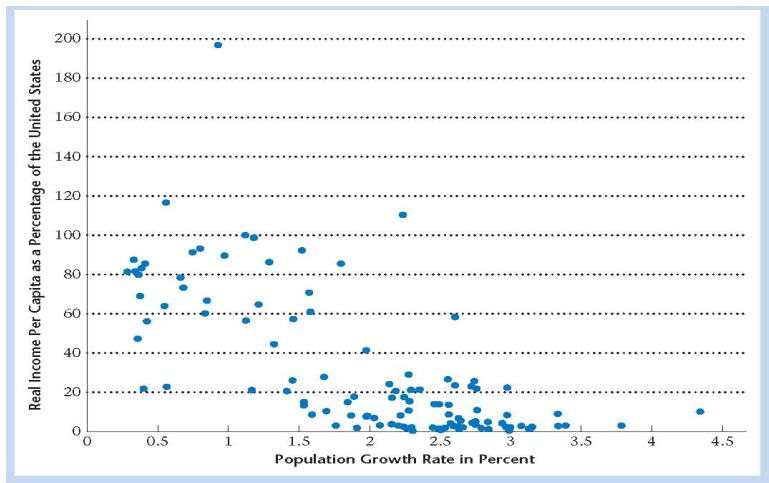
Suppose n changes: n_1 to n_2 and $n_2 > n_1$.



Steady state: k_1 decreases to k_2 .

Changing the Population Growth Rate

Recall: population and GDP level



Population is *negatively* correlative with GDP levels.

Changing δ and z

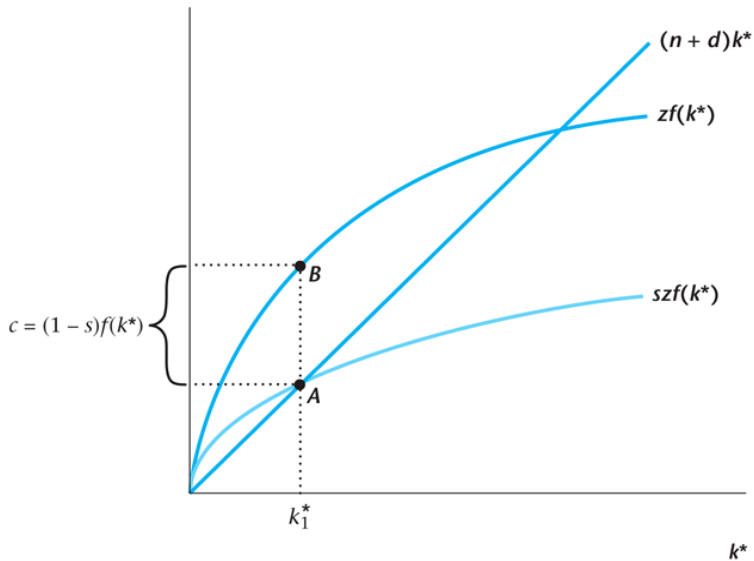
- ▶ Change δ is the same as change n .
- ▶ Change z is the same as change s .

How to Determine the Optimal Saving Rate?

From Perato Optimal:

- ▶ We need to determine the saving rate that maximizes a household's lifetime utility.
- ▶ In this model, households only care about consumption. i.e., we need to find a s to maximize lifetime utility.
 $\Rightarrow$ the golden rule.
- ▶ First, find the steady-state consumption per worker.

Steady State Consumption per Worker

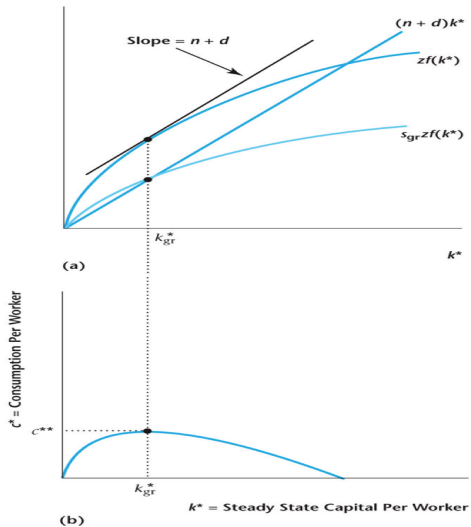


How to Determine the Optimal Saving Rate?

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 $\Rightarrow$ the golden rule.
- ▶ First, find the steady-state consumption per worker.
- ▶ Now, we can find the optimal s , saving rate, to maximize lifetime consumption.

The Golden Rule Quantity of Capital per Worker



Mathematically Determine the Optimal Saving Rate

- ▶ From PO, we need to determine the saving rate that maximizes a household's lifetime utility.
- ▶ In this model, households only care about consumption. i.e., we need to find a s to maximize lifetime utility.
 $\Rightarrow$ the golden rule.
- ▶ S-S consumption:
$$\max_s c^* = (1 - s)zf(k) = zf(k) - szf(k) = zf(k) - (n + \delta)k$$
$$\rightarrow \text{FOC} : \frac{d(zf(k))}{dk} - (n + \delta) = 0 \text{ implies } \frac{d(zf(k))}{dk} = n + \delta$$
- ▶ Intuition: when $MP_k = n + \delta$, household maximizes her lifetime utility. Now we can solve for S-S k^* by solving $MP_k = n + \delta$.
Recall: when we solve for golden-rule k^* , we do **not** use s , the saving rate.
- ▶ Once we know k^* , we know the optimal c^* . Now we can solve for the optimal $s^* = 1 - \frac{c^*}{zf(k^*)}$.
- ▶ Example: Assume $z = 1$ and $F(K, N) = K^\alpha N^{1-\alpha}$, solve for the optimal saving rate s on Solow growth model.

Growth Rates

- ▶ Question 1: how are consumption, output and capital growth over time?
- ▶ Question 2: how are consumption, output, and capital **per capita** growing over time?

Growth Rates

What is the growth rate of K in a steady state?

$$k' = k \quad \rightarrow \quad \frac{K'}{N'} = \frac{K}{N}$$

so that

$$K' = \frac{K}{N} N' = (1 + n)K$$

the growth rate is the population growth rate!

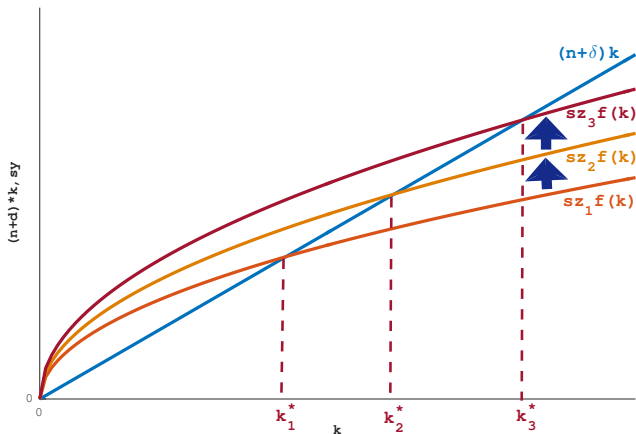
the growth rate is the population growth rate for C and Y . Why?

Growth Rates

- ▶ Question 1: how are consumption, output and capital growth over time?
 Y, C, K grow at rate n , the population growth rate.
- ▶ Question 2: how are consumption, output and capital **per capita** growing over time?
Q1 implies that K/N is constant over time

Sustaining Growth

To sustain growth over time we need something other than s and n : z !



Increasing z generates long-term growth!

The Solow Residual

- ▶ We have identified z as a key source for growth. z is sometimes called the **Solow Residual**.
- ▶ Key question: what is z in the data, and how do we calculate it?
- ▶ From equilibrium condition:

$$Y = zF(K, N)$$

hence

$$z = \frac{F(K, N)}{Y}$$

- ▶ We need information on GDP, easy to get; workers, easy to get too; production function, hard but assume Cobb-Douglas; capital, hard.

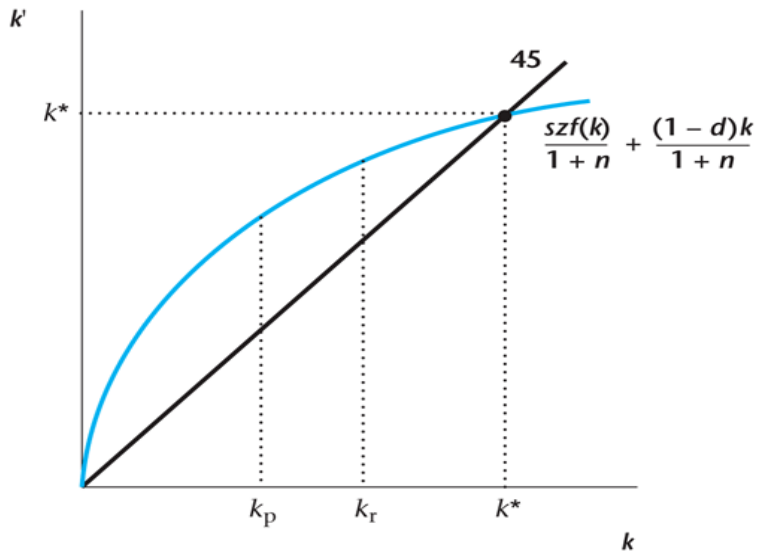
Cross Country Convergence

- ▶ Question: If the world was described as a Solow model, what would happen eventually to identical countries with different levels of GDP per capita y_{poor} and y_{rich} today?
- ▶ If today $y_{poor} < y_{rich}$ it implies $k_{poor} < k_{rich}$, what happens in steady state?
- ▶ Key equation:

$$sf(k) = (n + \delta)k$$

- ▶ In the transition, what happens?

Cross Country Convergence



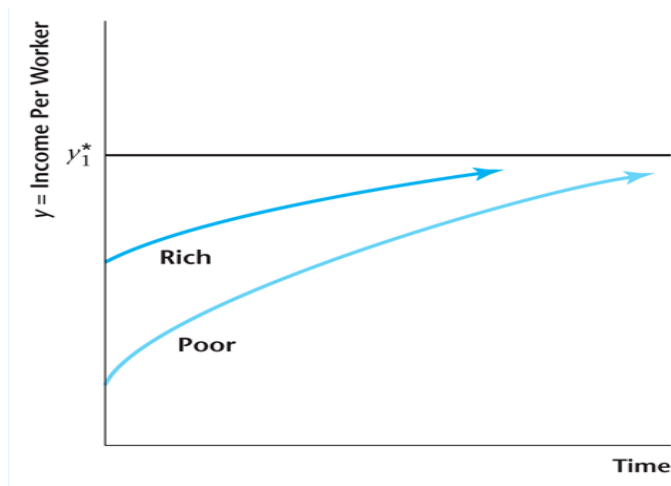
Cross Country Convergence

Why do poor countries grow faster than rich countries in the Solow growth model?

- ▶ The model predicts the poor country accumulates capital faster than the rich country.
- ▶ The poor country has a higher marginal product of capital than the rich country.

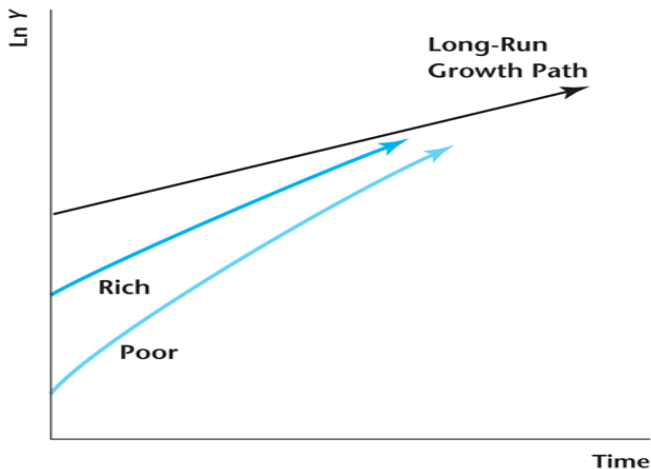
Cross Country Convergence

Answer: In the long run, they will have the same level of *GDP per capita*



Cross Country Convergence

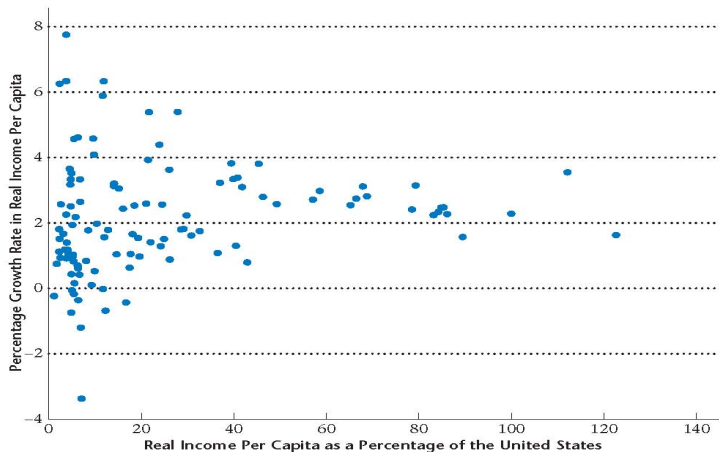
Answer: In the long run, they will have the same level of *GDP*



Is this happening?

Cross Country Convergence

Levels of GDP are not correlated with GDP growth rates.

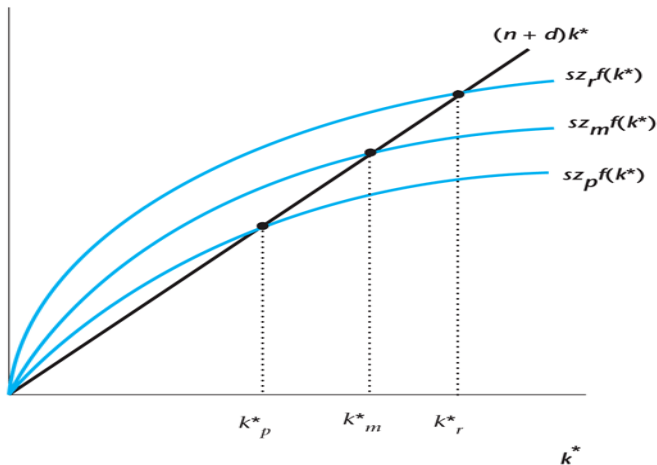


Convergence: An Explanation

What is *wrong* with the model?

- ▶ Maybe all the countries are not the same
- ▶ Barriers to investment
- ▶ Barriers to technology adoption

Cross Country Convergence



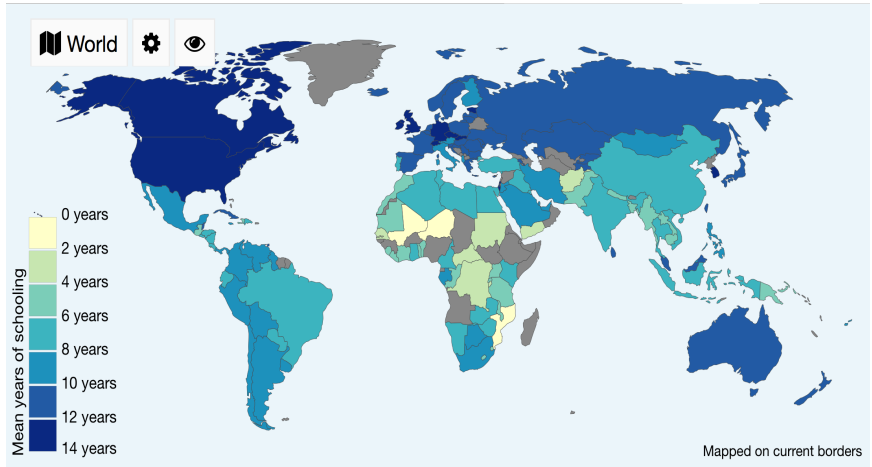
Differences in total factor productivity can explain the disparity in income per worker across countries.

Endogenous Growth

- ▶ Solow explains why we have growth: it's either z or n
- ▶ Does not tell us what to do to improve long-run growth: *i.e.* how does z go up?
- ▶ we need to go further and introduce human capital

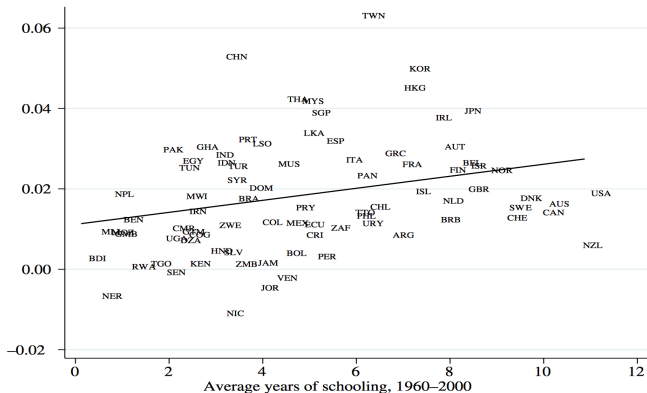
Years of Schooling

Mean Years of Schooling, 2010



Years of Schooling

Average growth rate of GDP per capita, 1960–2000



Endogenous Growth: Human Capital

Human capital: the stock of skills and education that workers have at a point in time

Properties:

- ▶ It grows
- ▶ It does not depreciate
- ▶ Technologies using it do not exhibit decreasing returns

A Simple Model: The Consumer

Suppose:

- ▶ There is no leisure: Workers divide their time between work and human capital accumulation. Let N denote time at work. $(1 - N)$ time at school.
- ▶ Human capital H^s increases effective time at work: more human capital = more output for the same amount of hours.
- ▶ There is no capital.

A Simple Model: The Consumer

Suppose:

- ▶ Budget constraint:

$$C = w \times N \times H$$

where $N \cdot H$ is the effective unit of labor

- ▶ Human capital accumulation

$$H' = b(1 - N)H$$

b is the efficiency of human capital accumulation (quality of schools) $1 - N$ is the intensity of human capital accumulation

A Simple Model: The Firm

Suppose:

- ▶ Firm maximize profits choosing effective labor $N \cdot H$

$$\max_{N \cdot H} zN \cdot H - wN \cdot H$$

Note: we assume linear firm technology

- ▶ Since the problem of the firm is linear:
 - ▶ Equilibrium wage is z .
 - ▶ Firm is indifferent in choice of $N \cdot H$.

A Simple Model: The Equilibrium

Let's determine the consumption growth rate.

- From budget constraint and firm maximization problem:

$$C = zN \cdot H$$

and

$$H' = b(1 - N)H \Rightarrow \frac{H'}{H} - 1 = b(1 - N) - 1$$

Note: we assume linear firm technology

- Computing growth rates of consumption:

$$\frac{C'}{C} - 1 = \frac{zNH'}{zNH} - 1 = \frac{H'}{H} - 1 = b(1 - N) - 1$$

A Simple Model: Summary

- ▶ Economy grows indefinitely because of human capital accumulation.
- ▶ Rate of growth determined by efficiency and intensity of human capital accumulation.

A Simple Model: Policy

What are good policies?

- ▶ Increase schooling
- ▶ Increase efficiency of schooling
- ▶ [World Bank](#)