

Name: _____ Student ID: _____ Seat No: _____

MA2001 Multi-variable Calculus and Linear Algebra

Semester A, 2023-24

Test (CA1)

SOLUTION

Date: Friday, 3 November 2023

Time allowed: 90 minutes (12:10 – 13:40)

Instructions to candidates:

This paper has 2 pages (including this cover page).

This paper has SEVEN questions.

The questions will begin on page 2.

Write your answers in the space provided under each question.

Answer ALL questions. Show ALL steps to get full mark.

Question	Max. mark	Your mark
Q1	25	
Q2	10	
Q3	10	
Q4	10	
Q5	20	
Q6	10	
Q7	15	
Total mark	100	

Question 1 [25 marks]

Given the matrix $A = \begin{pmatrix} 4 & 1 & 1 \\ -2 & 1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) Find the eigenvalues and the corresponding eigenvectors of A .

[17 marks]

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4-\lambda & 1 & 1 \\ -2 & 1-\lambda & -2 \\ 0 & 0 & 3-\lambda \end{vmatrix} \\ &= (3-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ -2 & 1-\lambda \end{vmatrix} \\ &= (3-\lambda) (\lambda^2 - 5\lambda + 6) \\ &= -(\lambda-3)^2 (\lambda-2) \end{aligned}$$

Set $|A - \lambda I| = 0 \Rightarrow \lambda = 2, 3, 3$ are the eigenvalues of A .

For $\lambda = 2$: $(A - 2I)\vec{x} = \vec{0}$

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ -2 & -1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow[R_3+R_2]{R_1-R_2} \left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\therefore \begin{cases} 2x_1 + x_2 = 0 \\ -x_3 = 0 \end{cases}$$

Let $x_1 = s$. Then $x_3 = 0$, $x_2 = -2s$.

$\therefore$ Eigenvectors corresponding to $\lambda = 2$ are

$$\vec{x} = s \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \quad s \in \mathbb{R}.$$

For $\lambda = 3$: $(A - 3I)\vec{x} = \vec{0}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ -2 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow{R_2+2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

$$\therefore x_1 + x_2 + x_3 = 0$$

Let $x_2 = s$, $x_3 = t$. Then $x_1 = -s - t$.

$\therefore$ Eigenvectors corresponding to $\lambda=3$ are

$$\vec{x} = s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad s^2 + t^2 \neq 0.$$

(b) Is A diagonalizable? If yes, express A in the form PDP^{-1} , where D is a diagonal matrix.

Clearly state the matrices P , D and P^{-1} in your answer.

[8 marks]

Since A has 3 linearly independent eigenvectors,

A is diagonalizable.

$$\text{Let } P = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

$$(P|I) \rightarrow \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_2+2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\xrightarrow[R_2+2R_3]{R_1+R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1-R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & -1 & 0 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{R_2 \times (-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & -1 & -1 \\ 0 & 1 & 0 & -2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\therefore P^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ -2 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore A = PDP^{-1} = \begin{pmatrix} 1 & -1 & -1 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} -1 & -1 & -1 \\ -2 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Question 2 [10 marks]

Determine whether the quadratic form

$$q = -2x_1^2 - 3x_2^2 - 2x_3^2 + 2x_1x_2 + 4x_2x_3$$

is positive definite, negative definite, positive semidefinite, negative semidefinite or indefinite.

$$q = (x_1 \ x_2 \ x_3) \begin{pmatrix} -2 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad [10 \text{ marks}]$$

The associated symmetric matrix is $A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{pmatrix}$.

Leading principal minors are

$$\text{1st order : } -2 < 0$$

$$\text{2nd order : } \begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} = 5 > 0$$

$$\begin{aligned} \text{3rd order : } \begin{vmatrix} -2 & 1 & 0 \\ 1 & -3 & 2 \\ 0 & 2 & -2 \end{vmatrix} &= -2 \begin{vmatrix} -3 & 2 \\ 2 & -2 \end{vmatrix} - 1 \begin{vmatrix} 1 & 2 \\ 0 & -2 \end{vmatrix} \\ &= -2(6-4) - (-2) \\ &= -2 < 0 \end{aligned}$$

$\therefore q$ is negative definite.

Question 3 [10 marks]

Does the limit

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{3x^2 + y^2}$$

exist? If yes, find the value of the limit. If no, show that the limit does not exist.

[10 marks]

As $(x,y) \rightarrow (0,0)$ along the x -axis ($y=0$),

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{3x^2 + y^2} = \lim_{x \rightarrow 0} \frac{0}{3x^2} = \lim_{x \rightarrow 0} 0 = 0$$

As $(x,y) \rightarrow (0,0)$ along the line $y=x$,

$$\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{3x^2 + y^2} = \lim_{x \rightarrow 0} \frac{2x^2}{4x^2} = \lim_{x \rightarrow 0} \frac{1}{2} = \frac{1}{2}$$

Since $\frac{2xy}{3x^2 + y^2}$ has different limits as $(x,y) \rightarrow (0,0)$

along different paths, the limit $\lim_{(x,y) \rightarrow (0,0)} \frac{2xy}{3x^2 + y^2}$

does not exist.

Question 4 [10 marks]

Suppose that $xy + xz^3 = 2yz$ defines z as an implicit function of x and y locally at $(x, y, z) = (-4, 2, -1)$. Compute the values of $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ at $(x, y, z) = (-4, 2, -1)$.

Partially differentiate both sides of $xy + xz^3 = 2yz$ [10 marks]
w.r.t. y :

$$x + 3xz^2 \frac{\partial z}{\partial y} = 2z + 2y \frac{\partial z}{\partial y} \quad \text{---} (*)$$

$$\Rightarrow (3xz^2 - 2y) \frac{\partial z}{\partial y} = 2z - x$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{2z - x}{3xz^2 - 2y}$$

$$\text{At } (-4, 2, -1), \quad \left. \frac{\partial z}{\partial y} \right|_{(-4, 2, -1)} = \frac{2(-1) - (-4)}{3(-4)(-1)^2 - 2(2)} = \frac{2}{-16} = -\frac{1}{8}$$

Partially differentiate both sides of $(*)$ w.r.t. y :

$$6xz \left(\frac{\partial z}{\partial y} \right)^2 + 3xz^2 \frac{\partial^2 z}{\partial y^2} = 2 \frac{\partial z}{\partial y} + 2 \frac{\partial z}{\partial y} + 2y \frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow (3xz^2 - 2y) \frac{\partial^2 z}{\partial y^2} = 4 \frac{\partial z}{\partial y} - 6xz \left(\frac{\partial z}{\partial y} \right)^2$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = \frac{4 \frac{\partial z}{\partial y} - 6xz \left(\frac{\partial z}{\partial y} \right)^2}{3xz^2 - 2y}$$

$$\text{At } (-4, 2, -1), \quad \left. \frac{\partial^2 z}{\partial y^2} \right|_{(-4, 2, -1)} = \frac{4 \left(-\frac{1}{8} \right) - 6(-4)(-1) \left(-\frac{1}{8} \right)^2}{3(-4)(-1)^2 - 2(2)} = \frac{-\frac{7}{8}}{-16} = \frac{7}{128}$$

Question 5 [20 marks]

Find all stationary points of the function $g(x, y) = x^3 - 3xy^2 + 4y^3 - 3x$ and determine their nature.

[20 marks]

$$g_x = 3x^2 - 3y^2 - 3$$

$$g_y = -6xy + 12y^2$$

$$\text{Set } \begin{cases} g_x = 0 \\ g_y = 0 \end{cases} \Rightarrow \begin{cases} 3x^2 - 3y^2 = 3 \\ 6y(2y - x) = 0 \end{cases} \Rightarrow \begin{cases} y = 0 \text{ or } x = 2y \\ x^2 - y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} y = 0 \\ x^2 = 1 \end{cases} \text{ or } \begin{cases} x = 2y \\ 3y^2 = 1 \end{cases}$$

$$\Rightarrow \begin{cases} y = 0 \\ x = \pm 1 \end{cases} \text{ or } \begin{cases} y = \pm \frac{1}{\sqrt{3}} \\ x = 2y \end{cases}$$

$\therefore (-1, 0), (1, 0), (-\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}), (\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}})$ are stationary points.

$$g_{xx} = 6x$$

$$g_{yy} = -6x + 24y$$

$$g_{xy} = -6y$$

Stationary point (x_0, y_0)	$g_{xx}(x_0, y_0)$	$g_{yy}(x_0, y_0)$	$g_{xy}(x_0, y_0)$	$g_{xx}(x_0, y_0)g_{yy}(x_0, y_0) - g_{xy}^2(x_0, y_0)$	Nature
$(-1, 0)$	-6	6	0	$-36 < 0$	Saddle point
$(1, 0)$	6	-6	0	$-36 < 0$	Saddle point
$(-\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$	$-\frac{12}{\sqrt{3}} (< 0)$	$-\frac{12}{\sqrt{3}}$	$\frac{6}{\sqrt{3}}$	$36 > 0$	local maximum
$(\frac{2}{\sqrt{3}}, \frac{1}{\sqrt{3}})$	$\frac{12}{\sqrt{3}} (> 0)$	$\frac{12}{\sqrt{3}}$	$-\frac{6}{\sqrt{3}}$	$36 > 0$	local minimum

Question 6 [10 marks]

Find the directional derivative of $f(x,y,z) = 3x^2 + 4y + xy^2z^3$ at $(1,-1,2)$ in the direction of $2\vec{i} + \vec{j} - 2\vec{k}$. [10 marks]

$$\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix} = \begin{pmatrix} 6x + y^2z^3 \\ 4 + 2xyz^3 \\ 3xy^2z^2 \end{pmatrix}$$

$$\nabla f(1,-1,2) = \begin{pmatrix} 6+8 \\ 4-16 \\ 3 \times 4 \end{pmatrix} = \begin{pmatrix} 14 \\ -12 \\ 12 \end{pmatrix}$$

Unit vector in the direction of $2\vec{i} + \vec{j} - 2\vec{k}$ is

$$\vec{u} = \frac{2\vec{i} + \vec{j} - 2\vec{k}}{3} = \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

Directional derivative of f at $(1,-1,2)$ in the direction of $\vec{u}$:

$$D_{\vec{u}}f(1,-1,2) = \nabla f(1,-1,2) \cdot \vec{u}$$

$$= \begin{pmatrix} 14 \\ -12 \\ 12 \end{pmatrix} \cdot \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

$$= \frac{28 - 12 - 24}{3}$$

$$= -\frac{8}{3}$$

Question 7 [15 marks]

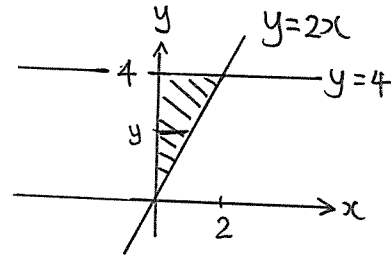
Evaluate the integral

$$\int_0^2 \left(\int_{2x}^4 e^{y^2} dy \right) dx$$

by changing the order of integration.

[15 marks]

$$D: \begin{cases} 0 \leq x \leq 2 \\ 2x \leq y \leq 4 \end{cases}$$



For every $0 \leq y \leq 4$,
 $0 \leq x \leq \frac{y}{2}$.

$$\begin{aligned} \therefore \int_0^2 \left(\int_{2x}^4 e^{y^2} dy \right) dx &= \int_0^4 \left(\int_0^{y/2} e^{y^2} dx \right) dy \\ &= \int_0^4 e^{y^2} [x]_0^{y/2} dy \\ &= \int_0^4 \frac{y}{2} e^{y^2} dy \\ &= \left[\frac{1}{4} e^{y^2} \right]_0^4 \\ &= \frac{1}{4} (e^{16} - 1) \end{aligned}$$

- End -