

City University of Hong Kong
Electrical Engineering
Semester A 2023-2024
EE2104 Introduction to Electromagnetics
Assignment 3

Deadline: 2 Nov 2023 at 1:00 pm Collection: YEUNG LT-4
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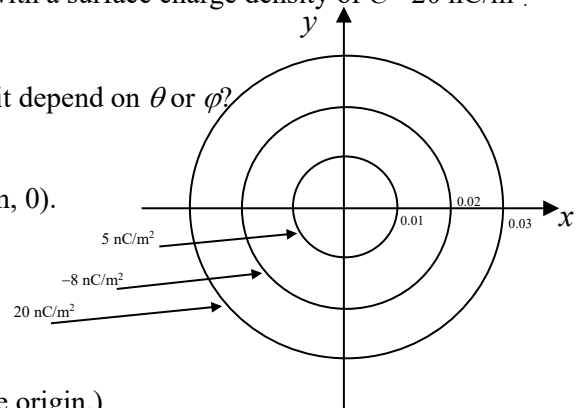
1. In the figure below, suppose that we have three concentric **spherical shells**. They are centered around the origin.

The innermost shell has a radius of $a = 0.01$ m with a surface charge density of $A = 5$ nC/m².

The middle shell has a radius of $b = 0.02$ m with a surface charge density of $B = -8$ nC/m².

The outermost shell has a radius of $c = 0.03$ m with a surface charge density of $C = 20$ nC/m².

- (a) What is the direction of $\mathbf{D}$?
- (b) Does the magnitude of $\mathbf{D}$ depend on r ? Does it depend on θ or ϕ ?
- (c) Derive $\mathbf{D}$ using Gauss's law.
- (d) Sketch the magnitude of $\mathbf{E}$ versus r .
- (e) Evaluate $\mathbf{E}$ at point $(x, y, z) = (0.02 \text{ m}, 0.02 \text{ m}, 0)$.



Solution:

- (a) $\mathbf{D}$ points along $\hat{\mathbf{r}}$ (which is the radial direction from the origin.)

- (b) By symmetry, $\mathbf{D}$ is dependent on r , independent of θ , and independent of ϕ .

- (c) To match the geometry of the structure, let us choose an imaginary sphere of radius r , centered at the origin as the gaussian surface S . Then, we have $\mathbf{D}(r, \theta, \phi) = D(r) \hat{\mathbf{a}}_\rho$.

From Gauss's law, we have $\oint_S \mathbf{D}(r, \theta, \phi) \cdot d\mathbf{S} = Q_{enc}$, where Q_{enc} is the enclosed charge within S .

$$\oint_S D(r) \hat{\mathbf{a}}_\rho \cdot (dS \hat{\mathbf{a}}_\rho) = Q_{enc}$$

$$\oint_S D(r) (\hat{\mathbf{a}}_\rho \cdot \hat{\mathbf{a}}_\rho) dS = Q_{enc}$$

$$\oint_S D(r) dS = Q_{enc}$$

$$D(r) \oint_S dS = Q_{enc}$$

$$D(r) (4\pi r^2) = Q_{enc}$$

$$\therefore D(r) = \frac{Q_{enc}}{4\pi r^2}$$

Therefore,

$$(1) \text{ for } 0 \leq r < 0.01, Q_{enc} = 0, D(r) = \frac{Q_{enc}}{4\pi r^2} = 0, E(r) = 0.$$

$$(2) \text{ for } 0.01 \leq r < 0.02, Q_{enc} = \rho_1 S_1 = 4\pi(0.01)^2 \rho_1 = 4\pi(0.01)^2(5 \times 10^{-9}) = 2\pi \times 10^{-12} \text{C}$$

So, we have

$$D(r) = \frac{Q_{enc}}{4\pi r^2} = \frac{2\pi \times 10^{-12}}{4\pi r^2} = \frac{5 \times 10^{-13}}{r^2} \text{ (C/m}^2\text{)}$$

$$E = \frac{5 \times 10^{-13}}{r^2 \epsilon_0} \text{ (V/m)}$$

$$(3) \text{ for } 0.02 \leq r < 0.03, Q_{enc} = \rho_1 S_1 + \rho_2 S_2 = 4\pi(0.01)^2(5 \times 10^{-9}) - 4\pi(0.02)^2(8 \times 10^{-9}) = -1.08\pi \times 10^{-11} \text{C}$$

So, we have

$$D(r) = \frac{Q_{enc}}{4\pi r^2} = \frac{-1.08\pi \times 10^{-11}}{4\pi r^2} = \frac{-2.7 \times 10^{-12}}{r^2} \text{ (C/m}^2\text{)}$$

$$E = \frac{-2.7 \times 10^{-12}}{r^2 \epsilon_0} \text{ (V/m)}$$

$$(4) \text{ for } r \geq 0.03, Q_{enc} = \rho_1 S_1 + \rho_2 S_2 + \rho_3 S_3 = 4\pi(0.01)^2(5 \times 10^{-9}) - 4\pi(0.02)^2(8 \times 10^{-9}) + 4\pi(0.03)^2(20 \times 10^{-9}) = 6.12\pi \times 10^{-11} \text{C}$$

So, we have

$$D(r) = \frac{Q_{enc}}{4\pi r^2} = \frac{6.12\pi \times 10^{-11}}{4\pi r^2} = \frac{1.53 \times 10^{-11}}{r^2} \text{ (C/m}^2\text{)}$$

$$E = \frac{1.53 \times 10^{-11}}{r^2 \epsilon_0} \text{ (V/m)}$$

In the end, we have

$$\mathbf{D}(r, \theta, \varphi) = 0, \quad \text{for } 0 \leq r < 0.01$$

$$\mathbf{D}(r, \theta, \varphi) = \frac{5 \times 10^{-13}}{r^2} \hat{\mathbf{a}}_r, \quad \text{for } 0.01 \leq r < 0.02$$

$$\mathbf{D}(r, \theta, \varphi) = \frac{-2.7 \times 10^{-12}}{r^2} \hat{\mathbf{a}}_r, \quad \text{for } 0.02 \leq r < 0.03$$

$$\mathbf{D}(r, \theta, \varphi) = \frac{1.53 \times 10^{-11}}{r^2} \hat{\mathbf{a}}_r, \quad \text{for } r \geq 0.03$$

(d) We have

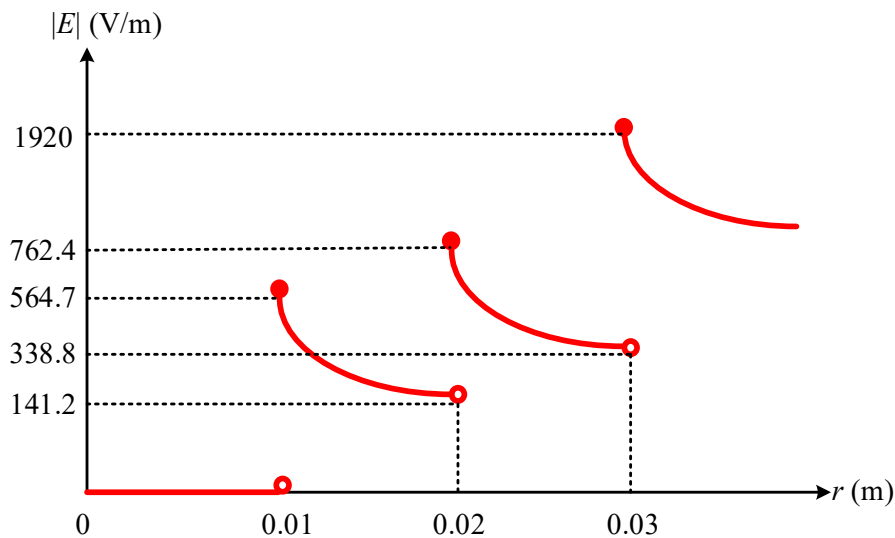
$$\mathbf{E}(r, \theta, \varphi) = 0, \quad \text{for } 0 \leq r < 0.01$$

$$\mathbf{E}(r, \theta, \varphi) = \frac{5 \times 10^{-13}}{r^2 \epsilon_0} \hat{\mathbf{a}}_r, \quad \text{for } 0.01 \leq r < 0.02$$

$$\mathbf{E}(r, \theta, \varphi) = \frac{-2.7 \times 10^{-12}}{r^2 \epsilon_0} \hat{\mathbf{a}}_r, \quad \text{for } 0.02 \leq r < 0.03$$

$$\mathbf{E}(r, \theta, \varphi) = \frac{1.53 \times 10^{-11}}{r^2 \epsilon_0} \hat{\mathbf{a}}_r, \quad \text{for } r \geq 0.03$$

The magnitude of the electric field $\mathbf{E}$ as a function of r can be plotted as:

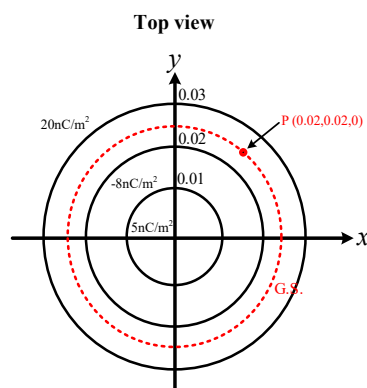


(e) The distance between point (0.02 m, 0.02 m, 0) and the origin is 0.028 mm ($0.02 < 0.028 < 0.3$). So, at

$$\text{point } (0.02 \text{ m}, 0.02 \text{ m}, 0), \mathbf{E}(r) = \frac{-2.7 \times 10^{-12}}{8 \times 10^{-4} \epsilon_0} = -381.36 \text{ V/m}$$

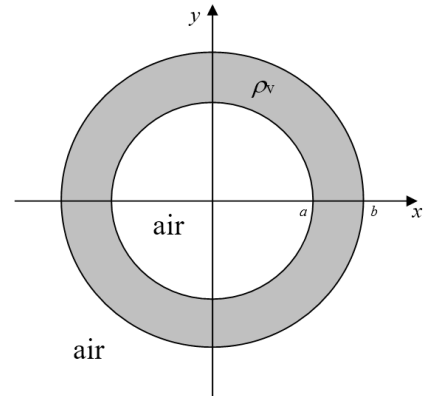
The direction of $\mathbf{E}(r)$ is parallel to the vector pointing from origin to P .

$$\mathbf{E} \text{ at point } (0.02 \text{ m}, 0.02 \text{ m}, 0) \text{ is } (-269.66 \hat{\mathbf{a}}_x - 269.66 \hat{\mathbf{a}}_y) \text{ V/m}$$



2. Consider a hollow **cylindrical** structure with inner radius a and outer radius b . The figure below shows the cross-section of the structure. It is parallel to the z -axis. In the cylindrical coordinate system, the charge density of the structure for $a < \rho < b$ is $A(a/\rho)$ for some constant A in C/m^3 .

Find the electric field $\mathbf{E}$. Then, sketch its magnitude as a function of the radial coordinate ρ .



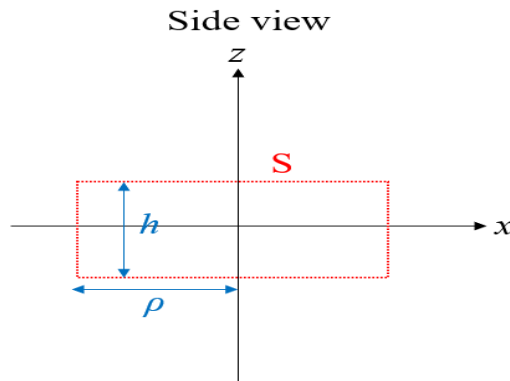
Solution:

(A) Based on the structure in the figure, we know that the charges are distributed symmetrically to the origin. So, $\mathbf{D}$ points along the radial direction and its magnitude only depends on the distance between the point and the origin. We can have $\mathbf{D}(\rho, \varphi, z) = \mathbf{D}(\rho) = D(\rho) \hat{\mathbf{a}}_\rho$.

(B) From Gauss's law, we have $\oint_S \mathbf{D}(\rho, \varphi, z) \cdot d\mathbf{S} = \oint_S \mathbf{D}(\rho) \cdot d\mathbf{S} = Q_{enc}$,

where Q_{enc} is the total charge enclosed by the Gaussian surface S .

As shown in the figure below, we can construct a Gaussian surface with a height of h and a radius of ρ .



Now

$$\oint_S \mathbf{D}(\rho) \cdot d\mathbf{S} = \int_{top} \mathbf{D}(\rho) \cdot d\mathbf{S} + \int_{bottom} \mathbf{D}(\rho) \cdot d\mathbf{S} + \int_{side} \mathbf{D}(\rho) \cdot d\mathbf{S}$$

Because $\mathbf{D}$ is along $\hat{\mathbf{a}}_\rho$, we have $\int_{top} \mathbf{D}(\rho) \cdot d\mathbf{S} = \int_{bottom} \mathbf{D}(\rho) \cdot d\mathbf{S} = 0$

As for the side surface, $\mathbf{D}$ is always parallel to $d\mathbf{S}$ and $|\mathbf{D}(\rho)| = D(\rho)$ is constant because ρ is constant. So

$$\oint_S \mathbf{D}(\rho) \cdot d\mathbf{S} = D(\rho) \int_{side} dS = 2\pi r h D(\rho) = 2\pi r h \epsilon_0 E(\rho) = Q_{enc}$$

$$\therefore E(\rho) = \frac{Q_{enc}}{2\pi \rho h \epsilon_0}$$

(C)

For $0 \leq \rho < a$, $Q_{enc} = 0$, $E(\rho) = \frac{Q_{enc}}{2\pi r h \epsilon_0} = 0$

For $a \leq \rho < b$,

$$\begin{aligned} Q_{enc} &= \int_a^\rho \int_0^{2\pi} \int_0^h \rho_v \rho \, d\rho \, d\varphi \, dz = \int_a^\rho \int_0^{2\pi} \int_0^h Aa \, d\rho \, d\varphi \, dz = Aa \int_a^\rho d\rho \int_0^{2\pi} d\varphi \int_0^h dz \\ &= Aa(\rho - a) \cdot 2\pi h = 2\pi Aah(\rho - a) \\ \therefore E(\rho) &= \frac{Q_{enc}}{2\pi \rho h \epsilon_0} = \frac{2\pi Aah(\rho - a)}{2\pi \rho h \epsilon_0} = \frac{Aa(\rho - a)}{\rho \epsilon_0} \end{aligned}$$

For $\rho \geq b$,

$$\begin{aligned} Q_{enc} &= \int_a^b \int_0^{2\pi} \int_0^h \rho_v \rho \, d\rho \, d\varphi \, dz = \int_a^b \int_0^{2\pi} \int_0^h Aa \, d\rho \, d\varphi \, dz = Aa \int_a^b d\rho \int_0^{2\pi} d\varphi \int_0^h dz \\ &= Aa(b - a) \cdot 2\pi h = 2\pi Aah(b - a) \\ \therefore E(\rho) &= \frac{Q_{enc}}{2\pi \rho h \epsilon_0} = \frac{2\pi Aah(b - a)}{2\pi \rho h \epsilon_0} = \frac{Aa(b - a)}{\rho \epsilon_0} \end{aligned}$$

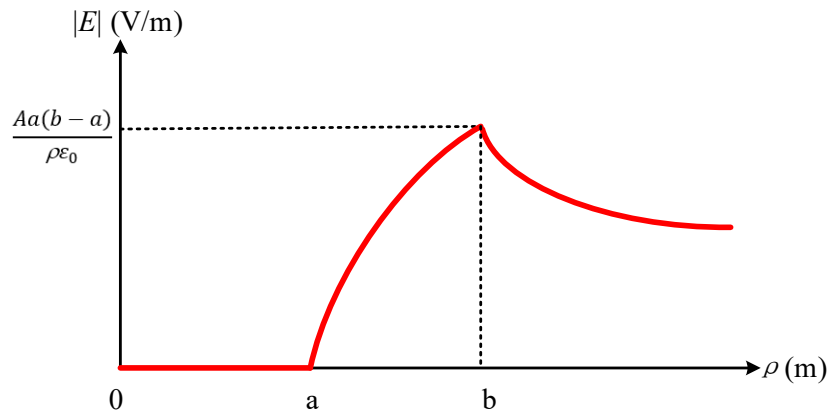
(D) Hence,

$$E(\rho, \theta, \varphi) = 0, \quad \text{for } 0 \leq \rho < a$$

$$E(\rho, \theta, \varphi) = \frac{Aa(\rho - a)}{\rho \epsilon_0} \hat{\mathbf{a}}_\rho, \quad \text{for } a \leq \rho < b$$

$$E(\rho, \theta, \varphi) = \frac{Aa(b - a)}{\rho \epsilon_0} \hat{\mathbf{a}}_\rho, \quad \text{for } \rho \geq b$$

The magnitude of the electric field E as a function of ρ can be plotted as:



3. In a dielectric material, the electric flux density is 1.7 times the polarization.
- (a) What is the electric susceptibility?
 - (b) What is the dielectric constant of the material?
 - (c) Name two different types of dielectric materials. Then, elaborate on their differences.

Solution:

- (a) Assuming that the dielectric constant (also called relative permittivity) of the material is ϵ_r , we have the relations as follows.

i: The electric flux density $\mathbf{D} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon_0 (\chi_e + 1) \mathbf{E}$

ii: The polarization $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$

Moreover, $\mathbf{D} = 1.7 \mathbf{P}$ is given.

Thus, $\epsilon_0 (\chi_e + 1) \mathbf{E} = 1.7 \epsilon_0 \chi_e \mathbf{E}$

$$\chi_e + 1 = 1.7 \chi_e$$

$$\chi_e = 1.43$$

The electric susceptibility is 1.43.

(b)

$$\epsilon_r = \chi_e + 1 = 2.43$$

The dielectric constant of the material is 2.43.

(c) Polar material and non-polar material.

Difference:

In a polar material, randomly oriented dipoles are aligned by an external electric field.

In a non-polar material, the dipoles are induced by an external electric field.

4. Consider a parallel-plate capacitor with a capacitance of $1 \mu\text{F}$. Its positive plate stores $3 \mu\text{C}$. Its negative plate stores $-3 \mu\text{C}$.
- (a) What is the potential difference between the two plates?
 - (b) How much electrical energy is stored by the capacitor?
 - (c) How much energy is needed to move $+0.3 \text{ pC}$ from the negative plate to the positive plate?
 - (d) Suppose the plates are separated by $500 \mu\text{m}$. What is the magnitude of the electric field?

Solution:

- (a) The potential difference between the two plates is

$$V = \frac{Q}{C} = \frac{3}{1} = 3 \text{ (V)}$$

- (b) The electrical energy stored by the capacitor is

$$W_e = \frac{1}{2} CV^2 = \frac{1}{2} \times 1 \times 10^{-6} \times 3^2 = 4.5 \times 10^{-6} \text{ (J)}$$

- (c) The energy needed to move $+0.3 \text{ pC}$ from the negative plate to the positive plate is

$$dW_e = dq \times V = 0.3 \times 10^{-12} \times 3 = 9 \times 10^{-13} \text{ (J)}$$

- (d) The magnitude of the electric field is

$$E = \frac{V}{d} = \frac{3}{5 \times 10^{-4}} = 6 \times 10^3 \text{ (V/m)}$$

5. In a copper wire, each copper atom contributes 3 electrons for conducting electricity. Its radius is 1 mm. When a current of 1 A is applied, how much time does it take for an electron to drift by 1 meter? Hint: There are about 8.5×10^{19} copper atom in every 1 mm^3 .

Solution:

The charge density in a copper wire is $\rho_v = \frac{3 \times 8.5 \times 10^{19} \times 10^9 \times 1.602 \times 10^{-19}}{1} = 40.851 \times 10^9 (\text{C/m}^3)$

The drift velocity $v_{drift} = \frac{J}{\rho_v} = \frac{I}{\rho_v S} = \frac{1}{40.85 \times 10^9 \pi (10^{-3})^2} = 7.79612 \times 10^{-6} \text{ m/s}$

To drift an electron by 1 meter, the cost time is $t = \frac{1}{7796.12 \times 10^{-6}} = 1.28 \times 10^5 \text{ s} = 1.78 \text{ days!}$

-Please staple the pages before submission-