

- Exam Time: 10:00 - 11:59, April 12, 2024
- Once you finish solving the exam, please submit your answer sheet to the TAs.
- Please make sure to write your name and student ID on top of the answer sheet.
- You don't need to return the question sheet.

1. (20 points) Solve the following Differential Equations (DE) using Matrix Exponential e^{At} .

(a)

$$x_1' = -2x_1 - 2x_2, \quad x_2' = -4x_1, \quad \text{where } x_1(0) = 1 \text{ and } x_2(0) = 0$$

(b)

$$x_1' = 7x_1 - 5x_2, \quad x_2' = 2x_1, \quad \text{where } x_1(0) = 1 \text{ and } x_2(0) = -1$$

2. (20 points) (a) Solve the following DEs using Variation of Parameters method.

$$\frac{d^2y}{dt^2} + 4 \cdot \frac{dy}{dt} + 4y = e^{-2t} \cdot \ln(t)$$

(b)

$$\frac{d^2y}{dt^2} - 2 \cdot \frac{dy}{dt} + y = \frac{1}{t} \cdot e^t, \quad t > 0$$

(c)

$$\frac{d^2x}{dt^2} - 3 \cdot \frac{dx}{dt} - 4x = 15 \cdot e^{4t} + 5 \cdot e^{-t}, \quad t > 0$$

3. (20 points) Please find the particular solution $x_p(t)$ for the following Mechanical Vibration System with two external forces

$$mx'' + cx' + kx = F_0 \cos(wt) + F_1 \cos(3wt), \quad \text{where } c > 0, \quad 2\beta \triangleq \frac{c}{m}, \quad w_0^2 \triangleq \frac{k}{m}$$

4. (20 points) Convert the following DEs into its matrix form $\frac{d\vec{X}}{dt} = \vec{A}\vec{X} + \vec{f}(t)$ and $\vec{X}(0) = \vec{X}_0$

(a)

$$\frac{d^2x}{dt^2} = -2 \cdot \frac{dx}{dt} - 5y + 3, \quad \frac{dy}{dt} = \frac{dx}{dt} + 2y, \quad \text{where } x(0) = 0, \quad x'(0) = 0, \quad y(0) = 1$$

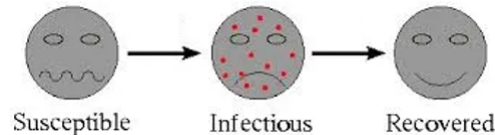
(Hint: Use $\vec{X} = [x_1(t), \quad x_2(t), \quad y_1(t)]^T$)

(b)

$$\frac{d^2y}{dt^2} + 5 \cdot \frac{dy}{dt} - 2t \cdot y = t^2 + 1, \quad \text{where } y(0) = 11, \quad y'(0) = 12$$

5. (20 points) Let us assume an country with population N has a disease spreading out. We will consider three classes; Susceptible class $s(t)$, Infected class $i(t)$ and Recovered class $r(t)$.

- Susceptible are all the individuals capable of becoming sick from an infection. Assuming that there is an infection that is capable of infecting everybody, so at the beginning everybody is going to be susceptible.
- As the individual in the population gets infected, it gets transitioned to the infected class.
- If the infected individual recovers from the infection, they fall into recovered class.



As we will consider a closed country, it is assumed that the three classes conform the entire population $s(t) + i(t) + r(t) = N$. In other words, no one can either enter or leave the country and we ignore any deaths that are not related to the disease. For simplicity, let us normalize the variables and use the fraction as follows.

- $S(t) = s(t)/N$ is the fraction of susceptible individuals
- $I(t) = i(t)/N$ is the fraction of infected individuals
- $R(t) = r(t)/N$ is the fraction of recovered individuals
- $1 = S(t) + I(t) + R(t)$ holds.

This model is known as SIR model and the variables are related through a system of DEs

$$\begin{aligned}\frac{dS}{dt} &= -\alpha \cdot S \cdot I, \\ \frac{dI}{dt} &= \alpha \cdot S \cdot I - \beta I, \\ \frac{dR}{dt} &= \beta I,\end{aligned}$$

where α and β are constants known as transmission rate and recovery rate, respectively. We will denote $R_0 = \frac{\alpha}{\beta}$ for simplicity.

- Use the given DEs to find I in terms of S , i.e., express $I = f(S)$ as a function of S .
- Similarly, find R in terms of S , i.e., express $R = g(S)$ as a function of S .
- At $t = 0$, we assume $I(0) = R(0) = 0$ and $S(0) = 1$. After a long time, we expect the fraction of infected individuals to approach zero $\lim_{t \rightarrow \infty} I(t) = I_\infty = 0$ and denote $\lim_{t \rightarrow \infty} S(t) = S_\infty$. Find the mathematical relationship between $R_0 = \frac{\alpha}{\beta}$ and S_∞ at $t \rightarrow \infty$ based on the prior results.