

EF2452: Mathematics for Economics and Finance

Lecture 2

Part I of II

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Main Topics

- Deriving and solving first-order recurrence equations of the form $y_t = ay_{t-1} + b$
- The time-independent solution, $y^* = \frac{b}{1-a}$, with $a \neq 1$.
- General solution, $y_t = y^* + (y_0 - y^*)a^t$
- The limiting behavior of a^t and the solution of the recurrence

Sequences and Recurrence Equations

- A sequence of numbers $y_0, y_1, y_2, \dots$ can be considered a description of how a variable quantity y evolves with respect to time.
- The general term y_t represents the value of y at the end of the t th time period.
- For example, y_t might represent the level of unemployment, income, exchange rate of a currency in terms of another currency, consumption, etc.
- y_t may depend on the value of y in the previous periods, y_{t-1}, y_{t-2} , etc.
- Generally, an equation which defines y_t in terms of y_{t-1}, y_{t-2} , and so on is known as a **recurrence equation**. It is also known as a **difference equation**.

First-Order Recurrence Equation

- A first-order recurrence is one in which y_t depends on y_{t-1} but no other previous values:

$$y_t = ay_{t-1} + b,$$

where a and b are given constants.

- We also say that the recurrence is linear and that it has constant coefficients.
- Our goal is to find a solution to this first-order or linear recurrence equation.

Solving a First-Order Recurrence Equation: STEP 1

$$y_t = ay_{t-1} + b$$

- The first step is to find a ***time-independent*** solution, that is, a value of y_t that does not depend on t .
- In other words, we try to find a constant y^* such that if $y_{t-1} = y^*$, then the recurrence equation ensures that y_t has the same value, y^* . This means

$$y^* = ay^* + b$$

Rearranging, we have:

$$(1 - a)y^* = b \quad \implies y^* = \frac{b}{1 - a}$$

Solving a First-Order Recurrence Equation: STEP 2

$$y_t = ay_{t-1} + b$$

- The second step is to write y_t as the sum of the constant y^* (time-independent solution) and another quantity z_t , that is,

$$y_t = y^* + z_t$$

- Substituting the expression for y_t and y_{t-1} into the first-order recurrence equation, we have

$$y^* + z_t = a(y^* + z_{t-1}) + b$$

- Note that y^* satisfies $y^* = ay^* + b$ (see last page). Then the above equation is reduced to

$$z_t = az_{t-1}$$

Solving a First-Order Recurrence Equation: STEP 3

$$z_t = az_{t-1}$$

- Now, given an initial value z_0 , we can write the above reduced equation as

$$z_t = az_{t-1} = a^2 z_{t-2} = a^3 z_{t-3} = \cdots = a^t z_0$$

- What is the expression for z_0 ? Use the definition (see last page):

$$y_0 = y^* + z_0 \quad \implies \quad z_0 = y_0 - y^*$$

- Use the definition of $z_t = y_t - y^*$ from last page and the above expression for z_0 and substitute into $z_t = a^t z_0$ to get

$$y_t - y^* = (y_0 - y^*)a^t \quad \implies \quad \boxed{y_t = y^* + (y_0 - y^*)a^t}$$

Solving a First-Order Recurrence Equation

- The general solution to the first-order recurrence equation $y_t = ay_{t-1} + b$ is given by

$$y_t = y^* + (y_0 - y^*)a^t$$

- Note that $y^* = \frac{b}{1-a}$, and we must have $a \neq 1$.
- What happens if $a = 1$? See under the “Special Cases” subsection later in the slides.

Solving a First-Order Recurrence Equation: An Example

$$y_t = 0.5y_{t-1} + 8000$$

$$y_0 = 32000$$

- Here, we have $a = 0.5$ and $b = 8000$. Therefore,

$$y^* = \frac{b}{1-a} = \frac{8000}{0.5} = 16000$$

- The general solution is

$$y_t = y^* + (y_0 - y^*)a^t$$

$$\implies y_t = 16000 + (32000 - 16000)(0.5)^t = 16000(1 + (0.5)^t)$$

Limits

$$y_t = y^* + (y_0 - y^*)a^t$$

- An important question to ask here: does y_t approach y^* as t increases?
- The behavior of y_t with increasing t depends on what happens to a^t . If $a^t \rightarrow 0$ as $t \rightarrow \infty$, then $y_t \rightarrow y^*$ as $t \rightarrow \infty$.
- For example, if $a = \frac{1}{2}$, then the sequence of values of a^t is

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \dots,$$

- If $a = 3$, then the sequence of values of a^t is

$$3, 9, 27, 81, 243, 729, 2187, \dots,$$

- The first sequence approaches zero as $t \rightarrow \infty$, but the second does not.

Limits: $a > 1$

$$y_t = y^* + (y_0 - y^*)a^t$$

- When $a > 1$, as t increases, a^t will eventually become greater than any given number, and we say that a^t tends to infinity as t tends to infinity:

$$a^t \rightarrow \infty \text{ as } t \rightarrow \infty \qquad \lim_{t \rightarrow \infty} a^t = \infty$$

- With $t \rightarrow \infty$, the behavior of y_t then is:

$$y_t \rightarrow \infty \text{ if } y_0 > y^*$$

$$y_t \rightarrow -\infty \text{ if } y_0 < y^*$$

Limits: $0 < a < 1$

$$y_t = y^* + (y_0 - y^*)a^t$$

- When $0 < a < 1$, as t increases, a^t will become smaller and smaller (the sequence is decreasing):

$$a^t \rightarrow 0 \text{ as } t \rightarrow \infty \qquad \lim_{t \rightarrow \infty} a^t = 0$$

- With $t \rightarrow \infty$, the behavior of y_t then is:

$$y_t \rightarrow y^*$$

Limits: $-1 < a < 0$

$$y_t = y^* + (y_0 - y^*)a^t$$

- When $-1 < a < 0$, the terms of the sequence are alternately positive and negative, and we say that the approach to zero is oscillatory.
- For example, when $a = -0.2$, the sequence of a^t is

$-0.2, 0.04, -0.008, 0.0016, -0.00032, 0.000064, -0.0000128, \dots$

$$a^t \rightarrow 0 \text{ as } t \rightarrow \infty$$

$$y_t \rightarrow y^* \text{ as } t \rightarrow \infty$$

Limits: $a < -1$

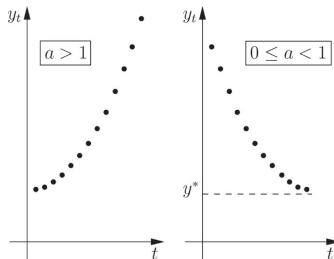
$$y_t = y^* + (y_0 - y^*)a^t$$

- When $a < -1$, the sequence is oscillatory, but it does not approach any limit, the terms being alternately large-positive and large-negative.
- For example, when $a = -2$, the sequence of a^t is

$$-2, 4, -8, 16, -32, 64, -128, 256, \dots$$

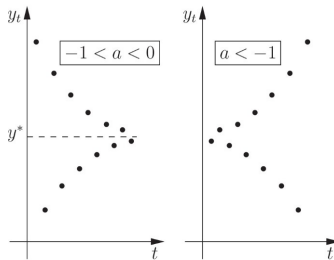
- In this case, a^t oscillates increasingly. y^t also oscillates increasingly.

The behavior of y_t (with $y_0 > y^*$)



(a)

(b)



(c)

(d)

Special Cases: $a = 1$

- When $a = 1$, our method of dealing with first-order recurrence equations breaks down completely because the expression $\frac{b}{1-a}$ for the time-independent solution is meaningless (undefined) if $a = 1$.
- However, we can solve the equation directly (and more easily)!
- With $a = 1$, we have

$$y_t = y_{t-1} + b$$

- Then, we have

$$y_1 = y_0 + b, \quad y_2 = y_1 + b = y_0 + 2b, \quad y_3 = y_2 + b = y_0 + 3b, \dots$$

Special Cases: $a = 1$

- The general solution then is: $y_t = y_0 + tb$.

If $b > 0, y_t \rightarrow \infty$

If $b < 0, y_t \rightarrow -\infty$

If $b = 0, y_t = y_0$ for all t

Special Cases: $a = -1$

- When $a = -1$,

$$y_t = -y_{t-1} + b$$

- The sequence of values is

$$y_0$$

$$y_1 = -y_0 + b$$

$$y_2 = -y_1 + b = -(-y_0 + b) + b = y_0$$

$$y_3 = -y_2 + b = -y_0 + b$$

$$y_4 = -y_3 + b = -(-y_0 + b) + b = y_0$$

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- Therefore, the values are y_0 and $-y_0 + b$ alternately, and the behavior is oscillatory. However, since the oscillations have constant magnitude, the sequence does not tend to a limit.

Example 1: Investment as a first-order recurrence

- Suppose that a closed economy produces an income Y_t in year t , of which a part C_t is consumed and the remainder I_t is invested; thus

$$Y_t = C_t + I_t$$

- Consumption C_t in year t is one half of the current year's income; that is,

$$C_t = \frac{1}{2}Y_t$$

- Current investment determines next year's income: next year's income is proportional to the current investment

$$Y_{t+1} = kI_t,$$

where k is a constant.

- We can show that investment I_t satisfies a first-order recurrence equation.

Example 1: Investment as a first-order recurrence

- We have the following three facts relating Y_t , C_t , and I_t :

$$Y_t = C_t + I_t, \quad C_t = \frac{1}{2}Y_t, \quad Y_{t+1} = kI_t$$

- Eliminating C_t from the first two equations, we get

$$Y_t = \frac{1}{2}Y_t + I_t$$

$$\implies I_t = \frac{1}{2}Y_t$$

- Replacing t by $t - 1$ in the third equation gives $Y_t = kI_{t-1}$.
Then,

$$I_t = \frac{1}{2}Y_t = \frac{1}{2}(kI_{t-1}) = \frac{k}{2}I_{t-1}$$

Example 1: Investment as a first-order recurrence

- The equation $I_t = \frac{k}{2}I_{t-1}$ is a first-order recurrence equation.
- To solve this, see that

$$I_1 = \frac{k}{2}I_0, \quad I_2 = \frac{k}{2}I_1 = \left(\frac{k}{2}\right)^2 I_0, \quad I_3 = \frac{k}{2}I_2 = \left(\frac{k}{2}\right)^3 I_0, \dots$$

- Therefore, we have

$$I_t = I_0 \left(\frac{k}{2}\right)^t$$

- We can also get the same solution by comparing it with the standard form of a first-order recurrence discussed earlier [recall in standard form: $y_t = ay_{t-1} + b$, the solution is $y_t = y^* + (y_0 - y^*)a^t$ where $y^* = \frac{b}{1-a}$. Here, $a = k/2$ and $b = 0$, so $y^* = 0$.]

Example 1: Investment as a first-order recurrence

$$I_t = I_0 \left(\frac{k}{2} \right)^t$$

- We can see that if $\frac{k}{2} > 1$ then I_t increases with t . So, if $k > 2$, investment rises from year to year. Since I_t determines Y_{t+1} , income also increases over time.

Example 2: A model of market disturbance

- A market is modeled by the demand and supply functions

$$q^D(p) = 4 - p \quad q^S(p) = p$$

- Solving for the equilibrium price and quantity, we obtain

$$4 - p^* = p^* \implies p^* = 2$$

- Substituting the value of $p^* = 2$ back into either the demand or supply function yields

$$q^* = 2$$

Example 2: A model of market disturbance

- Suppose that for some external reason, the market is disturbed and the actual price is reduced to $p_0 = 3/4$, so that demand exceeds supply.
- Assume that over a given time period (say a month), the resulting change in price is proportional to the excess demand:

$$p_t - p_{t-1} = c \left(q^D(p_{t-1}) - q^S(p_{t-1}) \right),$$

where c is a positive constant.

- We can derive a first-order recurrence equation for price. To do that, we first use the given expressions for q^D and q^S to get:

$$p_t - p_{t-1} = c \left((4 - p_{t-1}) - p_{t-1} \right) = c \left(4 - 2p_{t-1} \right)$$

Example 2: A model of market disturbance

$$\implies p_t = (1 - 2c)p_{t-1} + 4c$$

- This is a linear first-order recurrence in standard form, $y_t = ay_{t-1} + b$, with p_t in place of y_t , and $a = 1 - 2c$, $b = 4c$.
- The time-independent solution is

$$p^* = \frac{b}{1-a} = \frac{4c}{1-(1-2c)} = 2$$

- The general solution when $p_0 = 3/4$ is

$$p_t = p^* + (p_0 - p^*)a^t = 2 + \left(\frac{3}{4} - 2\right)(1 - 2c)^t$$

$$\implies p_t = 2 - \frac{5}{4}(1 - 2c)^t$$

Example 2: A model of market disturbance

$$p_t = 2 - \frac{5}{4}(1 - 2c)^t$$

- The behavior of p_t is determined by the behavior of $(1 - 2c)^t$, everything else in this expression being independent of t .
- $p_t \rightarrow 2$, the equilibrium price if and only if $(1 - 2c)^t \rightarrow 0$ as $t \rightarrow \infty$. This is the case when $-1 < a = 1 - 2c < 1$.
- Therefore, we need $c > 0$ and $c < 1$. It is already given that c is positive. Thus, we need $c < 1$ for $p_t \rightarrow 2$.

References

Following chapter of the textbook:

3 Sequences, Recurrences, Limits