

EE 3114 **Systems and Control**

10. State Space Design

Semester B, 2017-18

LTI State-Space Systems

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{y} = C\mathbf{x} + D\mathbf{u}$$

$\mathbf{x} \in \mathbb{R}^m$ is the state vector

$\mathbf{u} \in \mathbb{R}^n$ is the control vector

$\mathbf{y} \in \mathbb{R}^k$ is the output vector

$A_{m \times m}$ is the system matrix

$B_{m \times n}$ is the control matrix

$C_{k \times m}$ is the observation matrix

$D_{k \times n}$ is the input matrix (often, $D = 0$)

Now, we want to **design** the **controller** $\mathbf{u}(t)$ to achieve some goals, such as regulating the originally system so that the controlled system has desired poles (e.g., to stabilize an unstable system, or for better stability)

Note:

Only SISO systems are studied, but MIMO systems can be similarly handled

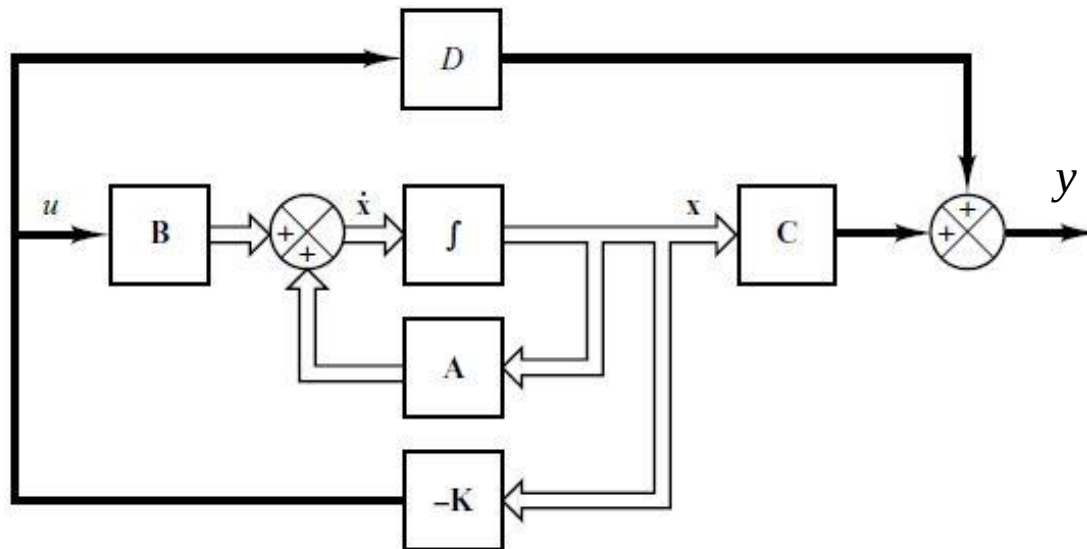
Regulation via State Feedback

Plant: $\dot{x} = Ax + Bu, \quad x(t_0) = x_0 \quad y = Cx + Du$

State-Feedback Controller: $u = -Kx$

Closed-Loop System:

$$\dot{x} = Ax + Bu = (A - BK)x$$



$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$y = Cx + Du$$

$$u = -Kx \quad \Rightarrow \quad \dot{x} = (A - BK)x$$

Idea:

$\dot{x} = ax + bu$ with $a > 0$ so the uncontrolled system is unstable

Design $u = -kx$ so as to have $\dot{x} = (a - bk)x$

Now, suppose we want the feedback-controlled system to have a pole -1

Then, $a - bk = -1 \rightarrow$ We can choose $k = \frac{a+1}{b}$

Negative state-feedback controller $u = -\frac{a+1}{b}x$ can guarantee the feedback-controlled system to be stable, with pole -1

$$\dot{x} = Ax + Bu, \quad x(t_0) = x_0$$

$$y = Cx + Du$$

$$u = -Kx \quad \longrightarrow \quad \dot{x} = (A - BK)x$$

$$\longrightarrow \quad X(s) = (A - BK)^{-1}x_0$$

$$\frac{Y(s)}{U(s)} = [C(A - BK)^{-1}B + D] \quad (\text{zero initial condition})$$

Theorem:

There exists a state feedback matrix K such that the closed-loop system has any desired poles $p_1, \dots, p_n$ if and only if (A, B) is controllable

Recall: State-Space Canonical Form

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ 0 & \vdots & \ddots & \vdots \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$u = -Kx = [-k_1 \quad -k_2 \quad \cdots \quad -k_n]x$$

$$A - BK = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -(a_1 + k_1) & -(a_2 + k_2) & \cdots & -(a_n + k_n) \end{bmatrix}$$

$$\det(sI - (A - BK)) = s^n + (a_n + k_n)s^{n-1} + \cdots + (a_1 + k_1)$$

$$\det(sI - (A - BK)) = s^n + (a_n + k_n)s^{n-1} + \dots + (a_1 + k_1)$$

Suppose the desired characteristic polynomial is

$$\prod_{i=1}^n (s - p_i) = s^n + a_{cn}s^{n-1} + \dots + a_{c1}$$

Comparing the coefficients of the above two formulas gives

$$k_n = a_{cn} - a_n$$

$$k_{n-1} = a_{cn-1} - a_{n-1}$$

$$\vdots$$

$$k_1 = a_{c1} - a_1$$

$$K = [a_{c1} - a_1 \quad a_{c2} - a_2 \quad \dots \quad a_{cn} - a_n]$$



Negative state-feedback controller $u = -Kx$

with the above gain matrix K will guarantee the controlled system to have the desired poles $p_1, \dots, p_n$

Example: (To stabilize an unstable system)

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\text{Poles: } \frac{1}{2} \pm \frac{\sqrt{13}}{2} \rightarrow \text{unstable}$$

We want the feedback-controlled system to have poles:

$$-1 \pm j2$$

Since the system is controllable (verify this!), it is possible to design

$$u = -K\mathbf{x} = -[k_1 \ k_2] \mathbf{x}$$

to achieve this goal of **stabilization**

(i.e., stabilizing an unstable system)

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

We want the feedback-controlled system to have poles:

$$p_{1,2} = -1 \pm j2$$

Formulas:

$$\det(sI - (A - BK)) = s^2 + (k_2 - 1)s + (k_1 - 3)$$

$$\prod_{i=1}^2 (s - p_i) = (s - 1 + j2)(s - 1 - j2) = s^2 + 2s + 5$$



Comparing coefficients $\rightarrow k_2 - 1 = 2$ and $k_1 - 3 = 5$

$$K = [k_1 \ k_2] = [8 \ 3]$$



$\dot{x} = (A - BK)x$ with desired poles: $-1 \pm j2$

Example: (to achieve better stability)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Although this system is stable (verify this!), we want the feedback-controlled system to have “more stable” poles:

$$s = -2 + j4, \quad s = -2 - j4, \quad s = -10$$

Since the system is controllable, it is possible to design

$$u = -K\mathbf{x}$$

to achieve this goal (of making the system more stable)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We want to design $u = -K\mathbf{x}$ such that the feedback-controlled system has poles: $s = -2 + j4$, $s = -2 - j4$, $s = -10$

Formulas:

$$\det(sI - (\mathbf{A} - \mathbf{B}K)) = s^3 + (6 + k_3)s^2 + (5 + k_2)s + (1 + k_1)$$

$$\prod_{i=1}^3 (s - \lambda_i) = s^3 + 14s^2 + 60s + 200$$

Comparing coefficients →



$$K = [a_{c1} - a_1 \quad a_{c2} - a_2 \quad a_{c3} - a_3] = [199 \quad 55 \quad 8]$$



$\dot{\mathbf{x}} = (\mathbf{A} - \mathbf{B}K)\mathbf{x}$ with desired poles:

$$s = -2 + j4, \quad s = -2 - j4, \quad s = -10$$

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We want to design $u = -K\mathbf{x}$ such that the feedback-controlled system has poles: $s = -2 + j4$, $s = -2 - j4$, $s = -10$

%MatLab code

```
A = [0 1 0; 0 0 1; -1 -5 -6];
```

```
B = [0; 0; 1];
```

```
J = [-2+j*4 -2-j*4 -10];
```

```
K = acker(A,B,J) %Or: K = place(A,B,J)
```

```
%
```

```
K =
```

```
199 55 8
```

Summary: SISO LTI systems

$$\dot{x} = Ax + Bu \quad x \in R^{n \times 1} \quad u \in R$$

$$y = Cx \quad y \in R \quad D = 0$$

Suppose that the system is controllable:

$$\text{State-feedback controller: } u = -Kx$$

$$\text{Closed-loop system: } \dot{x} = (A - BK)x$$

$$\text{Characteristic polynomial: } p(s) = \det(sI - A + BK)$$

$$\text{Desired poles: } p_1, \dots, p_n$$

$$\text{Desired characteristic polynomial: } (s - p_1) \cdots (s - p_n)$$

→ Choose gain matrix K such that

$$\det(sI - A + BK) = (s - p_1) \cdots (s - p_n)$$

Design Procedure for General Case

(non-canonical forms)

Obtain characteristic polynomial coefficients of the open-loop system:

$$p(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n$$

Form the controllability matrices $M = [B \ AB \ A^2 B \ \cdots \ A^{n-1} B]$ and

$$W = \begin{bmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-2} & \alpha_{n-1} \\ 0 & 1 & \alpha_1 & \alpha_2 & & \alpha_{n-2} \\ 0 & & 1 & \ddots & \ddots & \vdots \\ & 0 & & \ddots & \alpha_1 & \alpha_2 \\ & & & & 1 & \alpha_1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Find the coefficients of the desired closed-loop characteristic polynomial:

$$\begin{aligned} q(s) &= (s - p_1)(s - p_2) \cdots (s - p_n) \\ &= s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n \end{aligned}$$

Let $H = [a_1 - \alpha_1 \quad a_2 - \alpha_2 \quad \cdots \quad a_n - \alpha_n]$

Design the state-feedback gain by $K = H[MW]^{-1} \rightarrow u = -Kx$

State-Space Feedback Controller Design

To meet frequency-domain specifications

Given a state-space system in canonical form:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}\quad \begin{aligned}A &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ C &= [1 \quad 0 \quad 0]\end{aligned}$$

Design a state-feedback controller: $u = -Kx$

such that the closed-loop system: $\dot{x} = (A - BK)x$

satisfies:

- a) 2% settling time $< 0.5s$
- b) Overshoot of step response $< 10\%$
- c) Steady-state error $\rightarrow 0$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C = [1 \quad 0 \quad 0]$$

Transfer function: $G(s) = \frac{1}{s(s^2+3s+1)} = \frac{1}{s} \times (\text{standard 2nd-order})$

→ ζ and ω_n

Minimal realization: Yes (both controllable and observable)

Desired poles:

$$p_0 = \text{stable} \quad \text{and} \quad p_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \omega_n \quad (\text{standard 2nd-order})$$

To determine the desired poles to satisfy “overshoot” and “settling time” specs:

Recall:

$$\text{Overshoot (os)} = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \Rightarrow \zeta = \frac{-\ln(os)}{\sqrt{\pi^2 + \ln(os)^2}}$$

$$2\% \text{ Settling Time (Ts)} = \frac{-\ln(2\%)}{\zeta \omega_n}$$

Specs:

- a) 2% settling time < 0.5s
- b) Overshoot of step response < 10%
- c) Steady-state error $\rightarrow 0$

$$\text{a) Overshoot} < 0.1 = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$\rightarrow \zeta > \frac{-\ln(0.1)}{\sqrt{\pi^2 + \ln(0.1)^2}} = 0.5912$$

$$\text{b) 2\% Settling Time} < 0.5 = \frac{-\ln(0.02)}{\zeta\omega_n}$$

$$\rightarrow \omega_n > \frac{-\ln(0.02)}{0.5\zeta} = 13.2352$$

Poles should be placed further to the left of:

$$p_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right) \omega_n = -7.8240 \pm j10.6749$$

c) Stable pole can be arbitrarily chosen, e.g., $p_0 = -10$

$$\rightarrow e_{ss} \rightarrow 0 \text{ as } t \rightarrow \infty$$

Servo System via State Feedback

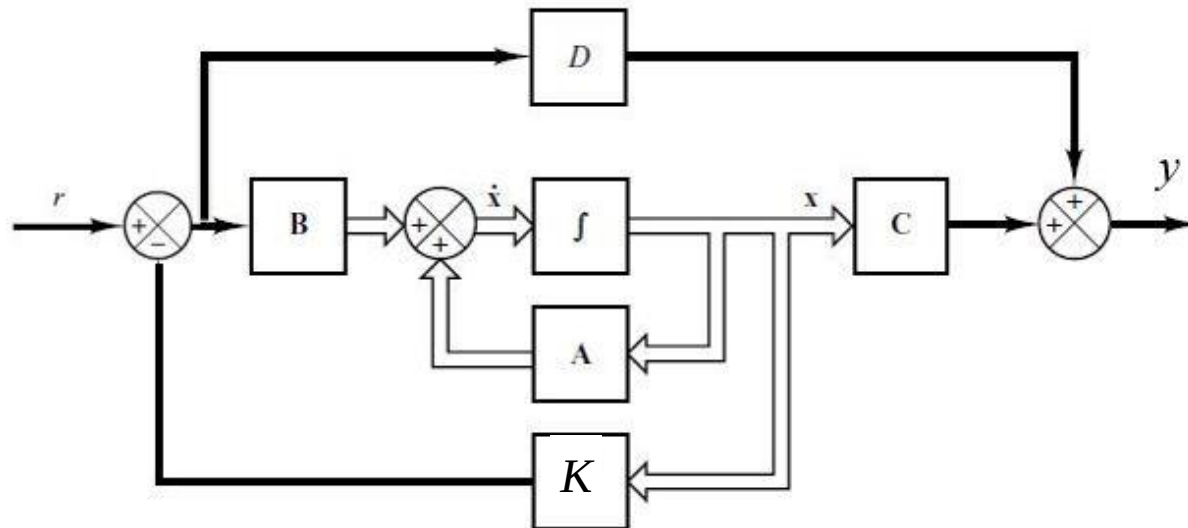
Plant: $\dot{x} = Ax + Bu$, $x(t_0) = x_0$ $y = Cx + Du$ $u \in R$

Goal: Also track a reference input: $y(\infty) = r$

State-Feedback Controller: $u = -Kx + r$

Closed-Loop System:

$$\dot{x} = Ax + Bu = (A - BK)x + Br$$



$$u(t) = -K\mathbf{x}(t) + r$$



$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t) = (A - BK)\mathbf{x}(t) + Br$$

In steady state (if $r(\infty) = \text{constant}$):

$$\dot{\mathbf{x}}(\infty) = (A - BK)\mathbf{x}(\infty) + Br$$

Since all poles of $(A - BK)$ are on LHP, $(A - BK)$ is stable, thus $\mathbf{x}(\infty) = \text{constant}$, and so $\dot{\mathbf{x}}(\infty) \rightarrow 0$ as $t \rightarrow \infty$

$$\longrightarrow 0 = (A - BK)\mathbf{x}(\infty) + Br$$

$$\longrightarrow \mathbf{x}(\infty) = -(A - BK)^{-1}Br$$

$$u(t) = -K\mathbf{x}(t) + r$$



$$u(\infty) = -K\mathbf{x}(\infty) + r = [K(A - BK)^{-1}B + 1]r$$

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t) = (A - BK)\mathbf{x}(t) + Br$$

In steady state:

$$\mathbf{x}(\infty) = -(A - BK)^{-1}Br$$

$$u(\infty) = -K\mathbf{x}(\infty) + r = [K(A - BK)^{-1}B + 1]r$$

$$y = C\mathbf{x} + Du$$



$$\begin{aligned} y(\infty) &= C\mathbf{x}(\infty) + Du(\infty) \\ &= [[-C(A - BK)^{-1} + DK(A - BK)^{-1}]B + D] r \\ &= [(DK - C)(A - BK)^{-1}B + D] r \end{aligned}$$

Goal: $y(\infty) = r$



$$[(DK - C)(A - BK)^{-1}B + D] = 1$$

Invariance of Controllability

$$\begin{array}{ccc} \dot{x} = Ax + Bu & \xrightarrow{u = -Kx} & \dot{x} = (A - BK)x \\ y = Cx & & y = Cx \end{array}$$

Theorem:

System (A, B) is controllable **if and only if** system $(A - BK, B)$ is controllable

This is the **invariance** property of LTI systems under state-feedback control

Note: Observability does **not** have such invariance

Example:

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \ 2] \mathbf{x}$$

$$[B \ AB] = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 7 & 4 \end{bmatrix}$$

→ controllable and observable

Design:

$$u = -[3 \ 1] \mathbf{x}$$

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



$$y = [1 \ 2] \mathbf{x}$$

$$[B \ AB] = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

→ controllable but **not** observable
(controllability is invariant)
(observability is **not** invariant)

Output-Feedback Observer Design

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

State-feedback controller: $u = -\mathbf{K}\mathbf{x} + r$

In practice, information of the (higher-dimensional) state vector $\mathbf{x}$ is usually not available or very expensive to obtain

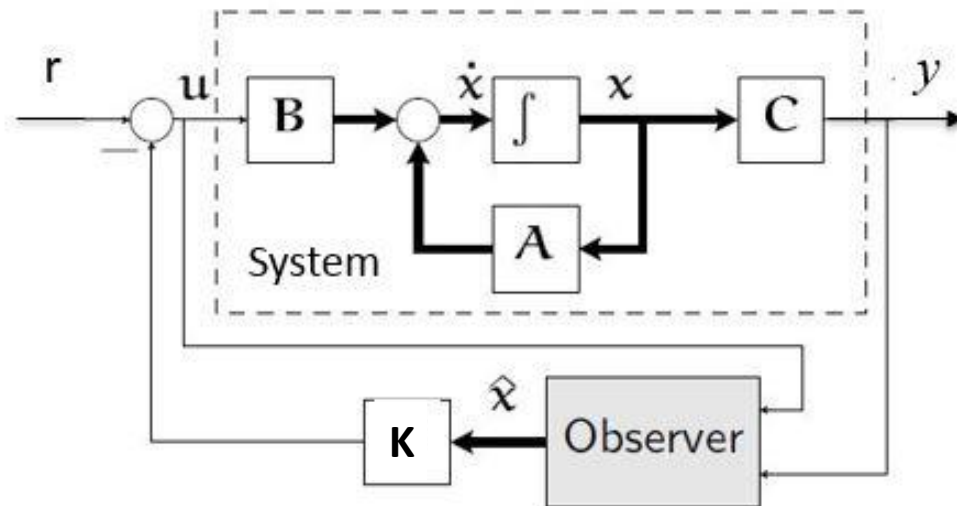
Since the (lower-dimensional) system output is y , which is available, especially if the system is observable, can we design the feedback controller u using y instead of $\mathbf{x}$?

→ Output-feedback controller

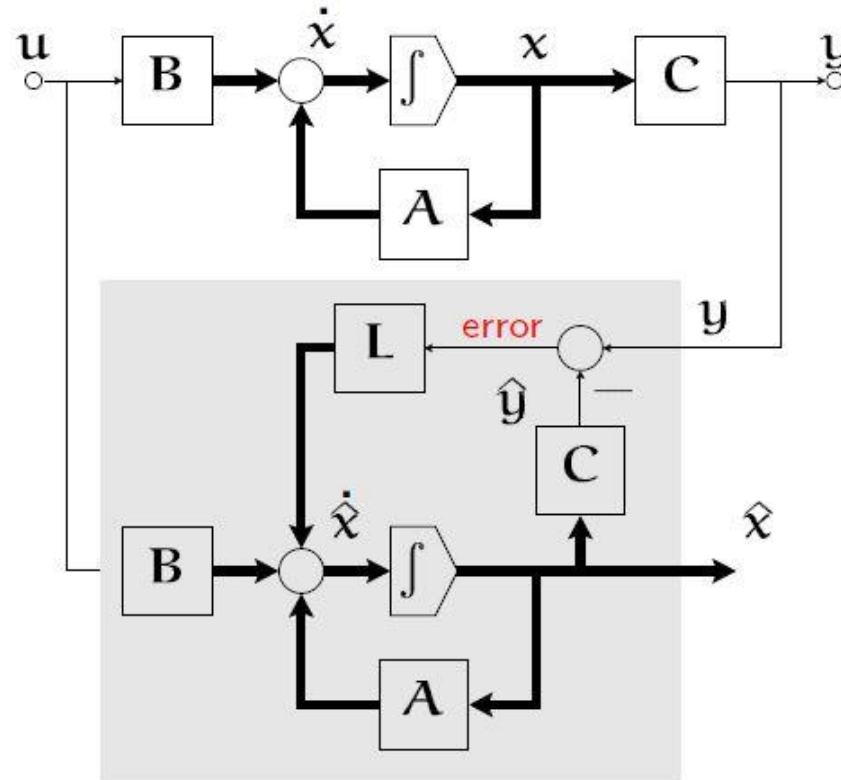
If the system is observable, then the state information can be well estimated by using an **observer**

An **observer** is a dynamic sub-system that estimates the system states by using the measurements of the system (single-)inputs and (single-)outputs: (u, y)

Note: MIMO systems can be similarly studied



Designing an Output-Feedback Observer



Determine the observer gain L satisfying $\dot{\hat{x}} = Ly + \dots$, such that $\hat{x} - x \rightarrow 0$ as $t \rightarrow \infty$ (then: $u = -K\hat{x} + r$)

Observer Design

Plant: $\dot{x} = Ax + Bu, \quad x(t_0) = x_0$
 $y = Cx$

Define: $\hat{x}$ be the observer state

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) & L: \text{Observer gain} \\ &= (A - LC)\hat{x} + Bu + Ly\end{aligned}$$

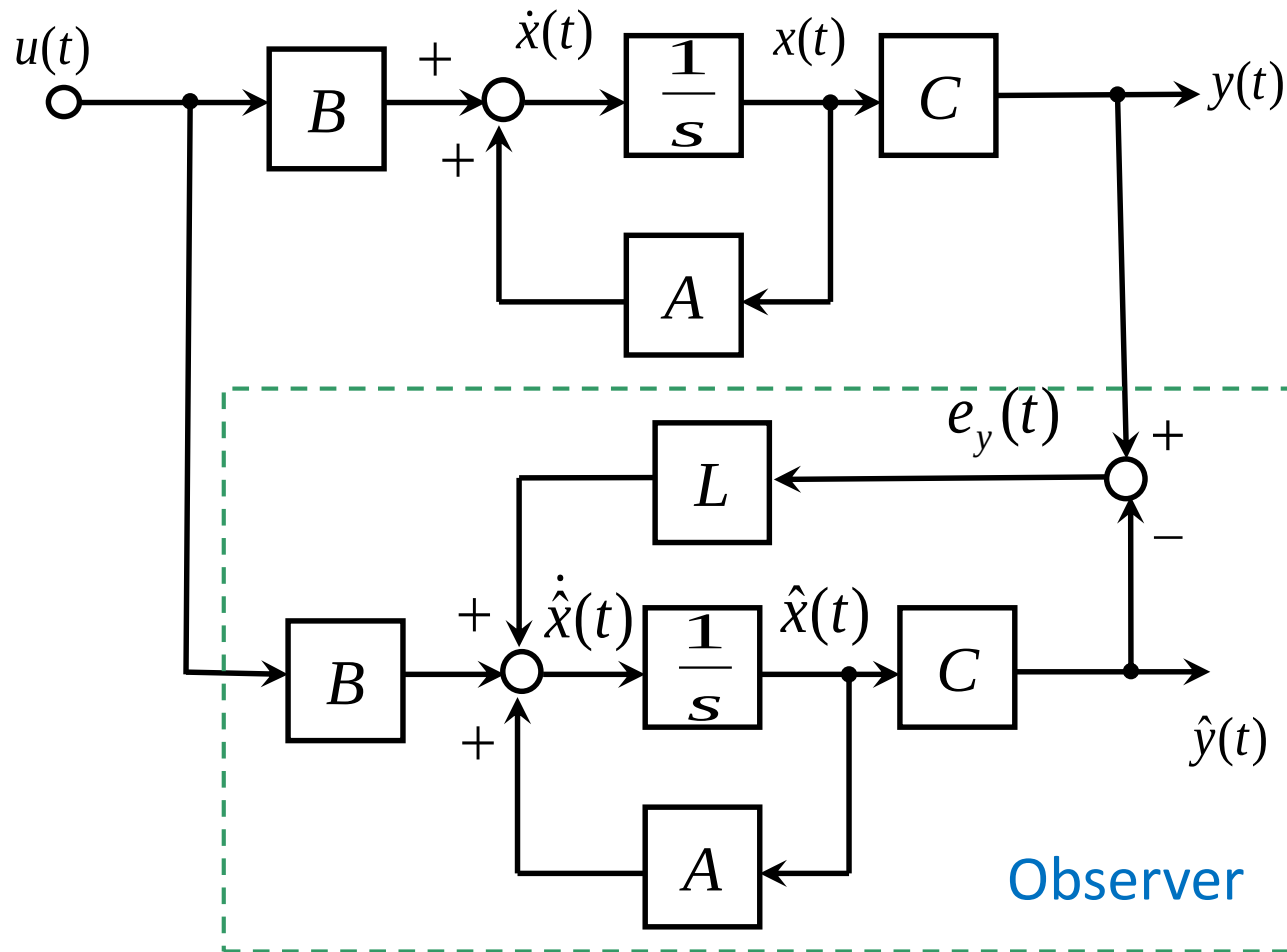
Estimation error:

$$e = x - \hat{x}$$

Error system:

$$\begin{aligned}\dot{e} &= \dot{x} - \dot{\hat{x}} = Ax + Bu - (A - LC)\hat{x} - Bu - Ly \\ &= (A - LC)e \quad (\text{Note: } y = Cx)\end{aligned}$$

Block Diagram of LTI System with an Observer



Error system: $\dot{e} = (A - LC)e$

→ If all eigenvalues of $(A - LC)$ are in LHP,
then the error system is stable

→ $e \rightarrow 0$, as $t \rightarrow \infty$

Duality between controllability and observability →

$$A_o = A_c^T \quad B_o = C_c^T \quad C_o = B_c^T$$

Theorem:

There exists an observer gain L such that

$$\det(sI - (A - LC)) = (s - p_1)(s - p_2) \cdots (s - p_n)$$

for any desirable poles p_i , $i = 1, 2, \dots, n$, if and only if
 (A, C) is observable.

Example:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = u$$

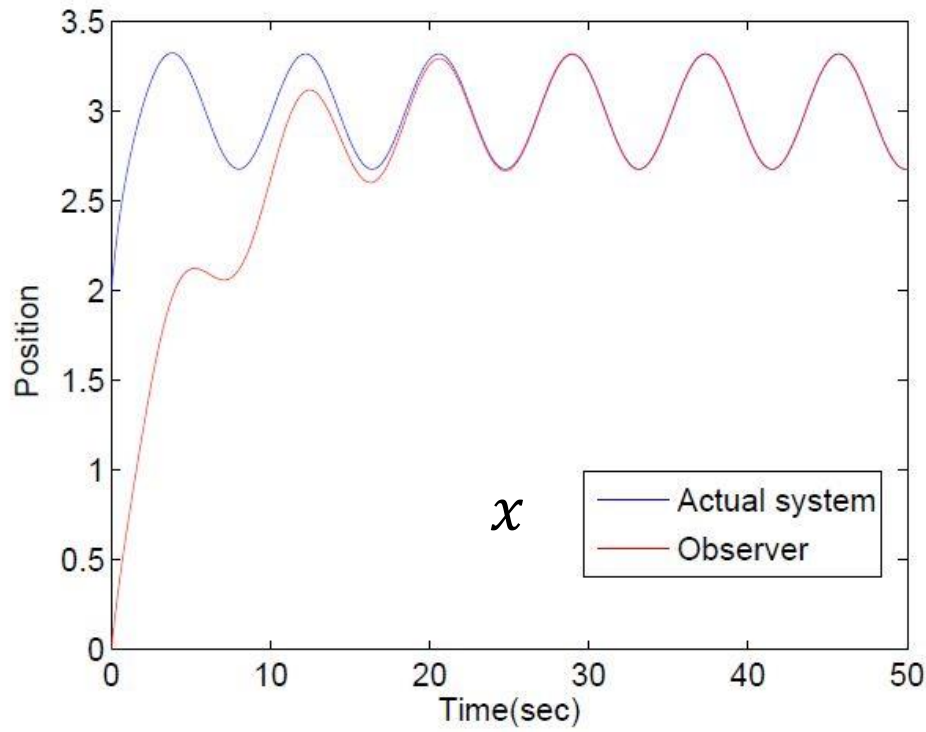
with $u = 3 + 0.5 \sin(0.75t)$ Suppose: $\zeta = 1, \omega_n = 1$

$$\frac{d}{dt} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$$

This system is observable (verify this!) → Design an observer:

$$\frac{d}{dt} \begin{pmatrix} \hat{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \dot{\hat{x}} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u - LC(\hat{x} - x)$$

Desired poles: $p_1 = -0.25, p_2 = -0.5$ → $L = \begin{pmatrix} 0.875 \\ -1.25 \end{pmatrix}$

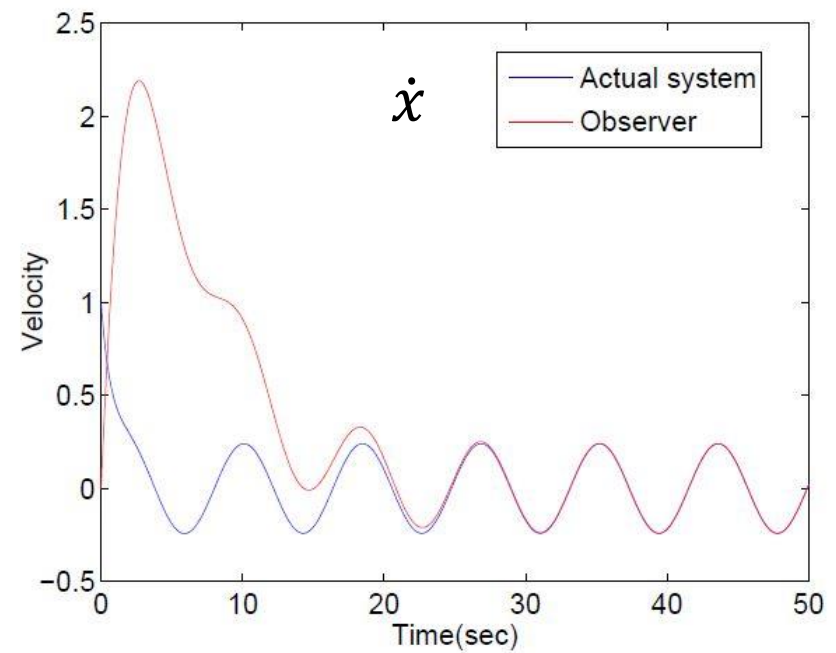


$$\frac{d}{dt} \begin{pmatrix} \hat{x} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \dot{\hat{x}} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u - LC(\hat{x} - x)$$

$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x} \\ \dot{\hat{x}} \end{pmatrix}$$

$$u = 3 + 0.5 \sin(0.75t) \quad L = \begin{pmatrix} 0.875 \\ -1.25 \end{pmatrix}$$

Simulation Results



Separation Principle

Plant:

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

State-feedback controller using estimated state: $u = -K\hat{x}$

State-space system:

$$\dot{x} = Ax + Bu = Ax - BK\hat{x} = (A - BK)x + BKe$$
$$e = x - \hat{x}$$

Observer:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) = (A - BK - LC)\hat{x} + Ly$$

Error system: $e = x - \hat{x}$

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)e$$

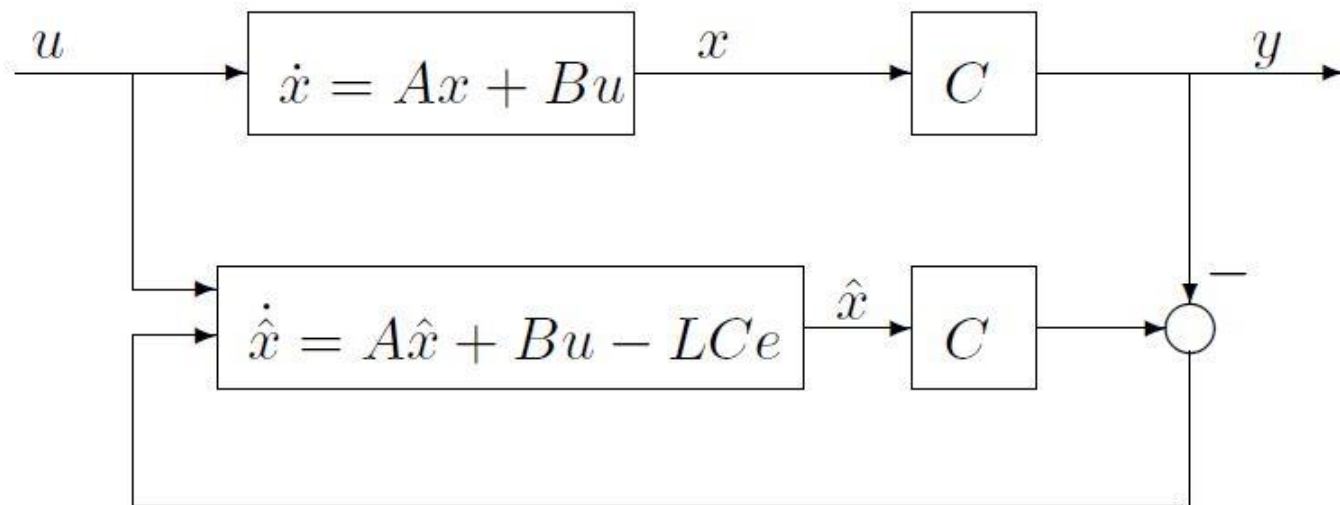
Putting together →

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ e \end{bmatrix}$$

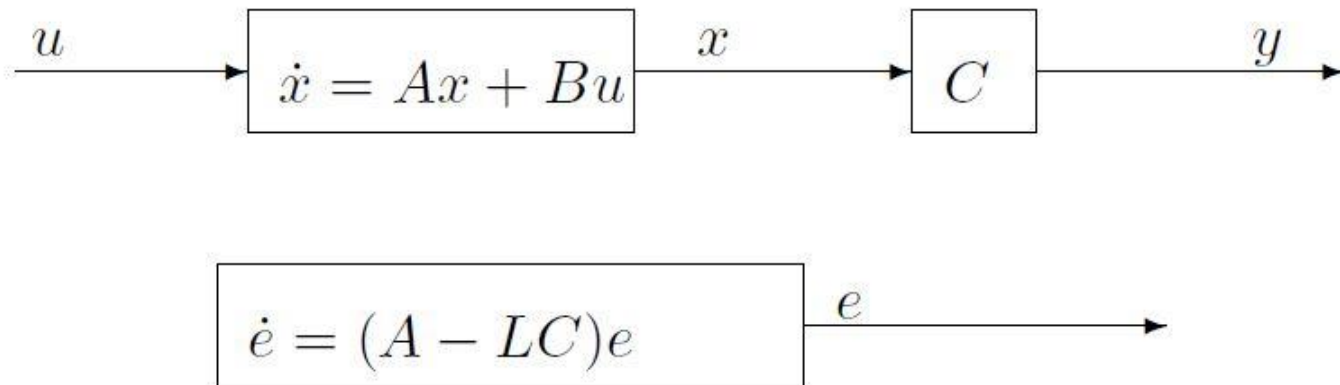
Poles:

$$\begin{aligned} |sI - \tilde{A}| &= \left| \begin{bmatrix} sI - (A - BK) & -BK \\ 0 & sI - (A - LC) \end{bmatrix} \right| \\ &= |sI - (A - BK)| \times |sI - (A - LC)| \end{aligned}$$

- Poles of the overall system
= Poles of $(A - BK)$ + Poles of $(A - LC)$
- Designing state-feedback controller gain and output-feedback observer gain can be done separately



(for design purpose)



Example:

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} x(t)\end{aligned}$$

We want the observer to have poles $-4, -3$

It can be verified that the system is observable, so this can be done

Let the observer gain be $L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}$

$$\rightarrow A - LC = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} -1 & -\frac{L_1}{2} \\ 1 & -1 - \frac{L_2}{2} \end{bmatrix}$$

$$\rightarrow \det(sI - (A - LC)) = s^2 + \left(2 + \frac{L_2}{2}\right)s + \left(1 + \frac{L_1 + L_2}{2}\right)$$

Desired poles $\rightarrow (s + 4)(s + 3) = s^2 + 7s + 12$

Comparing coefficients $\rightarrow L_1 = 12, L_2 = 10$

```
%MatLab code  
A=[-1 0; 1 -1];  
C=[0 0.5];  
J=[-4,-3];  
L=place(A',C',J)
```

Optimal Regulator Design

Consider an LTI system:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

with a negative state-feedback controller:

$$\mathbf{u} = -K\mathbf{x}$$

We want to control the system state $\mathbf{x}(t)$ from an initial state $\mathbf{x}(0)$ at $t = 0$ to a target state, say $\mathbf{x}^* = 0$: $\mathbf{x}(t) \rightarrow 0$ as $t \rightarrow \infty$

If this is not possible, then we want to

$$\min_{t \rightarrow \infty} |\mathbf{x}(t) - \mathbf{x}^*| = \min_{t \rightarrow \infty} |\mathbf{x}(t)| = \min_{t \rightarrow \infty} \mathbf{x}^T(t)\mathbf{x}(t)$$

Meanwhile, we want to minimize the control energy:

$$\min_{t \rightarrow \infty} \int_0^t |\mathbf{u}(\tau)|^2 d\tau = \min_{t \rightarrow \infty} \int_0^t \mathbf{u}^T(\tau)\mathbf{u}(\tau) d\tau$$

$$\min_{t \rightarrow \infty} | \mathbf{x}(t) - \mathbf{x}^* | = \min_{t \rightarrow \infty} | \mathbf{x}(t) | = \min_{t \rightarrow \infty} \mathbf{x}^T(t) \mathbf{x}(t)$$

$$\min_{t \rightarrow \infty} \int_0^t | \mathbf{u}(\tau) |^2 d\tau = \min_{t \rightarrow \infty} \int_0^t \mathbf{u}^T(\tau) \mathbf{u}(\tau) d\tau \quad \text{with} \quad \mathbf{u}(t) = -K \mathbf{x}(t)$$



$$\min_K \int_0^{\infty} [\mathbf{x}^T(\tau) Q \mathbf{x}(\tau) + \mathbf{u}^T(\tau) R \mathbf{u}(\tau)] d\tau$$

Here, $Q \geq 0$ and $R > 0$ are weight matrices

This is **Optimal Linear Quadratic Regulation** (**Optimal LQR Control**) problem

%MatLab code $\rightarrow$ `lqr(A,B,Q,R)`

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{u}(t) = -K\mathbf{x}(t)$$

$$\min_K \int_0^{\infty} [\mathbf{x}^T(\tau)Q\mathbf{x}(\tau) + \mathbf{u}^T(\tau)R\mathbf{u}(\tau)]d\tau$$



$$\dot{\mathbf{x}} = (A - BK)\mathbf{x}$$

$$\min_K \int_0^{\infty} [\mathbf{x}^T(\tau)Q\mathbf{x}(\tau) + \mathbf{x}^T(\tau)K^T RK\mathbf{x}(\tau)]d\tau$$

$$= \min_K \int_0^{\infty} \mathbf{x}^T(\tau)[Q + K^T RK]\mathbf{x}(\tau)d\tau$$

Set

$$\mathbf{x}^T[Q + K^T RK]\mathbf{x} = -\frac{d}{dt}\mathbf{x}^T P\mathbf{x} \quad (P \text{ is positive definite and symmetric})$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{u}(t) = -K\mathbf{x}(t)$$

$$\mathbf{x}^T(\tau)[Q + K^T R K]\mathbf{x}(\tau) = -\frac{d}{dt}\mathbf{x}^T(t)P\mathbf{x}(t)$$



$$\begin{aligned}\mathbf{x}^T[Q + K^T R K]\mathbf{x} &= -\frac{d}{dt}\mathbf{x}^T P \mathbf{x} = -\dot{\mathbf{x}}^T P \mathbf{x} - \mathbf{x}^T P \dot{\mathbf{x}} \\ &= -\mathbf{x}^T[(A - BK)^T P + P(A - BK)]\mathbf{x}\end{aligned}$$



$$(A - BK)^T P + P(A - BK) = -(Q + K^T R K)$$



$$K = R^{-1}B^T P$$

where P satisfies

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad (\text{Riccati equation})$$

→ $\mathbf{u} = -K\mathbf{x} = -R^{-1}B^T P\mathbf{x}$ (optimal controller) solves

$$\min_K \int_0^{\infty} [\mathbf{x}^T(\tau)Q\mathbf{x}(\tau) + \mathbf{u}^T(\tau)R\mathbf{u}(\tau)]d\tau$$

Example:

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \quad \min_K \int_0^{\infty} [\mathbf{x}^T(\tau)Q\mathbf{x}(\tau) + \mathbf{u}^T(\tau)R\mathbf{u}(\tau)]d\tau$$
$$\mathbf{u}(t) = -K\mathbf{x}(t)$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad R = [1]$$

(Riccati equation) $A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$



$$\begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



$$\begin{aligned} 1 - p_{12}^2 &= 0 \\ p_{11} - p_{12}p_{22} &= 0 \\ \mu + 2p_{12} - p_{22}^2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{\mu+2} & 1 \\ 1 & \sqrt{\mu+2} \end{bmatrix}$$
$$K = R^{-1}B^T P = 1 \cdot [0 \ 1] \cdot P = [1 \ \sqrt{\mu+2}]$$

%MatLab code (μ = number)

K=lqr(A,B,Q,R)

END