

Homework 2 Solutions

Question 1: Consider the following function

$$h(x) = \frac{x^3 + 1}{x^2 + 1}$$

- (a) (**15 points**) Use the quotient rule to find the derivative of the function, $h'(x)$.

Let $f(x) = x^3 + 1$, $g(x) = x^2 + 1$. Noting that $h(x) = f(x)/g(x)$, we have the following quotient rule

$$(3 \text{ points}) \quad h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

We have $f'(x) = 3x^2$, $g'(x) = 2x$. Then,

$$(8 \text{ points}) \quad h'(x) = \frac{3x^2(x^2 + 1) - (x^3 + 1)(2x)}{(x^2 + 1)^2}$$

$$(4 \text{ points}) \quad h'(x) = \frac{x^4 + 3x^2 - 2x}{(x^2 + 1)^2}$$

- (b) (**5 points**) What is the value of $h'(1)$?

$$(5 \text{ points}) \quad h'(1) = \frac{1 + 3 - 2}{4} = \frac{1}{2}$$

Question 2: Consider the following function

$$f(x) = (\ln x)(\log_2 x)$$

- (a) (**15 points**) Find $f'(x)$. (*Hint: If $\ln x = y$, then $e^y = x$; and if $\log_2 x = y$, then $2^y = x$).*

By product rule, we have

$$(5 \text{ points}) \quad f'(x) = (\log_2 x) \frac{d}{dx}(\ln x) + (\ln x) \frac{d}{dx}(\log_2 x)$$

$$(10 \text{ points}) \quad f'(x) = \frac{\log_2 x}{x} + \frac{\ln x}{x \ln 2}$$

Note that you may directly use $\frac{d}{dx}(\log_2 x) = \frac{1}{x \ln 2}$ as in the above expression or derive it using few more steps as described below:

Let $\log_2 x = y$, then $2^y = x$

Now, take $\ln$ of both sides to get

$$y \ln 2 = \ln x$$

Differentiating both sides w.r.t. x , we have

$$(\ln 2) \frac{dy}{dx} = \frac{1}{x} \implies \frac{dy}{dx} = \frac{1}{x \ln 2}$$

(b) (**5 points**) What is the value of $f'(2)$?

(**3 points**)

$$f'(2) = \frac{\log_2 2}{2} + \frac{\ln 2}{2 \ln 2}$$

(**2 points**)

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Question 3: Consider the function

$$f(x) = x^2 - 4x$$

Find the points (i.e., the values of x) in the given interval below where the function attains its maximum and minimum values.

Note that the F.O.C. is $f'(x) = 2x - 4 = 0$, which gives the critical point $x = 2$. Also, $f(2) = -4$. Note that there is a local minima at $x = 2$.

(a) (**5 points**) $[-1, 4]$

(**1 point**) $f(-1) = (-1)^2 - (4)(-1) = 5.$

(**1 point**) $f(4) = (4)^2 - (4)(4) = 0.$

Therefore, (**1.5 points**) $f(x)$ attains a maximum at $x = -1$

(**1.5 points**) $f(x)$ attains a minimum at $x = 2$.

(b) (**5 points**) $[1, 3]$

(1 point) $f(1) = (1)^2 - (4)(1) = -3.$

(1 point) $f(3) = (3)^2 - (4)(3) = -3.$

Therefore, (**2 points**) $f(x)$ attains a maximum at $x = 1$ and $x = 3$.

(1 point) $f(x)$ attains a minimum at $x = 2$.

(c) (**5 points**) $[3, 5]$

(1 point) $f(3) = (3)^2 - (4)(3) = -3.$

(1 point) $f(5) = (5)^2 - (4)(5) = 5.$

Therefore, (**1.5 points**) $f(x)$ attains a maximum at $x = 5$.

(1.5 points) $f(x)$ attains a minimum at $x = 3$.

(d) (**5 points**) $[0, \infty]$

(1 point) $f(0) = (0)^2 - (4)(0) = 0.$

(1 point) As $x \rightarrow \infty$, $f(x) \rightarrow \infty$.

(1.5 points) There is no (global) maximum attained by the function.

(1.5 points) $f(x)$ attains a minimum at $x = 2$.

Question 4: (**20 points**) The function f is given by

$$f(x) = (x^2 + 10x - 10)e^{-x}$$

Find the critical points of f and, for each, determine whether it is a maximum, minimum, or inflection point.

Using product rule, we have

(1 point)
$$f'(x) = (x^2 + 10x - 10) \frac{d}{dx} e^{-x} + e^{-x} \frac{d}{dx} (x^2 + 10x - 10)$$

$$= -(x^2 + 10x - 10)e^{-x} + e^{-x}(2x + 10)$$

(2 points)
$$= -e^{-x}(x^2 + 8x - 20)$$

To find critical points, we equate $f'(x)$ to zero. Thus, we have

(1 point)
$$f'(x) = -e^{-x}(x^2 + 8x - 20) = 0$$

(2 points) $= -e^{-x}(x^2 - 2x + 10x - 20) = -e^{-x}(x - 2)(x + 10) = 0$

(4 points) Therefore, $x = 2$ and $x = -10$ are the critical points.

Now,

(1 point) $f''(x) = -[(x^2 + 8x - 20)\frac{d}{dx}e^{-x} + e^{-x}\frac{d}{dx}(x^2 + 8x - 20)]$

$$= -[-e^{-x}(x^2 + 8x - 20) + e^{-x}(2x + 8)]$$

(2 points) $= e^{-x}(x^2 + 6x - 28)$

(2 points) $f''(2) = e^{-2}(2^2 + (6)(2) - 28) = -12e^{-2} < 0$

(2 points) $f''(-10) = e^{-10}((-10)^2 - (6)(10) - 28) = 12e^{-2} > 0$

(3 points) Therefore, there is a maximum at $x = 2$ and a minimum at $x = -10$.

Question 5: Suppose the demand function is given by

$$q^D(p) = 70 - 4p$$

(a) **(10 points)** Find the expression for price elasticity of demand.

The price elasticity of demand is given by

(2 point) $\varepsilon(p) = -\frac{q'p}{q}$

$$= -\frac{(-4)p}{70 - 4p}$$

(8 points) $= \frac{4p}{70 - 4p}$

(b) **(5 points)** For what range of values of p is the expression valid, i.e., find the economically meaningful values of p ? (*Hint: Use the restriction that p and q must be non-negative; similar to what we did on page 6 of Lecture 4 Part II of II.*)

(1 point) We must have $p \geq 0$. Also,

(1 point) $q \geq 0$

(1 point) $70 - 4p \geq 0$

(2 points) $p \leq \frac{70}{4}$

(c) (**5 points**) For what range of values of p is the demand inelastic?

Demand is inelastic when

(1 point) $\varepsilon(p) < 1$

(1 point) $\implies \frac{4p}{70 - 4p} < 1$

$\implies 4p < 70 - 4p \implies 8p < 70$

(2 points) $p < \frac{70}{8}$

(1 point) Therefore, demand is inelastic for $0 \leq p < \frac{70}{8}$.