
EE 3121
Differential Equations for Electrical Engineering

Tutorial 9: Partial Differential Equations



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Summary

This chapter discussed four of the most important PDEs in engineering. In two-dimensional Cartesian coordinates the heat equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{\alpha} \frac{\partial u}{\partial t} = 0.$$

- It is first order in time.
- The heat equation describes diffusion and its solutions smooth initial peaks and valleys.

Summary

In the same coordinates the wave equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = 0.$$

- It is second order in time.

The wave equation portrays oscillations propagating in space at a constant speed a . Again in the same coordinates Laplaces equation is

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

- Laplaces equation governs steady-state solutions of the heat and wave equation and situations involving a potential.

Summary

The beam equation

$$\frac{\partial^4 u}{\partial x^4} + \left(\frac{\rho}{EI} \right) \frac{\partial^2 u}{\partial t^2} = 0,$$

describes the lateral vibrations of a cantilever structural beam.

- It is fourth order in the spatial coordinate.
- These equations take different forms depending on geometry.
- We can solve boundary value problems using separation of variables, which lead us to important ODEs with variable coefficients: Bessel equations in cylindrical coordinates:

Summary

$$\frac{d^2v}{dr^2} + \frac{1}{r} \frac{dv}{dr} + \left(1 - \frac{m^2}{r^2}\right) v = 0.$$

Legendre equations in spherical coordinates (with $x = \cos \theta$)

$$(1 - x^2) \frac{d^2v}{dx^2} - 2x \frac{dv}{dx} + n(n + 1)v = 0.$$

We solve these ODEs using power series.

Summary

- When series expansions are about a regular singular point, solutions to the indicial equation determine the form of the power series.
- Solution of boundary value problems takes us into infinite series of eigenfunctions. There is a broad class of PDEs that, with the right boundary conditions, have eigenfunctions that are orthogonal and capable of representing well behaved functions (as initial conditions) in an infinite series.
- The additional independent variables in PDEs can be regarded as an infinite number of states, and that PDEs can be thought of as infinite dimensional systems.

Solved Problems

SP9.1: Solve

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \quad (1)$$

$$T(x, 0) = T_0 \sin^5(\pi x / L) \quad (2)$$

$$T(0, t) = 0 \quad (3)$$

$$T(L, t) = 0$$

Check your results by comparing your calculated value for $T(L/2, 0)$ with the given initial condition, which is T_0 .

Solution:

We try the solution

$$T(x, t) = \Phi(x)\Theta(t). \quad (4)$$

Solved Problems

Subsequent steps follow exactly the steps taken from Equations (15) to (31) in [Section 9.1.2](#), leading to the solution

$$T(x, t) = \sum_{n=1}^{\infty} c_n e^{-\alpha(n\pi/L)^2 t} \sin(n\pi x/L), \quad (5)$$

where

$$c_n = \frac{2}{L} \int_0^L T_0(x) \sin(n\pi x/L) dx = \frac{2}{L} \int_0^L T_0 \sin^5(\pi x/L) \sin(n\pi x/L) dx. \quad (6)$$

Solved Problems

Rather than computing the integral directly, however, we can simplify matters by looking for a geometric identity of the form

$$\sin^5 \theta = \sum_{n=1}^N a_n \sin n\theta. \quad (7)$$

We can derive the appropriate expression by hand using the geometric identities in [Section 1.1.3](#) and check our result by searching the internet, or vice versa. In any case, without difficulty we find that

$$\sin^5 \theta = (10 \sin \theta - 5 \sin 3\theta + \sin 5\theta)/16. \quad (8)$$

Solved Problems

From Equations (5), (6), and (8), and recalling that

$$\int_0^L \sin(n\pi x/L) \sin(m\pi x/L) dx = 0,$$

$$\int_0^L \sin^2(n\pi x/L) dx = L/2,$$

we determine that

$$c_1 = \left(\frac{10}{16}\right) T_0, \quad c_3 = \left(\frac{-5}{16}\right) T_0, \quad c_5 = \left(\frac{1}{16}\right) T_0, \quad (9)$$

and that all other c_n are zero.

Solved Problems

From (5) and (9), the solution to Equations (1 through (3 is

$$\begin{aligned} T(x, t) = (T_0/16) & \left(10e^{-\alpha(\pi/L)^2 t} \sin(\pi x/L) \right) \\ & - 5e^{-\alpha(3\pi/L)^2 t} \sin(3\pi x/L) \\ & + e^{-\alpha(5\pi/L)^2 t} \sin(5\pi x/L). \end{aligned} \quad (10)$$

Checking our results, we compare Equation (10) with the initial condition at $x = L/2$ and find

$$T(L/2, 0) = (T_0/16) [10 \sin(\pi/2) - 5 \sin(3\pi/2) + \sin(5\pi/2)],$$

$$T(L/2, 0) = (T_0/16)(10 \cdot (1) - 5 \cdot (-1) + (1)) = T_0,$$

as it should.

Solved Problems

SP9.2: [Section 9.1.1](#) introduced the kernel (also called the impulse response or Greens function) for the one-dimensional heat equation. Search the mathematical literature and, for the infinite space (i.e., unbounded) case, find the kernels for the two-dimensional and three-dimensional wave equations, which are

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2} = 0 \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2} = 0$$

Show that wave propagation in two dimensions (e.g., ripples on the surface when a pebble is dropped into a pond) is qualitatively different from that in three (e.g., an electromagnetic pulse from an omnidirectional antenna).

Solved Problems

Solution:

We conduct our search of the mathematical literature on the Internet and discover that the kernel for the wave equation in two dimensions is

$$K_2(x, y, t) = \frac{1}{2\pi((at)^2 - (x^2 + y^2))^{1/2}}, \quad (x^2 + y^2)^{1/2} < at$$

$$K_2(x, y, t) = 0, \quad (x^2 + y^2)^{1/2} > at$$

and the kernel for the wave equation in three dimensions is

$$K_3(x, y, z, t) = \frac{\delta(at - (x^2 + y^2 + z^2)^{1/2})}{4\pi(x^2 + y^2 + z^2)^{1/2}},$$

Solved Problems

where $\delta(w)$ is the Dirac delta function. (Recall [Section 7.6.3.1](#)).

- It is convenient here to consider the impulse response interpretation of these functions.

Consider an impulse at the spatial origin and at time equal to zero. That is, consider the equations

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{\alpha^2} \frac{\partial^2 u}{\partial t^2} = \delta(x)\delta(y)\delta(t),$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = \delta(x)\delta(y)\delta(z)\delta(t).$$

Solved Problems

Then, by definition of the impulse response, the solutions are, for the two-dimensional case,

$$u(x, y, t) = K_2(x, y, t) = \frac{1}{2\pi((at)^2 - (x^2 + y^2))^{1/2}}, \quad (x^2 + y^2)^{1/2} < at$$

$$u(x, y, t) = K_2(x, y, t) = 0, \quad (x^2 + y^2)^{1/2} > at$$

and for the three-dimensional case

$$u(x, y, z, t) = K_3(x, y, z, t) = \frac{\delta(at - (x^2 + y^2 + z^2)^{1/2})}{4\pi(x^2 + y^2 + z^2)^{1/2}}.$$

Solved Problems

In three dimensions there is a response to the impulse only at the wavefront; that is, at the range

$$(x^2 + y^2 + z^2)^{1/2} = at.$$

- In two dimensions there is no sharp trailing edge to the impulse response; the response at locations behind the wavefront lingers on.
- In three dimensions the response at a distant location will eventually cease, and it will cease in an instant. In two dimensions the response will die away slowly.

Solved Problems

SP9.3: Using the general rules in [Section 9.2.6.2](#), find a power series solution to the following equation about the point $x_0 = 0$. Find the recursive relation and calculate the first five terms in the power series

$$\begin{aligned}\frac{d^2y}{dx^2} + (x + 3)\frac{dy}{dx} + (x - 2)y &= 0 \\ y(0) &= y_0 \\ \frac{dy}{dx}(0) &= \dot{y}(0)\end{aligned}\tag{11}$$

Solved Problems

Solution:

According to the general rules, the first step is to determine whether the coefficients of dy/dx and y are analytic at the point $x = x_0$ when the coefficient of d^2y/dx^2 is unity. The coefficients are $x + 3$ and $x - 2$, respectively. Each can be expanded in a power series of about $x = 0$, so the point $x_0 = 0$ is an ordinary point and there exist two solutions of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^n. \quad (12)$$

Solved Problems

The key to finding power series solutions is to work the ODE from the form

$$\frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} c_n x^n \right) + p(x) \frac{d}{dx} \left(\sum_{n=0}^{\infty} c_n x^n \right) + q(x) \left(\sum_{n=0}^{\infty} c_n x^n \right) = 0,$$

into the form $\sum k_m x^m = 0$ and then find coefficients c_n such that $k_m = 0$ for each m . From Equation (12),

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad \frac{d^2 y}{dx^2} = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}. \quad (13)$$

Solved Problems

Then we have

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (x+3) \sum_{n=1}^{\infty} n c_n x^{n-1} + (x-2) \sum_{n=0}^{\infty} c_n x^n = 0,$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + 3 \sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^{n+1} - \sum_{n=0}^{\infty} 2c_n x^n = 0. \quad (14)$$

■ In the sum $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$, let $n-2 = m$;

■ In the sum $3 \sum_{n=1}^{\infty} n c_n x^{n-1}$, let $n-1 = m$;

Solved Problems

- In the sum $\sum_{n=0}^{\infty} c_n x^{n+1}$, let $n + 1 = m$;
- In the sum $\sum_{n=1}^{\infty} n c_n x^n$ and $\sum_{n=0}^{\infty} 2c_n x^n$, let $n = m$.

Then Equation (14) becomes

$$\begin{aligned} \sum_{m=1}^{\infty} ((m+2)(m+1)c_{m+2} + mc_m + 3(m+1)c_{m+1} + c_{m-1} - 2c_m)x^m \\ + (2(1)c_2 + 3(1)c_1 - 2c_0)x^{(0)} = 0, \end{aligned} \quad (15)$$

where the $x^{(0)}$ (constant) term stands alone because it was not included in two of the sums and isolating it allows us to write the rest of the left-hand side of Equation (31) in the desired form $\sum k_m x^m$. We must have the coefficient of x^m for each m equal to zero.

Solved Problems

Then, from Equation (15),

$$c_2 = c_0 - \left(\frac{3}{2}\right) c_1. \quad (16)$$

$$c_{m+2} = -\frac{1}{(m+1)(m+2)}(3(m+1)c_{m+1} + (m-2)c_m + c_{m+1}) \quad m \geq 1 \quad (17)$$

From the initial conditions and Equations (12) and (13) we determine that $c_0 = y_0$ and $c_1 = y_0$. From Equations (16) and (17) we find, after some simplification,

Solved Problems

$$\begin{aligned}c_2 &= y_0 - \left(\frac{3}{2}\right) \dot{y}_0, \\c_3 &= -\left(\frac{7}{6}\right) y_0 + \left(\frac{5}{3}\right) \dot{y}_0, \\c_4 &= \left(\frac{7}{8}\right) y_0 - \left(\frac{4}{3}\right) \dot{y}_0, \\c_5 &= -\left(\frac{31}{60}\right) y_0 + \left(\frac{19}{24}\right) \dot{y}_0.\end{aligned}\tag{18}$$

Solved Problems

SP9.4: Using the general rules in [Section 9.2.6.2](#), find a power series solution to the following equation about the point $x_0 = 0$. Find the recursive relation and calculate the first five terms in the power series.

$$x^2 \frac{d^2 y}{dx^2} + 5x \frac{dy}{dx} + (x + 3)y = 0 \quad (19)$$

Solution:

Following the general rules, we divide Equation(19) through by x^2 to make the coefficient of the highest derivative equal unity. The resulting coefficients $p(x)$ and $q(x)$ are not analytic, so we must examine the indicial equation. We have

Solved Problems

$$p_0 = \lim_{x \rightarrow 0} x p(x) = \lim_{x \rightarrow 0} x \left(\frac{5x}{x^2} \right) = 5, \quad (20)$$

$$q_0 = \lim_{x \rightarrow 0} x^2 q(x) = \lim_{x \rightarrow 0} x^2 \left(\frac{x+3}{x^2} \right) = 3.$$

Then the indicial equation is

$$\alpha(\alpha-1) + p_0\alpha + q_0 = \alpha(\alpha-1) + 5\alpha + 3 = \alpha^2 + 4\alpha + 3 = (\alpha+1)(\alpha+3) = 0. \quad (21)$$

The two roots of the indicial equation are $\alpha_1 = -1$, $\alpha_2 = -3$. The largest of these is α_2 . From the general rules, we are assured that there exists a solution of the form

$$y(x) = x^{\alpha_1} \sum_{n=0}^{\infty} c_n x^n = x^{-1} \sum_{n=0}^{\infty} c_n x^n. \quad (22)$$

Solved Problems

Because the difference between the two roots is an integer, the second solution may have one of two forms. We were only asked to find one solution, so our easiest option is to seek the solution in the form of Equation (22). Before we begin substituting this form into Equation (19) it is convenient to rewrite the form as

$$y(x) = x^{-1} \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n-1}. \quad (23)$$

Our objective is to work the ODE from the form

Solved Problems

$$\begin{aligned} \frac{d^2}{dx^2} \left(\sum_{n=0}^{\infty} c_n x^n \right) + p(x) \frac{d}{dx} \left(\sum_{n=0}^{\infty} c_n x^n \right) \\ + q(x) \left(\sum_{n=0}^{\infty} c_n x^n \right) = 0, \end{aligned} \quad (24)$$

into the form $\sum k_m x^m = 0$ and then find coefficients c_n such that $k_m = 0$ for each m . From Equation (23),

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} c_n (n-1) x^{n-2}, \quad (25)$$

$$\frac{d^2 y}{dx^2} = \sum_{n=0}^{\infty} c_n (n-1)(n-2) x^{n-3}.$$

Solved Problems

Substituting Equations (23) and (25) into Equation (19) we have

$$x^2 \left(\sum_{n=0}^{\infty} c_n (n-1)(n-2)x^{n-3} \right) + 5x \left(\sum_{n=0}^{\infty} c_n (n-1)x^{n-2} \right) + (x+3) \left(\sum_{n=0}^{\infty} c_n x^{n-1} \right) = 0.$$

$$\sum_{n=0}^{\infty} ((n-1)(n-2) + 5(n-1) + 3)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0. \quad (26)$$

The first sum in Equation 9.189 has an x^{-1} term and has different exponents than the second (i.e., each element in the sum is written in terms of x^{n-1} as compared to x^n). We isolate the x^{-1} term from the first sum and define $m = n - 1$ in the remainder of it, and then Equation (26) can be written as

Solved Problems

$$\begin{aligned} & ((0-1)(0-2) + 5(0-1) + 3)c_0x^{-1} \\ & + \sum_{n=0}^{\infty} ((n-1)(n-2) + 5(n-1) + 3)c_nx^{n-1} + \sum_{n=0}^{\infty} c_nx^n = 0. \end{aligned} \quad (27)$$

$$(0)c_0x^{-1} + \sum_{m=0}^{\infty} (m(m-1) + 5m + 3)c_{m+1}x^m + \sum_{n=0}^{\infty} c_nx^n = 0. \quad (28)$$

The coefficient of c_0 is zero by virtue of the indicial equation, which allows us a nonzero c_0 . Then we can change the index in the second sum from n to m and Equation (28) becomes

$$\sum_{m=0}^{\infty} ((m(m-1) + 5m + 3)c_{m+1} + c_m)x^m = 0. \quad (29)$$

Solved Problems

Equation 9.192 is in the form $\sum k_m x^m = 0$, and so we must now find coefficients c_n such that $k_m = 0$ for each m . This gives us the recurrence relation

$$(m(m-1) + 5m + 3)c_{m+1} + c_m = 0, \quad (30)$$

which can be simplified to

$$c_{m+1} = -\frac{c_m}{(m+1)(m+3)}. \quad (31)$$

Solved Problems

Then

$$\begin{aligned}c_1 &= -\frac{c_0}{3}, \\c_2 &= -\frac{c_1}{8} = \frac{c_0}{24}, \\c_3 &= -\frac{c_2}{15} = -\frac{c_0}{360}, \\c_4 &= -\frac{c_3}{24} = \frac{c_0}{8640},\end{aligned}\tag{32}$$

and

$$y(x) = \frac{c_0}{x} \left(1 - \frac{x}{3} + \frac{x^2}{24} - \frac{x^3}{360} + \frac{x^4}{8640} - \dots \right).\tag{33}$$

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