

EE3109 Test 1 Solution

1. (7 Marks) Given $\mathbf{A}(\mathbf{r}) = ax - by^{-2} \hat{\mathbf{z}}$ [Wb/m] over a region of free-space, find, over that region of space,

(a) The electric flux density (4)

(b) The magnetic flux density (3)

Include the appropriate units for both quantities. Assume a temporal variation with angular frequency ω .

Solution:

(b)

$$\mathbf{B} = \nabla \times \mathbf{A} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (0, 0, ax - by^{-2}) = (2by^{-3}, -a, 0) \text{ [Wb/m}^2\text{]}$$

(a)

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu} (2by^{-3}, -a, 0)$$
$$\mathbf{D} = \frac{1}{j\omega} \nabla \times \mathbf{H} = \frac{1}{j\omega\mu} \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times (2by^{-3}, -a, 0) = -j \frac{6b}{\omega\mu} y^{-4} \hat{\mathbf{z}} \text{ [V/m]}$$

2. (17 Marks) A plane wave travelling in a non-magnetic simple medium has a magnetic field

$$\mathbf{H}(\mathbf{r}, t) = 30 \exp(0.01(x + z)) \sin\left(10^9 t + \frac{10}{3}(x + z) - \frac{\pi}{4}\right) \hat{\mathbf{y}} \text{ [mA/m]}.$$

Find the following:

- (a) The frequency, wavelength, complex relative permittivity, relative permeability, conductivity and intrinsic impedance (9)
- (b) The electric field phasor (5)
- (c) The distance travelled by this plane wave by the time 1% of its power remains (3)

Solution:

(a)

$$\omega = 10^9 \Rightarrow f = \frac{10^9}{2\pi} \text{ [Hz]} = \frac{500}{\pi} \text{ [MHz]}; \mu_r = 1 \text{ (nonmagnetic);}$$

$$\alpha = 0.01\sqrt{2} \text{ [Np/m]}; \beta = \frac{10\sqrt{2}}{3} \text{ [rad/m]}; \lambda = \frac{2\pi}{\beta} = \frac{3\sqrt{2}\pi}{10} \text{ [m]}.$$

$$\alpha^2 - \beta^2 = -\frac{\omega^2}{c^2} \epsilon'_r \Rightarrow \epsilon'_r = \frac{c^2}{\omega^2} (\beta^2 - \alpha^2) = \frac{9 \times 10^{16}}{10^{18}} \left(\frac{200}{9} - 0.0001 \right) = 2.000$$

$$2\alpha\beta = \frac{\omega^2}{c^2} \epsilon''_r \Rightarrow \epsilon''_r = \frac{c^2}{\omega^2} (2\alpha\beta) = \frac{9 \times 10^{16}}{10^{18}} (4)(0.01) \left(\frac{10}{3} \right) = 0.012$$

Combine to get:

$$\epsilon_{rc} = \epsilon'_r - j\epsilon''_r = 2.000 - j0.012$$

$$\sigma = \omega \epsilon_0 \epsilon''_r = 10^9 (8.854 \times 10^{-12}) (0.012) = 1.062 \times 10^4 \text{ [S/m]}$$

$$\eta = \sqrt{\frac{\mu_0}{\epsilon_0 \epsilon_{rc}}} \cong \frac{\eta_0}{\sqrt{\epsilon_{rc}}} = \frac{120\pi}{\sqrt{2.000}} = 60\sqrt{2}\pi \text{ [\Omega]}$$

(b)

$$\mathbf{H}(\mathbf{r}) = 0.03 e^{0.01(x+z)} e^{j\frac{10}{3}(x+z)} e^{-j\frac{3\pi}{4}} \hat{\mathbf{y}} = H(\mathbf{r}) \hat{\mathbf{y}} \text{ [A/m]}$$

$$\mathbf{E}(\mathbf{r}) = \eta H(\mathbf{r}) \hat{\mathbf{y}} \times \hat{\mathbf{k}} = \eta H(\mathbf{r}) (0, 1, 0) \times \left(-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}} \right)$$

$$= 1.800\pi e^{0.01(x+z)} e^{j\frac{10}{3}(x+z)} e^{-j\frac{3\pi}{4}} (-\hat{\mathbf{x}} + \hat{\mathbf{z}}) \text{ [V/m]}$$

(c)

$$\alpha = 0.01\sqrt{2} \text{ [Np/m]} = 0.1228 \text{ [dB/m]}$$

$$1\% \text{ power in dB} = 10 \log 0.01 = -20 \text{ dB}$$

$$\text{Distance travelled} = \frac{20 \text{ [dB]}}{0.1228 \text{ [dB/m]}} = 162.8 \text{ [m]}$$

3. (16 Marks) A 60 GHz LHC plane wave travels the +x-direction and has $\mathbf{E} = 1 \text{ [V/m]} \hat{\mathbf{y}}$ at $t = 0, \mathbf{r} = (x, y, z) = (33, 23, 13) \text{ [mm]}$. Its wavelength is 2 mm
- (a) Find the wavenumber and the refractive index (3)
 - (b) Write the electric field in time domain (5)
 - (c) Plot the time evolution of the electric field (5)
 - (d) At $t = 4.17 \text{ ps}$, to which directions will the electric and magnetic fields point? (3)

If you wish, you may solve for a different version of the question, where $\mathbf{k}$ points at the z-direction. **Please indicate clearly if you wish to do so by saying “Answer alternative question” – by doing so you will forfeit the original marking scheme.** Instead you will get a maximum score of 3, 3, 3 and 2 for Parts a-d respectively.

Solution:

(a)

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{0.002} = 1000\pi \text{ [rad/m]}$$

$$k = \frac{\omega}{v_p} = \frac{\omega}{c} n \Rightarrow n = \frac{ck}{\omega} = \frac{(3 \times 10^8)(1000\pi)}{2\pi \times 60 \times 10^9} = 2.5$$

- (b) The polarization plot is drawn as on the right, with $\{Y, Z, X\}$ forming a right-handed triplet. (This is transformed from the usual $\{X, Y, Z\}$ co-ordinate triplet covered in class. Basically, Y takes the place of X, Z takes the place of Y, etc.) The +x-direction points out of the page as this is the wave's travel direction.

Under this context we write:

$$\mathbf{E}(\mathbf{r}, t) = a_y \cos(\omega t - kx + \phi) \hat{\mathbf{y}} + a_z \cos(\omega t - kx + \phi + \delta) \hat{\mathbf{z}}$$

Because we deal with an LHC wave,

$$a_y = a_z = 1; \delta = \frac{\pi}{2}.$$

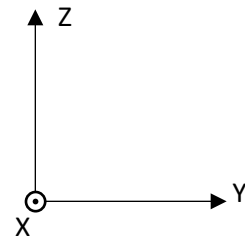
At $t = 0$ and $x = 33 \text{ mm}$, $\mathbf{E}$ is fully in the +y-direction, so

$$\omega t - kx + \phi = 0$$

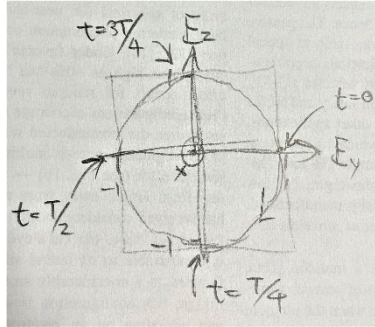
$$\Rightarrow \phi = kx = (1000\pi)(0.033) = 33\pi = \pi$$

Therefore,

$$\mathbf{E}(\mathbf{r}, t) = -\cos(1.2\pi \times 10^{11}t - 1000\pi y) \hat{\mathbf{y}} + \cos\left(1.2\pi \times 10^{11}t - 1000\pi y - \frac{\pi}{2}\right) \hat{\mathbf{z}}$$



(c)



(d) At $t_1 = 4.17 \text{ ps}$, $\omega t_1 = (1.2\pi \times 10^{11})(4.17 \times 10^{-12}) \cong 0.5\pi = T/4$.

Hence the $\mathbf{E}$ -vector has travelled a quarter of a cycle / period from its point at $t = 0$.
From the plot, $\mathbf{E}$ points in the $-z$ -direction.

Using the right hand rule on the known directions of $\mathbf{E}$ and $\mathbf{k}$, we find that $\mathbf{H}$ points in the $+y$ -direction.