

EE3101 Tutorial 9 Solutions

$$(1) \quad a(t) = At^2 \rightarrow a(t) = \frac{dv(t)}{dt}$$

$$\text{i.e.} \quad \int_{t=0}^{t=t} dv(t) = \int_{t=0}^{t=t} At^2 dt = \frac{A}{3}t^3$$

$$v(t) = \frac{A}{3}t^3 + v(0) \quad \text{where } v(0) \text{ is the initial relative velocity at } t = 0$$

The Doppler frequency shift is given by:

$$\delta\omega = \frac{\omega_0}{c}v(t) = \frac{\omega_0}{c}\left[\frac{A}{3}t^3 + v(0)\right]$$

The corresponding expression for the time varying phase shift will be:

$$\delta\omega = \frac{d\theta(t)}{dt} \rightarrow \int_{t=0}^{t=t} d\theta(t) = \int_{t=0}^{t=t} \delta\omega dt = \frac{\omega_0}{c} \int_{t=0}^{t=t} \left[\frac{A}{3}t^3 + v(0)\right] dt$$

$$\text{i.e.} \quad \theta(t) - \theta(0) = \frac{\omega_0}{c} \left[\frac{A}{12}t^4 + v(0)t\right]$$

Since the receiver is initially in phase lock, i.e. $\theta(0) = 0$.

$$\text{The Fourier transform of } \theta(t) \text{ is: } \Theta(\omega) = \frac{\omega_0}{c} \left[\frac{2A}{(j\omega)^5} + \frac{v(0)}{(j\omega)^2} \right]$$

From equation (14) of chapter 3 lecture notes,

$$\lim_{t \rightarrow \infty} e(t) = \lim_{j\omega \rightarrow 0} \frac{(j\omega)^2}{j\omega + K_0 F(\omega)} \Theta(\omega) = \lim_{j\omega \rightarrow 0} \frac{\omega_0}{c} \frac{\frac{2A}{(j\omega)^3} + v(0)}{j\omega + K_0 F(\omega)}$$

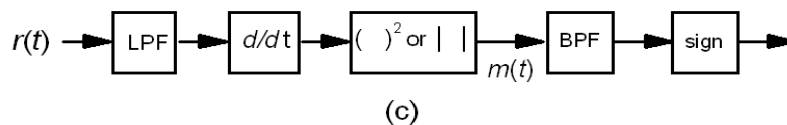
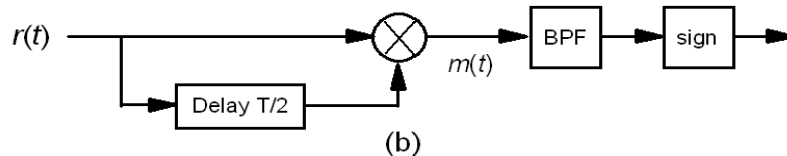
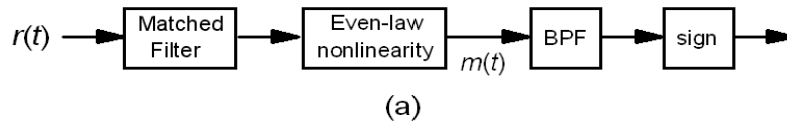
In order to maintain frequency lock, $F(\omega)$ must be at least 3rd order,

and in order to maintain phase lock, $F(\omega)$ must be at least 4th order.

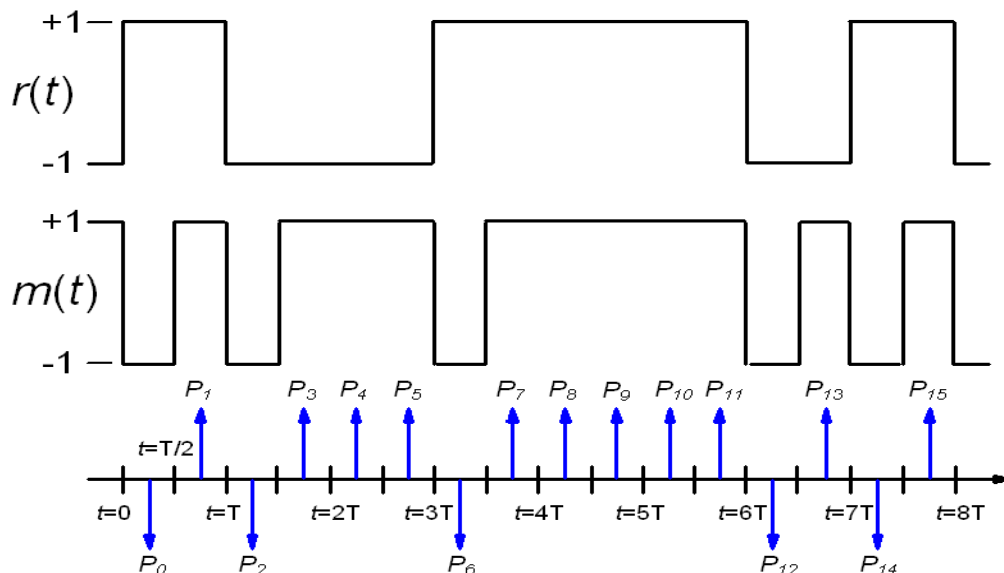
The corresponding PLL should have at least 4th and 5th order respectively.

(2) Both (a) & (b) will have a very low level signal output and may not be very stable because the PSD of them have a null at the symbol frequency.

(3)



In order to have a better understanding of the characteristics of $m(t)$, we draw the waveform of $m(t)$ for input sequence of 1000111101. $m(t)$ will be a sequence of rectangular pulse with $T/2$ duration. According to the current time index, it is always a positive pulse when time index is an odd number because both $r(t)$ and $r(t-T/2)$ have the same value.



Note: $1 \times 1 = 1$; $(-1) \times (-1) = 1$; $1 \times (-1) = -1$; $(-1) \times (1) = -1$

If i =odd number P_i is always positive;

If i =even number, P_i has the 50% positive and 50% negative.

$$\begin{aligned}
R_0 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{i=-N}^N P_i P_i = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left[\sum_{i \text{ is odd}} P_i P_i + \sum_{i \text{ is even}} P_i P_i \right] \\
&= \frac{1}{2} (1)^2 + \frac{1}{2} [(1)^2 \frac{1}{2} + (-1)^2 \frac{1}{2}] \\
&= 1
\end{aligned}$$

$$\begin{aligned}
R_1 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{i=-N}^N P_i P_{i+1} = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left[\sum_{i \text{ is odd}} P_i P_{i+1} + \sum_{i \text{ is even}} P_i P_{i+1} \right] \\
&= \frac{1}{2} [(1)(-1) \frac{1}{2} + (1)(1) \frac{1}{2}] + \frac{1}{2} [(-1)(1) \frac{1}{2} + (1)(1) \frac{1}{2}] \\
&= 0
\end{aligned}$$

Similarly, $R_r = R_{-r} = 0$ when $r = \text{odd number}$

$$\begin{aligned}
R_2 &= \lim_{N \rightarrow \infty} \frac{1}{2N+1} \sum_{i=-N}^N P_i P_{i+2} = \lim_{N \rightarrow \infty} \frac{1}{2N+1} \left[\sum_{i \text{ is odd}} P_i P_{i+2} + \sum_{i \text{ is even}} P_i P_{i+2} \right] \\
&= \frac{1}{2} [(1)(1)] + \frac{1}{2} [(-1)(1) \frac{1}{4} + (-1)(-1) \frac{1}{4} + (1)(1) \frac{1}{4} + (1)(-1) \frac{1}{4}] \\
&= \frac{1}{2}
\end{aligned}$$

Similar, $R_r = R_{-r} = \frac{1}{2}$ when $r = \text{even number}$

$$H(\omega) = T' \text{Sa}\left(\frac{\omega T'}{2}\right) \quad T' = T/2$$

$$P(\omega) = \frac{1}{T'} \sum_{r=-\infty}^{\infty} R_r \exp(-jr\omega T')$$

$$\begin{aligned}
\text{i.e. } P(\omega) &= \frac{1}{2T'} + \frac{1}{2T'} \sum_{r=-\infty}^{\infty} \exp(-j2r\omega T') \\
&= \frac{1}{T} + \frac{1}{T} \sum_{r=-\infty}^{\infty} \exp(-jr\omega T)
\end{aligned}$$

$$\text{as } \sum_{r=-\infty}^{\infty} \exp(-jr\omega T) = \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_T) \quad \text{where } \omega_T = \frac{2\pi}{T}$$

$$\Rightarrow P(\omega) = \frac{1}{T} + \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_T)$$

$$\begin{aligned}
 S(\omega) &= \left[\frac{1}{T} + \frac{1}{T} \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_T) \right] \left[\left(\frac{T}{2}\right) \text{Sa}\left(\frac{\omega T}{4}\right) \right]^2 \\
 &= \left[1 + \sum_{m=-\infty}^{\infty} \delta(\omega - m\omega_T) \right] \frac{T}{4} \text{Sa}^2\left(\frac{\omega T}{4}\right)
 \end{aligned}$$

$S(\omega)$ has a peak at $\omega = \omega_T$