

MA2001 Differential Calculus of Multivariable Functions

1 Introduction

Quantities of interest in real problems usually depend on more than one variable.

Examples

- (a) The height of the ground at map coordinates (x, y) might be expressed mathematically as

$$\text{Height} = h(x, y).$$

- (b) The temperature in a lecture room varies with position and time, Temperature = $T(x, y, z, t)$.

- (c) The volume of a rectangular box of length x , width y and height z is $V(x, y, z) = xyz$.

- (d) The magnitude of the electric potential $u(x, y, z)$ due to a uniformly charged sphere with total

$$\text{charge } Q \text{ is } u(x, y, z) = \frac{Q}{4\pi\epsilon_0(x^2 + y^2 + z^2)}, \quad x^2 + y^2 + z^2 > a^2, \text{ where } a \text{ is the radius of the sphere.}$$

□

These are all real-valued functions of real variables. In general we have $z = f(x_1, x_2, \dots, x_n) \equiv f(\vec{x}^T)$,

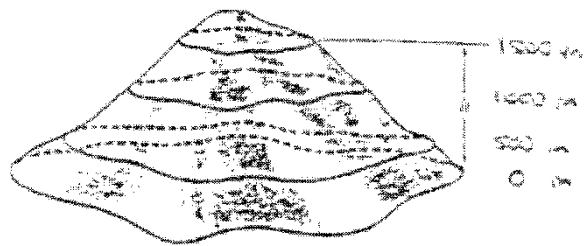
where $x_1, x_2, \dots, x_n$ are the independent variables and z is the dependent variable. $f(\vec{x}^T)$ is a scalar field.

It is possible to visualize a function of two variables (a surface in 3-D space) by drawing its graph. One way is to draw a contour map. A level curve of $f(x, y)$ is a set of points (x, y) for which $f(x, y) = c$, where c is a constant.

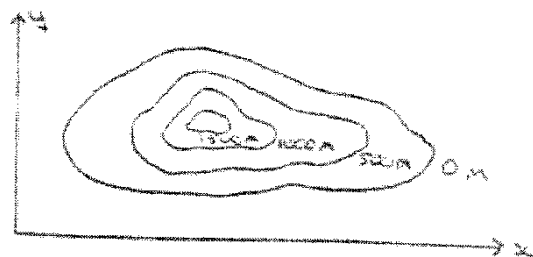
A collection of level curves then forms a contour map (as in the familiar contour maps produced by map makers).

Examples

- (a)

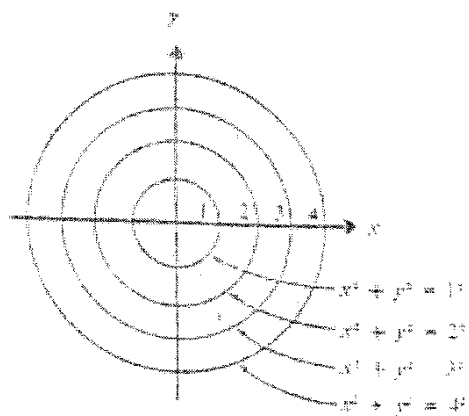


Level curves of a function are defined in the same manner as one obtains contour lines for a topographical map.



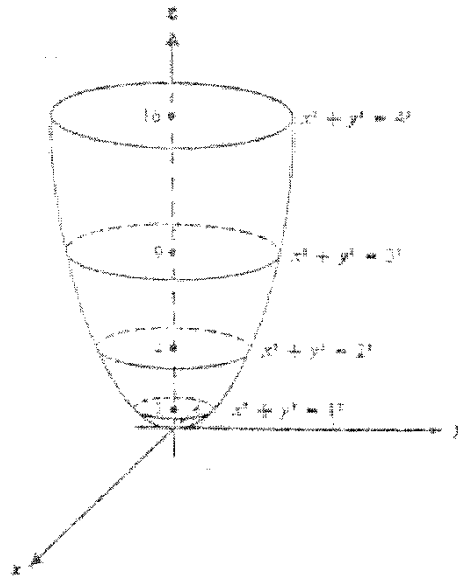
Contour map

(b)



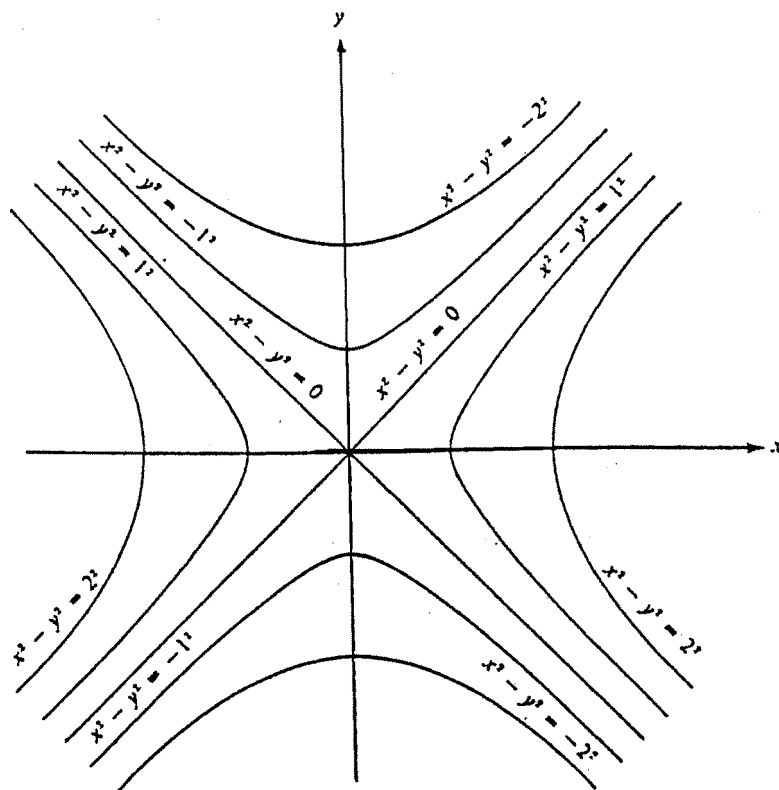
Contour map

Same level curves for the function $f(x, y) = x^2 + y^2$

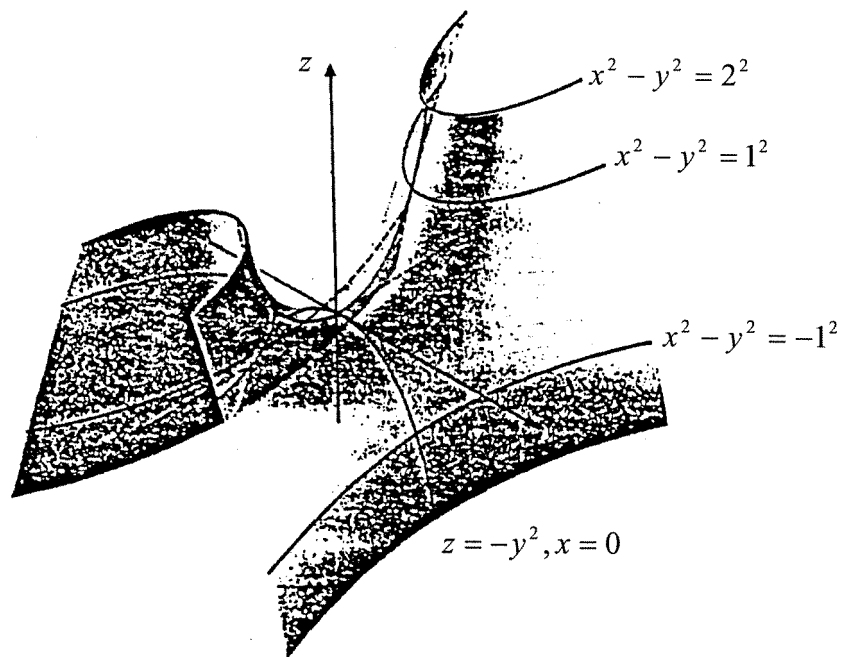


Level curves shown on the graph of $f(x, y) = x^2 + y^2$

(c)



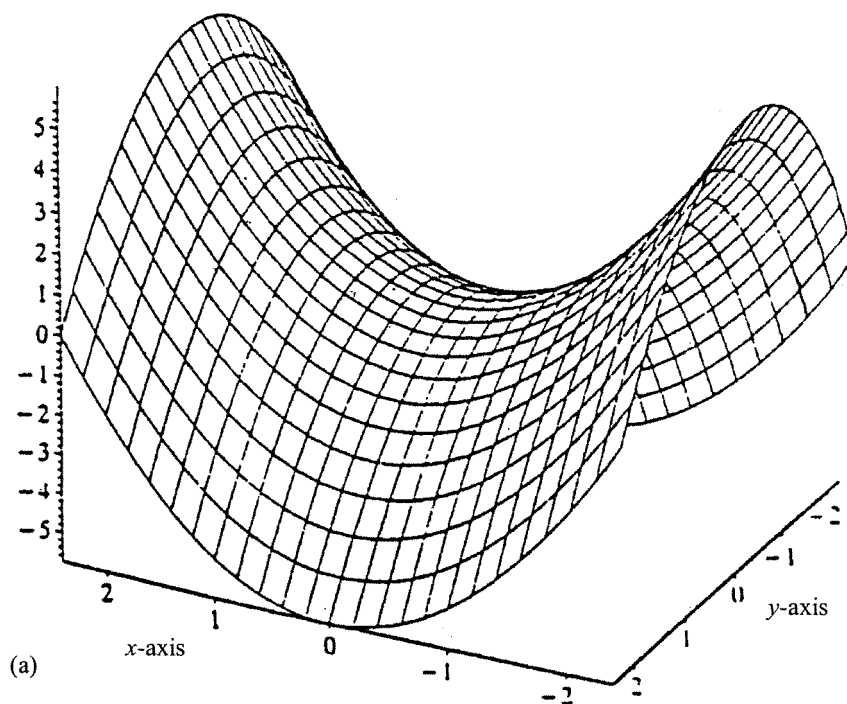
Level curves for the function $f(x, y) = x^2 - y^2$

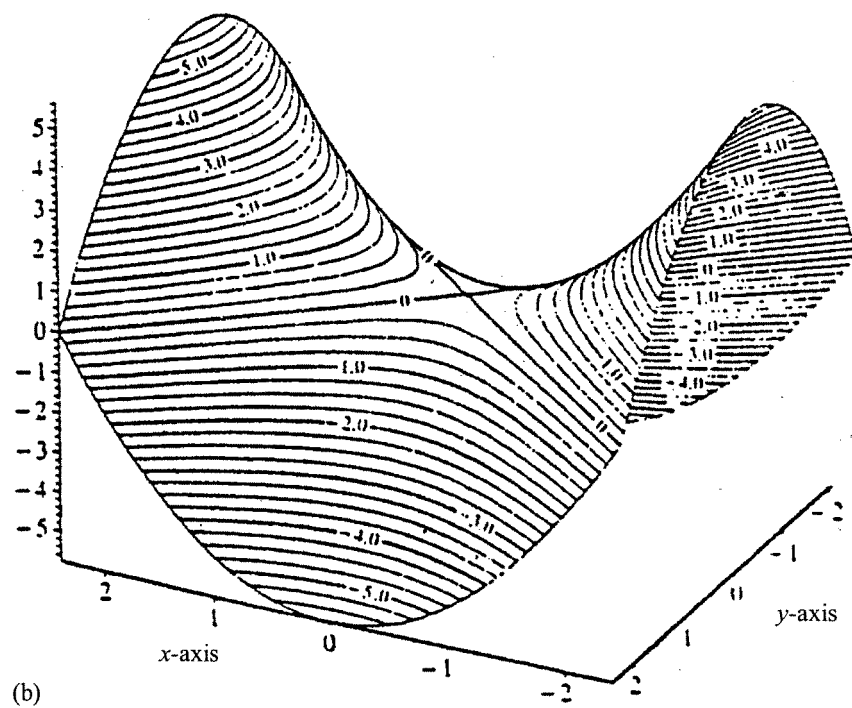


Some level curves on the graph of $f(x, y) = x^2 - y^2$

Computer generated graphs usually show lines for which x is constant and lines for which y is constant in order to give a picture of the surface.

Examples





Part (a) shows a computer-generated graph of $z = x^2 - y^2$. Part (b) shows this graph with level curves lifted to it.

□

2 Limit, Continuity and Partial Derivative

The concepts of limit, continuity and derivative naturally generalize from the one variable situation to two or more variables.

A function $f(x, y)$ is said to possess a limit f_0 at the point (x_0, y_0) if for any $\varepsilon > 0$ it is possible to

find a $\delta > 0$ such that whenever $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$, that is, (x, y) is in disc of radius δ

centered at (x_0, y_0) , we have $|f(x, y) - f_0| < \varepsilon$. We write $\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f_0$ or

$$\lim_{\substack{x \rightarrow x_0 \\ y \rightarrow y_0}} f(x, y) = f_0.$$

Intuitively, the above says that when (x, y) is approaching (x_0, y_0) , $f(x, y)$ is approaching f_0 then the function $f(x, y)$ is said to possess a limit at the point (x_0, y_0) .

Note that the limit exists only if, $f(x, y)$ is approaching f_0 independently of the manner in which (x, y) is approaching (x_0, y_0) .

Example

$$\lim_{(x, y) \rightarrow (1, 2)} (x^2 + 3y^2 + xy) = 1^2 + 3 \times 2^2 + 1 \times 2 = 15.$$

□

Example

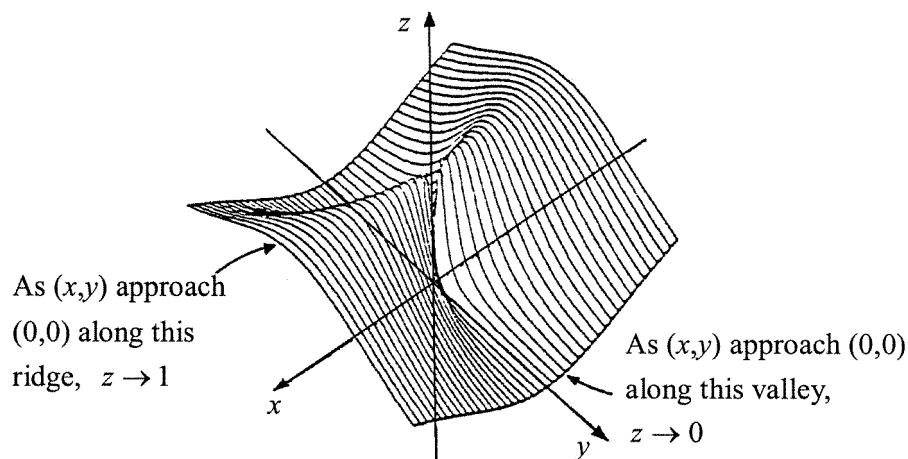
Show that $f(x, y) = \frac{x^2}{x^2 + y^2}$ has no limit at the origin.

Solution:

As $(x, y) \rightarrow (0, 0)$ along the line $y = 0$, $f(x, 0) = \frac{x^2}{x^2} = 1$ and the limiting value is 1.

As $(x, y) \rightarrow (0, 0)$ along the line $x = 0$, $f(0, y) = 0$ and the limiting value is 0.

Hence $\lim_{(x, y) \rightarrow (0, 0)} \frac{x^2}{x^2 + y^2}$ does not exist.



$$z = \frac{x^2}{x^2 + y^2}, \quad -1 \leq x \leq 1, \quad -1 \leq y \leq 1$$

□

Example

Show $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} = 0$

Solution:

Now,

$$0 \leq \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} \leq \frac{x^2 + y^2}{(x^2 + y^2)^{\frac{1}{2}}} = (x^2 + y^2)^{\frac{1}{2}}.$$

We observe that $\lim_{(x,y) \rightarrow (0,0)} (x^2 + y^2)^{\frac{1}{2}} = 0$. Therefore, $\lim_{(x,y) \rightarrow (0,0)} \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} = 0$

□

A function $f(x, y)$ is said to be continuous at the point (x_0, y_0) if and only if it possesses a limit at

(x_0, y_0) and $\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

Examples

(a) $f(x, y) = x^2 + 3y^2 + xy$ is continuous everywhere.

Solution:

$$\lim_{(x,y) \rightarrow (x_0, y_0)} f(x, y) = \lim_{(x,y) \rightarrow (x_0, y_0)} (x^2 + 3y^2 + xy) = x_0^2 + 3y_0^2 + x_0 y_0.$$

(b) $f(x, y) = \frac{x^2}{x^2 + y^2}$ is continuous everywhere except at the origin where $f(x, y)$ is not defined.

Solution:

If $(x_0, y_0) \neq (0, 0)$ then $f(x_0, y_0)$ is defined at (x_0, y_0) and $f(x_0, y_0) = \frac{x_0^2}{x_0^2 + y_0^2}$. In addition,

$$\lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = \lim_{(x, y) \rightarrow (x_0, y_0)} \frac{x^2}{x^2 + y^2} = \frac{x_0^2}{x_0^2 + y_0^2} = f(x_0, y_0).$$

$$(c) \quad f(x, y) = \begin{cases} \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} & (x, y) \neq (0, 0) \\ A & (x, y) = (0, 0) \end{cases} \text{ is continuous everywhere iff } A = 0.$$

Solution:

$$\text{Since } \lim_{(x, y) \rightarrow (0, 0)} \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} = 0, \quad f(x, y) = \begin{cases} \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}} & (x, y) \neq (0, 0) \\ A & (x, y) = (0, 0) \end{cases} \text{ is continuous everywhere iff}$$

$$A = 0.$$

□

Example

Give an example of a function $f(x, y)$ such that $f(x, y)$ has the same limit as $f(0, 0)$ when (x, y) approaches $(0, 0)$ along any straight line passing through the origin, however, the function is not continuous at $(0, 0)$.

Solution:

Consider the function:

$$f(x, y) = \begin{cases} \frac{2xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}.$$

Suppose (x, y) is approaching $(0, 0)$ along the line $x = t \cos \theta, y = t \sin \theta$ where θ is an angle made counterclockwise from the positive direction of x -axis to the line $x = t \cos \theta, y = t \sin \theta$ and t is any real number.

Then if $\cos \theta \neq 0$

$$\begin{aligned} \lim_{\substack{(x,y) \rightarrow (0,0) \\ x=t \cos \theta \\ y=t \sin \theta}} f(x,y) &= \lim_{t \rightarrow 0} f(t \cos \theta, t \sin \theta) = \lim_{t \rightarrow 0} \frac{2t^3 \cos \theta \sin^2 \theta}{t^2 \cos^2 \theta + t^4 \sin^4 \theta} = \lim_{t \rightarrow 0} \frac{\frac{2t^3 \cos \theta \sin^2 \theta}{t^2}}{\frac{t^2 \cos^2 \theta + t^4 \sin^4 \theta}{t^2}} \\ &= \lim_{t \rightarrow 0} \frac{2t \cos \theta \sin^2 \theta}{\cos^2 \theta + t^2 \sin^4 \theta} = \frac{0}{\cos^2 \theta} = 0 = f(0,0) \end{aligned}$$

If $\cos \theta = 0$, then this is the case that (x, y) is approaching $(0, 0)$ along y -axis.

It is obvious that $\lim_{\substack{(x,y) \rightarrow (0,0) \\ x=0}} f(x,y) = \lim_{y \rightarrow 0} f(0,y) = \lim_{y \rightarrow 0} 0 = 0 = f(0,0)$.

However, if (x, y) is approaching $(0, 0)$ along the curve $x = t^2, y = t$ where t is any real number, we

$$\text{have } \lim_{\substack{(x,y) \rightarrow (0,0) \\ x=t^2 \\ y=t}} f(x,y) = \lim_{t \rightarrow 0} f(t^2, t) = \lim_{t \rightarrow 0} \frac{2t^4}{t^4 + t^4} = 1 \neq 0 = f(0,0).$$

□

3 Partial Differentiation

Recall that for a function $f(x)$ of one variable

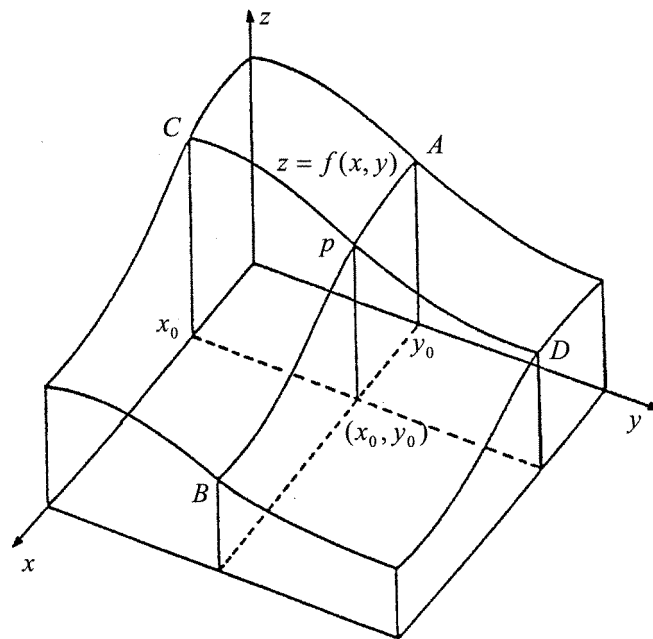
$$\frac{df}{dx}(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{or} \quad \frac{df}{dx}(x_0) = \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

Now consider the function $f(x, y)$ of two variables. If y is held constant, say $y = y_0$, then $f(x, y_0)$ becomes a function of one variable, x and we may define its derivative at x_0 , the partial derivative of f with respect to x at the point (x_0, y_0) as

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \quad \text{or} \quad \frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \rightarrow x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0}$$

Similarly, the partial derivative of f with respect to y at the point (x_0, y_0) as

$$\frac{\partial f}{\partial y}(x_0, y_0) = \lim_{k \rightarrow 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} \quad \text{or} \quad \frac{\partial f}{\partial y}(x_0, y_0) = \lim_{y \rightarrow y_0} \frac{f(x_0, y) - f(x_0, y_0)}{y - y_0}$$



AB – curve of $f(x, y_0)$

CD – curve of $f(x_0, y)$

Note:

- (a) A partial derivative is just an ordinary derivative with respect to the variable which is not held constant.
(b) For $z = f(x, y)$, alternative notations are:

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \frac{\partial z}{\partial x}(x, y) = z_x(x, y)$$

and similarly for $\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \frac{\partial z}{\partial y}(x, y) = z_y(x, y)$.

- (c) The definition of a partial derivative only requires the idea of a limit of a function of a single variable.

Example

Find the first partial derivatives of $f(x, y) = x^2y + y^3 \sin x$.

Solution:

$$f(x, y) = x^2y + y^3 \sin x \Rightarrow \frac{\partial f}{\partial x} = 2xy + y^3 \cos x, \quad \frac{\partial f}{\partial y} = x^2 + 3y^2 \sin x$$

□

Example

Evaluate $\frac{\partial f}{\partial x}(1, 2)$ for $f(x, y) = x^2y + y^3 \sin x$.

Solution:

Method 1:

Consider $f(x, 2) = 2x^2 + 8 \sin x$. Then $\frac{\partial f}{\partial x}(1, 2) = \frac{d}{dx}(2x^2 + 8 \sin x)|_{x=1} = (4x + 8 \cos x)|_{x=1} = 4 + 8 \cos 1$.

Method 2:

$$f(x, y) = x^2y + y^3 \sin x \Rightarrow \frac{\partial f}{\partial x} = 2xy + y^3 \cos x. \text{ Then } \frac{\partial f}{\partial x}(1, 2) = (2xy + y^3 \cos x) \Big|_{\substack{x=1 \\ y=2}} = 4 + 8 \cos 1.$$

□

Example

Find the first partial derivatives of $f(x, y, z) = e^{2z} \cos xy$.

Solution:

$$f(x, y, z) = e^{2z} \cos xy \Rightarrow \frac{\partial f}{\partial x} = -e^{2z} y \sin xy, \quad \frac{\partial f}{\partial y} = -e^{2z} x \sin xy, \quad \frac{\partial f}{\partial z} = 2e^{2z} \cos xy.$$

□

We may differentiate the first partial derivatives of a function again to obtain second partial derivatives.

$$f_{xx} = \frac{\partial^2 f}{\partial x \partial x} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right), f_{yy} = \frac{\partial^2 f}{\partial y \partial y} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right), f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right), f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right).$$

Higher order partial derivatives may be similarly defined.

Example

Let $f(x, y) = \log(x^2 + y^2)^{\frac{1}{2}}$, find its partial derivatives up to order two.

Solution:

$$\begin{aligned} f(x, y) &= \log(x^2 + y^2)^{\frac{1}{2}} \Rightarrow f_x = \frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2}, f_y = \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2} \\ \Rightarrow f_{xx} &= \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \\ , f_{yy} &= \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{x^2 - y^2}{(x^2 + y^2)^2}, f_{xy} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) = \frac{-2xy}{(x^2 + y^2)^2}, \\ f_{yx} &= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{y}{x^2 + y^2} \right) = \frac{-2xy}{(x^2 + y^2)^2} \end{aligned}$$

□

Note that in the above example $f_{xy} = f_{yx}$, which is the case for most functions that are useful in engineering and science. However, it is not always true.

Example

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

- If $(x, y) \neq (0, 0)$, find $f_{xy}(x, y)$, $f_{yx}(x, y)$ and verify that they are equal.
- Show that $f_x(0, 0) = f_y(0, 0)$.
- Show that $f_{xy}(0, 0)$, $f_{yx}(0, 0)$ exist but they are not equal.

Solution:

(a)

There is a theorem saying that if $f_{xy}(x, y)$ & $f_{yx}(x, y)$ exist at (x_0, y_0) and nearby and

$f_{xy}(x, y)$ & $f_{yx}(x, y)$ both are continuous at (x_0, y_0) then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.

Now, if $(x, y) \neq (0, 0)$

$$f_{xy}(x, y) = \frac{\partial}{\partial y} \left(\frac{\partial}{\partial x} \left[\frac{xy(x^2 - y^2)}{x^2 + y^2} \right] \right) = \frac{\partial}{\partial y} \left[\frac{4x^2y^3 + x^4y - y^5}{(x^2 + y^2)^2} \right] = \frac{-y^6 - 9x^2y^4 + 9x^4y^2 + x^6}{(x^2 + y^2)^3}$$

$$f_{yx}(x, y) = \frac{\partial}{\partial x} \left[\frac{-4y^2x^3 - y^4x + x^5}{(x^2 + y^2)^2} \right] = \frac{-y^6 - 9x^2y^4 + 9x^4y^2 + x^6}{(x^2 + y^2)^3}$$

So $f_{xy}(x, y) = f_{yx}(x, y)$

(b)

$$f_x(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0, \quad f_y(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0$$

(c)

$$\text{If } y \neq 0, \quad f_x(0, y) = \lim_{x \rightarrow 0} \frac{f(x, y) - f(0, y)}{x} = \lim_{x \rightarrow 0} \frac{\frac{xy(x^2 - y^2)}{x^2 + y^2} - 0}{x} = \lim_{x \rightarrow 0} \frac{y(x^2 - y^2)}{x^2 + y^2} = -y$$

$$\text{Thus, } f_x(0, y) = \begin{cases} -y & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases} = -y, \forall y$$

$$f_{xy}(0, 0) = \lim_{y \rightarrow 0} \frac{f_x(0, y) - f_x(0, 0)}{y} = \lim_{y \rightarrow 0} \frac{-y - 0}{y} = -1$$

Similarly,

$$\text{If } x \neq 0, \quad f_y(x, 0) = \lim_{y \rightarrow 0} \frac{f(x, y) - f(x, 0)}{y} = \lim_{y \rightarrow 0} \frac{\frac{xy(x^2 - y^2)}{x^2 + y^2} - 0}{y} = \lim_{y \rightarrow 0} \frac{x(x^2 - y^2)}{x^2 + y^2} = x$$

Thus, $f_y(x, 0) = x$

$$f_{yx}(0, 0) = \lim_{x \rightarrow 0} \frac{f_y(x, 0) - f_y(0, 0)}{x} = \lim_{x \rightarrow 0} \frac{x - 0}{x} = 1$$

We find that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$. Why does this happen?

$$\text{For } f_{xy}(x, y) = \begin{cases} \frac{-y^6 - 9x^2y^4 + 9x^4y^2 + x^6}{(x^2 + y^2)^3} & \text{if } (x, y) \neq (0, 0) \\ -1 & \text{if } (x, y) = (0, 0) \end{cases}$$

,we observe that along $y = 0$, $\lim_{(x,y) \rightarrow (0,0)} f_{xy}(x, y) = \lim_{x \rightarrow 0} f_{xy}(x, 0) = 1$.

However, along $x = 0$, $\lim_{(x,y) \rightarrow (0,0)} f_{xy}(x, y) = \lim_{y \rightarrow 0} f_{xy}(0, y) = -1$.

It concludes that $f_{xy}(x, y)$ is not continuous at $(0, 0)$.

□

4 The Chain Rule

For functions of one variable, if $y = f(x)$ and $x = x(t)$, then $y = f[x(t)]$ and $\frac{dy}{dt} = \frac{df}{dx} \frac{dx}{dt}$.

If $f = f(u)$ and $u = u(x, y)$ then $\frac{\partial f}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x}$ and $\frac{\partial f}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y}$.

Example

If $f = f(u) = \log u$ and $u = (x^2 + y^2)^{\frac{1}{2}}$, evaluate $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{df}{du} \frac{\partial u}{\partial x} = \frac{1}{u} \times \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \times 2x = \frac{1}{(x^2 + y^2)^{\frac{1}{2}}} \times \frac{1}{2} (x^2 + y^2)^{-\frac{1}{2}} \times 2x = \frac{x}{x^2 + y^2}.$$

$$\text{And } \frac{\partial f}{\partial y} = \frac{df}{du} \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}.$$

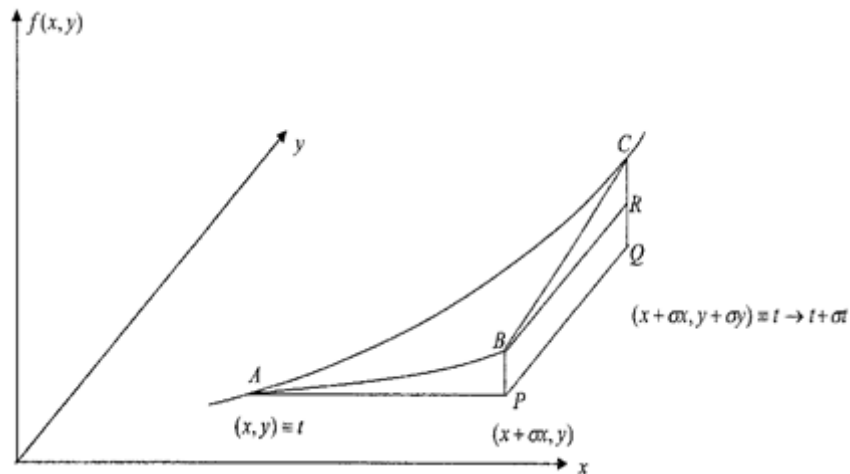
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For a function of two variables we have

Theorem

If $z = f(x, y)$ and $x = x(t), y = y(t)$ then $\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$.

Proof:



$$\frac{dz}{dt} = \lim_{\delta t \rightarrow 0} \frac{z(t + \delta t) - z(t)}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{f[x(t + \delta t), y(t + \delta t)] - f[x(t), y(t)]}{\delta t}.$$

Now as t change to $t + \delta t$, x and y change by δx and δy respectively, where $\delta x = x(t + \delta t) - x(t)$,

$\delta y = y(t + \delta t) - y(t)$. Thus the change in f may be expressed as

$$f[x(t + \delta t), y(t + \delta t)] - f[x(t), y(t)] = f(x + \delta x, y + \delta y) - f(x, y)$$

$$= QC = QR + RC = PB + RC$$

$$= [f(x + \delta x, y) - f(x, y)] + [f(x + \delta x, y + \delta y) - f(x + \delta x, y)]$$

, and

$$\begin{aligned} \frac{dz}{dt} &= \lim_{\delta t \rightarrow 0} \frac{z(t + \delta t) - z(t)}{\delta t} = \lim_{\delta t \rightarrow 0} \frac{f[x(t + \delta t), y(t + \delta t)] - f[x(t), y(t)]}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \frac{f(x + \delta x, y) - f(x, y) + f(x + \delta x, y + \delta y) - f(x + \delta x, y)}{\delta t} \\ &= \lim_{\delta t \rightarrow 0} \left[\frac{f(x + \delta x, y) - f(x, y)}{\delta x} \frac{\delta x}{\delta t} + \frac{f(x + \delta x, y + \delta y) - f(x + \delta x, y)}{\delta y} \frac{\delta y}{\delta t} \right] \\ &= \lim_{\delta x \rightarrow 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x} \lim_{\delta t \rightarrow 0} \frac{\delta x}{\delta t} + \lim_{\delta y \rightarrow 0} \frac{f(x + \delta x, y + \delta y) - f(x + \delta x, y)}{\delta y} \lim_{\delta t \rightarrow 0} \frac{\delta y}{\delta t} \\ &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \end{aligned}$$

Example

Let $z = f(x, y) = x^2 + 2xy$ and $x = \sin t$ and $y = \cos t$, find $\frac{dz}{dt}$.

Solution:

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} = (2x + 2y) \cos t + 2x(-\sin t) = (2 \sin t + 2 \cos t) \cos t + 2 \sin t(-\sin t) \\ &= \sin 2t + 2 \cos 2t \end{aligned}$$

This may be confirmed by direct substitution:

$$z = f(\sin t, \cos t) = \sin^2 t + 2 \sin t \cos t \Rightarrow \frac{dz}{dt} = 2 \sin t \cos t - 2 \sin^2 t + 2 \cos^2 t = \sin 2t + 2 \cos 2t$$

□

A special case is when $z = f(x, y)$ and $y = y(x)$, that is, $x = x$ and hence z is a function of x only.

$$\text{Then } \frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}.$$

Example

$$z = f(x, y) = \tan^{-1}\left(\frac{x}{y}\right), y = \sin x, \text{ find } \frac{dz}{dx}.$$

Solution:

$$\frac{dz}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx} = \frac{y}{x^2 + y^2} - \frac{x}{x^2 + y^2} \cos x = \frac{\sin x}{x^2 + \sin^2 x} - \frac{x \cos x}{x^2 + \sin^2 x} = \frac{\sin x - x \cos x}{x^2 + \sin^2 x}.$$

□

Example (Chain rule involving second derivatives)

$$\text{Let } z = f(x, y), x = x(t), y = y(t), \text{ evaluate } \frac{d^2 z}{dt^2}.$$

Solution:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}.$$

$$\frac{d^2 z}{dt^2} = \frac{d}{dt} \left(\frac{dz}{dt} \right) = \frac{d}{dt} \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) = \frac{\partial f}{\partial x} \frac{d^2 x}{dt^2} + \frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{d^2 y}{dt^2} + \frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) \frac{dy}{dt}.$$

We observe that $\frac{\partial f}{\partial x}$ is depending on x and y directly, depending indirectly on t through x and y respectively.

To evaluate $\frac{d}{dt} \left(\frac{\partial f}{\partial x} \right)$ we need the Chain Rule and we have

$$\frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \frac{dx}{dt} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \frac{dy}{dt} = \frac{\partial^2 f}{\partial x^2} \frac{dx}{dt} + \frac{\partial^2 f}{\partial y \partial x} \frac{dy}{dt}.$$

$$\text{For the same reason, } \frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \frac{dx}{dt} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \frac{dy}{dt} = \frac{\partial^2 f}{\partial x \partial y} \frac{dx}{dt} + \frac{\partial^2 f}{\partial y^2} \frac{dy}{dt}.$$

Then it follows that

$$\begin{aligned} \frac{d^2 z}{dt^2} &= \frac{\partial f}{\partial x} \frac{d^2 x}{dt^2} + \frac{d}{dt} \left(\frac{\partial f}{\partial x} \right) \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{d^2 y}{dt^2} + \frac{d}{dt} \left(\frac{\partial f}{\partial y} \right) \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \frac{d^2 x}{dt^2} + \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \frac{dx}{dt} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \frac{dy}{dt} \right] \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{d^2 y}{dt^2} + \left[\frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \frac{dx}{dt} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \frac{dy}{dt} \right] \frac{dy}{dt} \\ &= \frac{\partial f}{\partial x} \frac{d^2 x}{dt^2} + \frac{\partial^2 f}{\partial x^2} \left(\frac{dx}{dt} \right)^2 + \frac{\partial^2 f}{\partial y \partial x} \frac{dy}{dt} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{d^2 y}{dt^2} + \frac{\partial^2 f}{\partial x \partial y} \frac{dx}{dt} \frac{dy}{dt} + \frac{\partial^2 f}{\partial y^2} \left(\frac{dy}{dt} \right)^2 \end{aligned}$$

□

Example

$z = f(x, y), x = t^2, y = t$. Suppose

$$\frac{\partial f}{\partial x}(1,1) = \frac{\partial f}{\partial y}(1,1) = \frac{\partial^2 f}{\partial x^2}(1,1) = \frac{\partial^2 f}{\partial y^2}(1,1) = e, \frac{\partial^2 f}{\partial x \partial y}(1,1) = \frac{\partial^2 f}{\partial y \partial x}(1,1) = 2e. \text{ Evaluate } \frac{d^2 z}{dt^2} \text{ for } t = 1$$

Solution:

$$x = t^2 \Rightarrow \frac{dx}{dt} = 2t, \frac{d^2 x}{dt^2} = 2, y = t \Rightarrow \frac{dy}{dt} = 1, \frac{d^2 y}{dt^2} = 0. \text{ If } t = 1, \text{ then } x = 1, y = 1.$$

So

$$\begin{aligned} \frac{d^2 z}{dt^2} \Big|_{t=1} &= \frac{\partial f}{\partial x}(1,1) \frac{d^2 x}{dt^2} \Big|_{t=1} + \frac{\partial^2 f}{\partial x^2}(1,1) \left(\frac{dx}{dt} \right)^2 \Big|_{t=1} + \frac{\partial^2 f}{\partial y \partial x}(1,1) \frac{dy}{dt} \Big|_{t=1} \frac{dx}{dt} \Big|_{t=1} \\ &+ \frac{\partial f}{\partial y}(1,1) \frac{d^2 y}{dt^2} \Big|_{t=1} + \frac{\partial^2 f}{\partial x \partial y}(1,1) \frac{dx}{dt} \Big|_{t=1} \frac{dy}{dt} \Big|_{t=1} + \frac{\partial^2 f}{\partial y^2}(1,1) \left(\frac{dy}{dt} \right)^2 \Big|_{t=1} \\ &= e \times 2 + e \times 4 + 2e \times 1 \times 2 + e \times 0 + 2e \times 2 \times 1 + e \times 1 = 15e \end{aligned}$$

□

Theorem (Change of Variable)

If $z = f(x, y)$ and $x = x(s, t), y = y(s, t)$, find $\frac{\partial z}{\partial s}, \frac{\partial z}{\partial t}$.

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.$$

□

Example

If $z = f(x, y) = e^{2xy}$ and there is the change of variables $x = r \cos \theta, y = r \sin \theta$, find $\frac{\partial z}{\partial r}, \frac{\partial z}{\partial \theta}$.

Solution:

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = 2ye^{2xy} \cos \theta + 2xe^{2xy} \sin \theta = 2re^{r^2 \sin 2\theta} \sin 2\theta \text{ and}$$

$$\begin{aligned} \frac{\partial z}{\partial \theta} &= \frac{\partial f}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial \theta} = 2ye^{2xy}(-r \sin \theta) + 2xe^{2xy}r \cos \theta \\ &= 2r^2 e^{r^2 \sin 2\theta} \cos 2\theta. \end{aligned}$$

Again these results may be checked by substitution and direct partial differentiation.

□

Example (Chain rule involving second partial derivatives)

Let $z = f(x, y)$, $x = x(s, t)$, $y = y(s, t)$, evaluate $\frac{\partial^2 z}{\partial t^2}$.

Solution:

$$\frac{\partial z}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}.$$

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial z}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \right) = \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \frac{\partial y}{\partial t} \right) \\ &= \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right) \end{aligned}$$

We observe that both $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ are depending on x and y directly, depending indirectly on t through x and

y respectively. To evaluate $\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right)$, $\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right)$ we need the Chain Rule and we have

$$\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) \frac{\partial y}{\partial t} = \frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial t}$$

$$\frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) \frac{\partial x}{\partial t} + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial t} = \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial t}$$

It follows that

$$\begin{aligned} \frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial x} \right) \frac{\partial x}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial}{\partial t} \left(\frac{\partial x}{\partial t} \right) + \frac{\partial}{\partial t} \left(\frac{\partial f}{\partial y} \right) \frac{\partial y}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial}{\partial t} \left(\frac{\partial y}{\partial t} \right) \\ &= \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial t} \right) \frac{\partial x}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial t^2} + \left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial t} \right) \frac{\partial y}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial t^2} \\ &= \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial t} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial t^2} \\ &= \frac{\partial^2 f}{\partial x^2} \left(\frac{\partial x}{\partial t} \right)^2 + 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial t} \frac{\partial y}{\partial t} + \frac{\partial^2 f}{\partial y^2} \left(\frac{\partial y}{\partial t} \right)^2 + \frac{\partial f}{\partial x} \frac{\partial^2 x}{\partial t^2} + \frac{\partial f}{\partial y} \frac{\partial^2 y}{\partial t^2} \end{aligned}$$

□

Example

Consider $z = f(x, y)$, suppose $x = r \cos \theta$, $y = r \sin \theta$. Show that

$$\frac{\partial^2 z}{\partial r^2} = \cos^2 \theta \frac{\partial^2 f}{\partial x^2} + 2 \sin \theta \cos \theta \frac{\partial^2 f}{\partial x \partial y} + \sin^2 \theta \frac{\partial^2 f}{\partial y^2}.$$

Solution:

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial r} = \cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y}.$$

$$\begin{aligned}
\frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial r} \right) = \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial f}{\partial x} + \sin \theta \frac{\partial f}{\partial y} \right) = \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial x} \right) + \sin \theta \frac{\partial}{\partial r} \left(\frac{\partial f}{\partial y} \right) \\
&= \cos \theta \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y \partial x} \frac{\partial y}{\partial r} \right) + \sin \theta \left(\frac{\partial^2 f}{\partial x \partial y} \frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2} \frac{\partial y}{\partial r} \right) \\
&= \cos^2 \theta \frac{\partial^2}{\partial x^2} f(x, y) + 2 \sin \theta \cos \theta \frac{\partial^2}{\partial x \partial y} f(x, y) + \sin^2 \theta \frac{\partial^2}{\partial y^2} f(x, y)
\end{aligned}$$

□

Example

Show that in polar coordinates (r, θ) in R^2 , Laplace's equation $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ takes the form

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0.$$

Solution:

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial r} \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial \theta}
\end{aligned}$$

$$x = r \cos \theta, y = r \sin \theta \Rightarrow \frac{\partial x}{\partial r} = \cos \theta, \frac{\partial y}{\partial r} = \sin \theta, \frac{\partial x}{\partial \theta} = -r \sin \theta, \frac{\partial y}{\partial \theta} = r \cos \theta$$

Thus,

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x} \cos \theta + \frac{\partial u}{\partial y} \sin \theta \\
\frac{\partial u}{\partial \theta} &= \frac{\partial u}{\partial x} (-r \sin \theta) + \frac{\partial u}{\partial y} r \cos \theta
\end{aligned}$$

Therefore,

$$\begin{aligned}
\begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{pmatrix} &= \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -r \sin \theta & r \cos \theta \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} &= \begin{pmatrix} \cos \theta & -\frac{\sin \theta}{r} \\ \sin \theta & \frac{\cos \theta}{r} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial r} \\ \frac{\partial u}{\partial \theta} \end{pmatrix}
\end{aligned}$$

$$\text{So } \frac{\partial u}{\partial x} = \cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \dots (*), \quad \frac{\partial u}{\partial y} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \dots (**)$$

From (**), we learn that $\frac{\partial(f)}{\partial y} = \sin \theta \frac{\partial(f)}{\partial r} + \frac{\cos \theta}{r} \frac{\partial(f)}{\partial \theta}$. Notice that the f in the left side of (**) is a

function of x, y , however, the f in the right side of (**) after x, y being replaced by $r \cos \theta$ and $r \sin \theta$, respectively, is a function of r, θ . Let $\frac{\partial u}{\partial y} = f$, therefore, we have

$$\text{In addition, we notice that } \frac{\partial u}{\partial y} = \sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta}$$

Therefore,

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \sin \theta \frac{\partial}{\partial r} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right) + \frac{\cos \theta}{r} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &= \sin \theta \left(\sin \theta \frac{\partial^2 u}{\partial r^2} + \frac{\cos \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} - \frac{\cos \theta}{r^2} \frac{\partial u}{\partial \theta} \right) + \frac{\cos \theta}{r} \left(\cos \theta \frac{\partial u}{\partial r} + \sin \theta \frac{\partial^2 u}{\partial \theta \partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial r} + \frac{\cos \theta}{r} \frac{\partial^2 u}{\partial \theta \partial r} \right) \\ &\dots (1) \end{aligned}$$

Similarly,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = \cos \theta \frac{\partial}{\partial r} \left(\frac{\partial u}{\partial x} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{\partial u}{\partial x} \right) \\ &= \cos \theta \frac{\partial}{\partial r} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\cos \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &= \cos \theta \left(\cos \theta \frac{\partial^2 u}{\partial r^2} + \frac{\sin \theta}{r^2} \frac{\partial u}{\partial \theta} - \frac{\sin \theta}{r} \frac{\partial^2 u}{\partial r \partial \theta} \right) - \frac{\sin \theta}{r} \left(\cos \theta \frac{\partial^2 u}{\partial \theta \partial r} - \sin \theta \frac{\partial u}{\partial r} - \frac{\sin \theta}{r} \frac{\partial^2 u}{\partial \theta^2} - \frac{\cos \theta}{r} \frac{\partial u}{\partial \theta} \right) \\ &\dots (2) \end{aligned}$$

$$\text{Finally, (1) + (2) we have } \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r} \frac{\partial u}{\partial r} = 0.$$

□

(For reference only)

Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$. We say $f(x, y)$ is differentiable at (x_0, y_0) if and only if there exists a 1×2 matrix

$$T \text{ such that } \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - T \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

Observe that if $T = (a, b)$ then

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - T \begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0 = \lim_{(x,y) \rightarrow (x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - a(x-x_0) - b(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Theorem

Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at (x_0, y_0) , then the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at

(x_0, y_0) and the 1×2 matrix T is given by $T = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$.

Theorem

Suppose $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is differentiable at (x_0, y_0) , then $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is continuous at (x_0, y_0) .

Example

$$\text{Consider the function } f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

Show that

$$(a) \quad \frac{\partial f}{\partial x}(0, 0), \quad \frac{\partial f}{\partial y}(0, 0) \text{ exist.}$$

Proof:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{0 - 0}{x - 0} = 0; \quad \frac{\partial f}{\partial y}(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{0 - 0}{y - 0} = 0$$

(b)

Show that $f(x, y)$ is not differentiable at $(0, 0)$.

Proof:

We observe that if (x, y) is on the line $y = t, x = t$, then if $t \neq 0$

$$\frac{f(x, y) - f(0, 0) - \left[\frac{\partial f}{\partial x}(0, 0) \right](x - 0) - \left[\frac{\partial f}{\partial y}(0, 0) \right](y - 0)}{\sqrt{(x - 0)^2 + (y - 0)^2}} = \frac{f(t, t)}{\sqrt{t^2 + t^2}} = \frac{t^3}{t^2 + t^2} = \frac{1}{2\sqrt{2}} \frac{t}{|t|}$$

Suppose now (x, y) approaches $(0, 0)$ along the straight line $y = t, x = t$, $\lim_{t \rightarrow 0} \frac{1}{2\sqrt{2}} \frac{t}{|t|}$ does not exist.

It follows that

$$\lim_{(x, y) \rightarrow (x_0, y_0)} \frac{f(x, y) - f(x_0, y_0) - \left[\frac{\partial f}{\partial x}(x_0, y_0) \right](x - x_0) - \left[\frac{\partial f}{\partial y}(x_0, y_0) \right](y - y_0)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \text{ does not exist.}$$

So $f(x, y)$ is not differentiable at $(0, 0)$.

□

5 Implicit Functions

Let $f(x, y)$ be a function of two independent variables and suppose $f_x(x, y)$ & $f_y(x, y)$ exist and are continuous at (x_0, y_0) and nearby, in addition, $f(x_0, y_0) = 0$ & $f_y(x_0, y_0) \neq 0$. Then about (x_0, y_0) the equation $f(x, y) = 0$ determines an implicit function $y = y(x)$ about x_0 with $y_0 = y(x_0)$ and $y = y(x)$ being differentiable at x_0 .

The above is also applicable to system of equations:

The system of equations
$$\begin{cases} f_1(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \\ \vdots \\ f_m(x_1, \dots, x_n, y_1, \dots, y_m) = 0 \end{cases}$$
 determine locally $y_1, \dots, y_m$ as implicit functions

of $x_1, \dots, x_n$ and the partial derivatives of $y_1, \dots, y_m$ with respect to $x_i, 1 \leq i \leq n$ can be found.

Example

The equation $x^3 - xy^2 + yz^2 - z^3 = 5$ determine z as a function of x, y . Find $\frac{\partial z}{\partial x}$.

Solution:

Differentiating $x^3 - xy^2 + yz^2 - z^3 = 5$ with respect to x gives

$$3x^2 - y^2 + 2zy \frac{\partial z}{\partial x} - 3z^2 \frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial z}{\partial x} = \frac{y^2 - 3x^2}{2zy - 3z^2} \quad \text{if} \quad 2zy - 3z^2 \neq 0.$$

□

Example

Find the gradient of the tangent $\frac{dy}{dx}$ at a point (x, y) of the conic

$$f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Solution:

y is defined as an implicit function of x by the equation $z = f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

In this case $\frac{dz}{dx} = 0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \frac{dy}{dx}$ and we have $\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ whenever $\frac{\partial f}{\partial y} \neq 0$;

$$\frac{dy}{dx} = -\frac{2ax + 2hy + 2g}{2hx + 2by + 2f} \quad \text{whenever} \quad 2hx + 2by + 2f \neq 0.$$

□

Example

If $z = e^x \cos y$, while x and y are implicit functions of t defined by the equations $x^3 + e^x - t^2 - t = 1$, $yt^2 + y^2t - t + y = 0$ then find $\frac{dz}{dt}$ for $t = 0$.

Solution:

Differentiate $z = e^x \cos y$ with respect to t , we have $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$

The equation $x^3 + e^x - t^2 - t = 1$ defines a function $x = x(t)$ implicitly.

Differentiate both sides of $x^3 + e^x - t^2 - t = 1$ with respect to t .

Then $3x^2 \frac{dx}{dt} + e^x \frac{dx}{dt} - 2t - 1 = 0$. And we have $\frac{dx}{dt} = \frac{2t+1}{3x^2 + e^x}$, if $3x^2 + e^x \neq 0$.

We observe that for $x^3 + e^x - t^2 - t = 1$, $yt^2 + y^2t - t + y = 0$, if $t = 0$ then $x = 0$.

Therefore, $\left. \frac{dx}{dt} \right|_{t=0} = \left. \frac{2t+1}{3x^2 + e^x} \right|_{\substack{t=0 \\ x=0}} = 1$

Similarly, The equation $yt^2 + y^2t - t + y = 0$ defines a function $y = y(t)$ implicitly. And we have

$2yt + t^2 \frac{dy}{dt} + 2yt \frac{dy}{dt} + y^2 - 1 + \frac{dy}{dt} = 0$. This implies $\frac{dy}{dt} = \frac{1 - y^2 - 2yt}{1 + t^2 + 2yt}$.

For $yt^2 + y^2t - t + y = 0$ if $t = 0$ then $y = 0$. Therefore, $\left. \frac{dy}{dt} \right|_{t=0} = \left. \frac{1 - y^2 - 2yt}{1 + t^2 + 2yt} \right|_{\substack{t=0 \\ y=0}} = 1$

Also, $\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} = e^x \cos y \frac{dx}{dt} - e^x \sin y \frac{dy}{dt}$.

Finally, $\left. \frac{dz}{dt} \right|_{t=0} = e^x \cos y \left. \frac{dx}{dt} \right|_{\substack{x=0 \\ y=0}} - e^x \sin y \left. \frac{dy}{dt} \right|_{\substack{x=0 \\ y=0}} = 1$

□

Example

Suppose we solve $\begin{cases} x^2 + y^2 = \frac{1}{2} z^2 \\ x + y + z = 2 \end{cases}$ uniquely for x, y as functions of z near $x = 1, y = -1$. Find

$\frac{dx}{dz}, \frac{dy}{dz}, \frac{d^2x}{dz^2}, \frac{d^2y}{dz^2}$ when $x = 1, y = -1, z = 2$.

Solution:

$$\begin{cases} x^2 + y^2 = \frac{1}{2}z^2 \\ x + y + z = 2 \end{cases} \Rightarrow \begin{cases} 2x \frac{dx}{dz} + 2y \frac{dy}{dz} = z \dots (1) \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \dots (2) \end{cases} \Rightarrow \begin{cases} 2\left(\frac{dx}{dz}\right)^2 + 2x \frac{d^2x}{dz^2} + 2\left(\frac{dy}{dz}\right)^2 + 2y \frac{d^2y}{dz^2} = 1 \dots (3) \\ \frac{d^2x}{dz^2} + \frac{d^2y}{dz^2} = 0 \dots (4) \end{cases}$$

Put $x = 1, y = -1, z = 2$ in (1) & (2), we have

$$\begin{cases} 2x \frac{dx}{dz} + 2y \frac{dy}{dz} = z \dots (1) \\ \frac{dx}{dz} + \frac{dy}{dz} + 1 = 0 \dots (2) \end{cases} \Rightarrow 4 \frac{dx}{dz} = 0, 4 \frac{dy}{dz} = -4 \Rightarrow \frac{dx}{dz} = 0, \frac{dy}{dz} = -1$$

Put $x = 1, y = -1, z = 2$ and $\frac{dx}{dz} = 0, \frac{dy}{dz} = -1$ in (3) & (4), we have $\frac{d^2x}{dz^2} = -\frac{1}{4}, \frac{d^2y}{dz^2} = \frac{1}{4}$.

□

Example

Suppose that near the point $(x, y, u, v) = (1, 1, 1, 1)$ we can solve $\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$ uniquely for u and v as

functions $u(x, y), v(x, y)$ of x and y . Compute $\left(\frac{\partial u}{\partial x}\right)(1, 1)$.

Solution:

Differentiate all equations with respect to x of $\begin{cases} xu + yvu^2 = 2 \\ xu^3 + y^2v^4 = 2 \end{cases}$. We have

$$\begin{cases} u + x \frac{\partial u}{\partial x} + yv2u \frac{\partial u}{\partial x} + yu^2 \frac{\partial v}{\partial x} = 0 \\ u^3 + x3u^2 \frac{\partial u}{\partial x} + y^2 4v^3 \frac{\partial v}{\partial x} = 0 \end{cases}$$

Since $x = 1, y = 1, u = 1, v = 1$, we have

$$\begin{cases} 1 + 3 \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \\ 1 + 3 \frac{\partial u}{\partial x} + 4 \frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow \begin{cases} 4 + 12 \frac{\partial u}{\partial x} + 4 \frac{\partial v}{\partial x} = 0 \\ 1 + 3 \frac{\partial u}{\partial x} + 4 \frac{\partial v}{\partial x} = 0 \end{cases} \Rightarrow 9 \frac{\partial u}{\partial x} = -3 \Rightarrow \frac{\partial u}{\partial x}(1, 1) = -\frac{1}{3}$$

□

Example (for reference only)

Functions $x = x(\rho, \phi, \theta)$, $y = y(\rho, \phi, \theta)$, $z = z(\rho, \phi, \theta)$ can be regarded as equations involving

$x, y, z, \rho, \phi, \theta$. Assume $x = x(\rho, \phi, \theta)$, $y = y(\rho, \phi, \theta)$, $z = z(\rho, \phi, \theta)$ with appropriate differentiability may define ρ, ϕ, θ each as implicit functions of x, y, z .

The determinant $J \equiv \frac{\partial(y_1, \dots, y_n)}{\partial(x_1, \dots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \dots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \dots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$, is called the Jacobian of $y_1, \dots, y_n$ with respect to $x_1, \dots, x_n$.

(a) Using Cramer's Rule, deduce that $\frac{\partial \rho}{\partial x} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\phi, \theta)}$, $\frac{\partial \phi}{\partial x} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\theta, \rho)}$, $\frac{\partial \theta}{\partial x} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\rho, \phi)}$, where

$$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}.$$

(b) Guess $\frac{\partial \rho}{\partial y}$, $\frac{\partial \rho}{\partial z}$, then prove your guess at $\frac{\partial \rho}{\partial y}$.

(c) Let $\nabla \rho \equiv \frac{\partial \rho}{\partial x} \vec{i} + \frac{\partial \rho}{\partial y} \vec{j} + \frac{\partial \rho}{\partial z} \vec{k}$ and $\frac{\partial \vec{r}}{\partial \phi} \equiv \frac{\partial x}{\partial \phi} \vec{i} + \frac{\partial y}{\partial \phi} \vec{j} + \frac{\partial z}{\partial \phi} \vec{k}$, $\frac{\partial \vec{r}}{\partial \theta} \equiv \frac{\partial x}{\partial \theta} \vec{i} + \frac{\partial y}{\partial \theta} \vec{j} + \frac{\partial z}{\partial \theta} \vec{k}$, show that

$$\nabla \rho = \frac{\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}}{J}.$$

(d) Let $x = x(\rho, \phi, \theta) = \rho \sin \phi \cos \theta$, $y = y(\rho, \phi, \theta) = \rho \sin \phi \sin \theta$, $z = z(\rho, \phi, \theta) = \rho \cos \phi$.

Evaluate $J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}$, $\frac{\partial \rho}{\partial x}$, $\frac{\partial \rho}{\partial y}$, $\frac{\partial \rho}{\partial z}$.

Solution:

(a)

$$x = x(\rho, \phi, \theta), \quad y = y(\rho, \phi, \theta), \quad z = z(\rho, \phi, \theta)$$

$$\Rightarrow \begin{cases} 1 = \frac{\partial x}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial x}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial x}{\partial \theta} \frac{\partial \theta}{\partial x} \\ 0 = \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial y}{\partial \theta} \frac{\partial \theta}{\partial x} \\ 0 = \frac{\partial z}{\partial \rho} \frac{\partial \rho}{\partial x} + \frac{\partial z}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial x} \end{cases} \xRightarrow{\text{Cramer's Rule}} \frac{\partial \rho}{\partial x} = \frac{\det \begin{pmatrix} 1 & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ 0 & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ 0 & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}}{\det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}} = \frac{\frac{\partial(y, z)}{\partial(\phi, \theta)}}{\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\phi, \theta)}$$

$$\frac{\partial \phi}{\partial x} = \frac{\det \begin{pmatrix} \frac{\partial x}{\partial \rho} & 1 & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & 0 & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & 0 & \frac{\partial z}{\partial \theta} \end{pmatrix}}{\det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}} = \frac{-\frac{\partial(y, z)}{\partial(\rho, \theta)}}{\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\theta, \rho)}, \quad \frac{\partial \theta}{\partial x} = \frac{\det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & 1 \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & 0 \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & 0 \end{pmatrix}}{\det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\rho, \phi)}$$

(b)

$$\frac{\partial \rho}{\partial y} = \frac{\frac{\partial(z, x)}{\partial(\phi, \theta)}}{\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}} = \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)}, \quad \frac{\partial \rho}{\partial z} = \frac{\frac{\partial(x, y)}{\partial(\phi, \theta)}}{\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}} = \frac{1}{J} \frac{\partial(x, y)}{\partial(\phi, \theta)}.$$

$$x = x(\rho, \phi, \theta), y = y(\rho, \phi, \theta), z = z(\rho, \phi, \theta)$$

$$\Rightarrow \begin{cases} 0 = \frac{\partial x}{\partial \rho} \frac{\partial \rho}{\partial y} + \frac{\partial x}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial x}{\partial \theta} \frac{\partial \theta}{\partial y} \\ 1 = \frac{\partial y}{\partial \rho} \frac{\partial \rho}{\partial y} + \frac{\partial y}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial y}{\partial \theta} \frac{\partial \theta}{\partial y} \\ 0 = \frac{\partial z}{\partial \rho} \frac{\partial \rho}{\partial y} + \frac{\partial z}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial z}{\partial \theta} \frac{\partial \theta}{\partial y} \end{cases} \xRightarrow{\text{Cramer's Rule}} \frac{\partial \rho}{\partial y} = \frac{\det \begin{pmatrix} 0 & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ 1 & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ 0 & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}}{\det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix}} = \frac{-\frac{\partial(x, z)}{\partial(\phi, \theta)}}{\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}} = \frac{\frac{\partial(z, x)}{\partial(\phi, \theta)}}{\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)}} = \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)}$$

(c)

$$\begin{aligned} \nabla \rho &= \frac{\partial \rho}{\partial x} \vec{i} + \frac{\partial \rho}{\partial y} \vec{j} + \frac{\partial \rho}{\partial z} \vec{k} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\phi, \theta)} \vec{i} + \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)} \vec{j} + \frac{1}{J} \frac{\partial(x, y)}{\partial(\phi, \theta)} \vec{k} \\ &= \frac{\frac{\partial(y, z)}{\partial(\phi, \theta)} \vec{i} + \frac{\partial(z, x)}{\partial(\phi, \theta)} \vec{j} + \frac{\partial(x, y)}{\partial(\phi, \theta)} \vec{k}}{J} = \frac{\begin{vmatrix} \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} \vec{i} + \begin{vmatrix} \frac{\partial z}{\partial \phi} & \frac{\partial x}{\partial \phi} \\ \frac{\partial z}{\partial \theta} & \frac{\partial x}{\partial \theta} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} \end{vmatrix} \vec{k}}{J} \\ &= \frac{\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial \phi} & \frac{\partial y}{\partial \phi} & \frac{\partial z}{\partial \phi} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix}}{J} = \frac{\left(\frac{\partial x}{\partial \phi} \vec{i} + \frac{\partial y}{\partial \phi} \vec{j} + \frac{\partial z}{\partial \phi} \vec{k} \right) \times \left(\frac{\partial x}{\partial \theta} \vec{i} + \frac{\partial y}{\partial \theta} \vec{j} + \frac{\partial z}{\partial \theta} \vec{k} \right)}{J} = \frac{\frac{\partial \vec{r}}{\partial \phi} \times \frac{\partial \vec{r}}{\partial \theta}}{J} \end{aligned}$$

(d)

$$\begin{aligned} J &= \frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{pmatrix} = \begin{vmatrix} \sin \phi \cos \theta & \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ \cos \phi & -\rho \sin \phi & 0 \end{vmatrix} \\ &= \cos \phi \begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} - (-\rho \sin \phi) \begin{vmatrix} \sin \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \sin \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix} \\ &= \cos \phi (\rho^2 \cos \phi \sin \phi \cos^2 \theta + \rho^2 \cos \phi \sin \phi \sin^2 \theta) + \rho \sin \phi (\rho \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta) \\ &= \rho^2 \cos^2 \phi \sin \phi + \rho^2 \sin^3 \phi = \rho^2 \sin \phi \end{aligned}$$

$$\frac{\partial \rho}{\partial x} = \frac{1}{J} \frac{\partial(y, z)}{\partial(\phi, \theta)} = \frac{\begin{vmatrix} \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \\ -\rho \sin \phi & 0 \end{vmatrix}}{\rho^2 \sin \phi} = \frac{\rho^2 \sin^2 \phi \cos \theta}{\rho^2 \sin \phi} = \sin \phi \cos \theta,$$

$$\frac{\partial \rho}{\partial y} = \frac{1}{J} \frac{\partial(z, x)}{\partial(\phi, \theta)} = \frac{\begin{vmatrix} -\rho \sin \phi & 0 \\ \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \end{vmatrix}}{\rho^2 \sin \phi} = \frac{\rho^2 \sin^2 \phi \sin \theta}{\rho^2 \sin \phi} = \sin \phi \sin \theta,$$

$$\begin{aligned} \frac{\partial \rho}{\partial z} &= \frac{1}{J} \frac{\partial(x, y)}{\partial(\phi, \theta)} = \frac{\begin{vmatrix} \rho \cos \phi \cos \theta & -\rho \sin \phi \sin \theta \\ \rho \cos \phi \sin \theta & \rho \sin \phi \cos \theta \end{vmatrix}}{\rho^2 \sin \phi} = \frac{\rho^2 \sin \phi \cos \phi \cos^2 \theta + \rho^2 \sin \phi \cos \phi \sin^2 \theta}{\rho^2 \sin \phi} \\ &= \frac{\rho^2 \sin \phi \cos \phi}{\rho^2 \sin \phi} = \cos \phi \end{aligned}$$

□

6 Taylor's Theorem

If a function $f(x)$ of one variable is differentiable through order N in $a-l < x < a+l$ where $l > 0$, then for any real number h such that $a-l < a+h < a+l$, we have

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{N-1}}{(N-1)!} f^{(N-1)}(a) + R_N(h) \text{ where}$$

$$R_N(h) = \frac{h^N}{N!} f^{(N)}(a + \theta h) \text{ for some } 0 < \theta < 1.$$

OR

If a function $f(x)$ of one variable is differentiable through order $n+1$ in $a-l < x < a+l$ where $l > 0$, then for any real number x such that $a-l < x < a+l$, we have

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(N-1)}(a)}{(N-1)!}(x-a)^{N-1} + R_N(x) \text{ where}$$

$$R_N(x) = \frac{f^{(N)}[a + \theta(x-a)]}{N!}(x-a)^N \text{ for some } 0 < \theta < 1.$$

If $a = 0$ we have Maclaurin's Theorem:

$$f(h) = f(0) + \frac{h}{1!} f'(0) + \frac{h^2}{2!} f''(0) + \cdots + \frac{h^{N-1}}{(N-1)!} f^{(N-1)}(0) + R_N(h) \text{ with } R_N(h) = \frac{h^N}{N!} f^{(N)}(\theta h) \text{ for}$$

some $0 < \theta < 1$.

OR

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(N-1)}(0)}{(N-1)!}x^{N-1} + R_N(x) \text{ with } R_N(x) = \frac{f^{(N)}(\theta x)}{N!}x^N \text{ for some}$$

$0 < \theta < 1$.

Example

Express the Maclaurin series of $\cos x$ (up to x^4) and $\frac{1}{1+y}$, hence find the Maclaurin series of $\frac{1}{\cos x}$ (up to x^4).

Solution:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots, \quad \frac{1}{1+y} = 1 - y + y^2 - y^3 + y^4 - \cdots.$$

$$\begin{aligned}
\frac{1}{\cos x} &= \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots} = \frac{1}{1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)} \\
&= 1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right) + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)^2 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \dots \right)^3 + \dots \\
&= 1 - \left(-\frac{x^2}{2!} + \frac{x^4}{4!} \right) + \left(\frac{x^4}{4} + \dots \right) + \dots = 1 + \frac{x^2}{2!} + \frac{5x^4}{24} + \dots
\end{aligned}$$

(up to x^4).

□

Taylor's Theorem generalizes to a function of two variables $f(x, y)$ which has continuous partial derivatives of all orders up to N inside the rectangle with vertices $(a-p, b-q)$, $(a-p, b+q)$, $(a+p, b-q)$, $(a+p, b+q)$, where $p, q > 0$.

At the point $(a+h, b+k)$, where $a-p < a+h < a+p$, $b-q < b+k < b+q$, we have

$$\begin{aligned}
f(a+h, b+k) &= f(a, b) + \frac{1}{1!} [hf_x(a, b) + kf_y(a, b)] + \frac{1}{2!} [h^2 f_{xx}(a, b) + 2hkf_{xy}(a, b) + k^2 f_{yy}(a, b)] \\
&+ \frac{1}{3!} [h^3 f_{xxx}(a, b) + 3h^2 kf_{xxy}(a, b) + 3hk^2 f_{xyy}(a, b) + k^3 f_{yyy}(a, b)] + \dots + R_N(h, k) \\
&= \sum_{i=0}^{N-1} \frac{1}{i!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^i f(a, b) + R_N(h, k)
\end{aligned}$$

where

$$\begin{aligned}
R_N(h, k) &= \frac{1}{N!} \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^N f(a + \theta h, b + \theta k) = \frac{1}{N!} \left[\sum_{i=0}^N \binom{N}{i} h^i k^{N-i} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta h, b + \theta k) \right] \\
&= \frac{1}{N!} \left[\sum_{i=0}^N \frac{N(N-1)\dots(N-i+1) h^i k^{N-i}}{i!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta h, b + \theta k) \right] \\
&= \sum_{i=0}^N \frac{h^i k^{N-i}}{i!(N-i)!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta h, b + \theta k)
\end{aligned}$$

for some $0 < \theta < 1$.

If $R_N(h, k) \rightarrow 0$ as $N \rightarrow \infty$ we have the Taylor Series representation of $f(a+h, b+k)$.

OR

For the case of two variables:

$$\begin{aligned}
f(x, y) &= f(a, b) + \frac{1}{1!} \left[(x-a) f_x(a, b) + (y-b) f_y(a, b) \right] + \\
&\frac{1}{2!} \left[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b) + (y-b)^2 f_{yy}(a, b) \right] + \cdots + R_N(x, y) \\
&= \sum_{i=0}^{N-1} \frac{1}{i!} \left((x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right)^i f(a, b) + R_N(x, y)
\end{aligned}$$

Where

$$\begin{aligned}
R_N(x, y) &= \frac{1}{N!} \left[(x-a) \frac{\partial}{\partial x} + (y-b) \frac{\partial}{\partial y} \right]^N f[a + \theta(x-a), b + \theta(y-b)] \\
&= \frac{1}{N!} \left[\sum_{i=0}^N \binom{N}{i} (x-a)^i (y-b)^{N-i} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta(x-a), b + \theta(y-b)) \right] \\
&= \frac{1}{N!} \left[\sum_{i=0}^N \frac{N(N-1)\cdots(N-i+1)(x-a)^i (y-b)^{N-i}}{i!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta(x-a), b + \theta(y-b)) \right] \\
&= \sum_{i=0}^N \frac{(x-a)^i (y-b)^{N-i}}{i!(N-i)!} \frac{\partial^N f}{\partial x^i \partial y^{N-i}}(a + \theta(x-a), b + \theta(y-b))
\end{aligned}$$

If $R_N(x, y) \rightarrow 0$ as $N \rightarrow \infty$ we have the Taylor Series representation of $f(x, y)$.

Successive terms given by Taylor's Theorem give the tangent plane approximation to the surface $z = f(x, y)$ (Taking the first two terms of the series), a quadratic surface approximation (the first three terms), a cubic surface (the first four terms) etc, all meeting $z = f(x, y)$ at the point (a, b) .

Example

Find the tangent plane and quadratic surface approximations given by Taylor's Theorem at the point

$(1, 0)$ for the cone $f(x, y) = (x^2 + y^2)^{\frac{1}{2}}$. Hence estimate $f(0.9, 0.1)$.

Solution:

$$\begin{aligned}
f_x(x, y) &= \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}, f_y(x, y) = \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}, \\
f_{xx}(x, y) &= \frac{y^2}{(x^2 + y^2)^{\frac{3}{2}}}, f_{xy}(x, y) = \frac{-xy}{(x^2 + y^2)^{\frac{3}{2}}}, f_{yy}(x, y) = \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}},
\end{aligned}$$

which exist everywhere except at the origin $(0, 0)$.

The tangent plane approximation is:

$$p_1(x, y) = f(1, 0) + \frac{1}{1!}[(x-1)f_x(1, 0) + (y-0)f_y(1, 0)] \Rightarrow p_1(x, y) = x.$$

The quadratic surface approximation is:

$$\begin{aligned} p_2(x, y) &= f(1, 0) + \frac{1}{1!}[(x-1)f_x(1, 0) + (y-0)f_y(1, 0)] \\ &+ \frac{1}{2!}[(x-1)^2 f_{xx}(1, 0) + 2(x-1)(y-0)f_{xy}(1, 0) + (y-0)^2 f_{yy}(1, 0)] \\ &\Rightarrow p_2(x, y) = x + \frac{1}{2}y^2 \end{aligned}$$

Hence, using the tangent plane $f(0.9, 0.1) \approx p_1(0.9, 0.1) = 0.9$. And, using the quadratic surface

$f(0.9, 0.1) \approx p_2(0.9, 0.1) = 0.905$. The correct value is $f(0.9, 0.1) = 0.9055$ (to 4 s.f)

□

Example

$z^3 - 2xz + y = 0$ determines an implicit function $z = z(x, y)$ of x and y about the point $(1, 1)$. Use the

Taylor theorem to expand $z = z(x, y)$ about the point $(1, 1)$ up to the linear terms.

Solution:

Note that from $z^3 - 2xz + y = 0$ when $x = 1, y = 1$ we have $z = 1$.

$$z^3 - 2xz + y = 0 \Rightarrow 3z^2 \frac{\partial z}{\partial x} - 2z - 2x \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{2z}{3z^2 - 2x} \& \frac{\partial z}{\partial x} \bigg|_{\substack{x=1 \\ y=1}} = 2.$$

$$z^3 - 2xz + y = 0 \Rightarrow 3z^2 \frac{\partial z}{\partial y} - 2x \frac{\partial z}{\partial y} + 1 = 0 \Rightarrow \frac{\partial z}{\partial y} = \frac{-1}{3z^2 - 2x} \& \frac{\partial z}{\partial y} \bigg|_{\substack{x=1 \\ y=1}} = -1$$

$$z(x, y) = z(1, 1) + \left[\frac{\partial z}{\partial x} \bigg|_{\substack{x=1 \\ y=1}} (x-1) + \frac{\partial z}{\partial y} \bigg|_{\substack{x=1 \\ y=1}} (y-1) \right] + \dots$$

So $z(x, y) = 1 + 2(x-1) - 1(y-1) + \dots$

□

Example

Let $f(x)$ be a smooth function, that is, $f(x)$ has continuous derivatives up to any order.

Then the Taylor series of $f(x)$ about the point a up to order 2 is

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots$$

Hence or otherwise, find the Taylor series of $g(x, y, z) = \cos(x+y+z) - \cos x \cos y \cos z$ about the

point $\left(0, \frac{\pi}{2}, \frac{\pi}{2}\right)$ up to order 2.

Solution:

$$\begin{aligned}\cos(x+y+z) &= \cos x \cos(y+z) - \sin x \sin(y+z) \\ &= \cos x (\cos y \cos z - \sin y \sin z) - \sin x (\sin y \cos z + \cos y \sin z) \\ &= \cos x \cos y \cos z - \cos x \sin y \sin z - \sin x \sin y \cos z - \sin x \cos y \sin z\end{aligned}$$

Observe that $\cos x = 1 - \frac{x^2}{2} + \dots$, $\cos x = -\sin\left(x - \frac{\pi}{2}\right) + \dots$, $\sin x = x - \dots$, $\sin x = 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \dots$

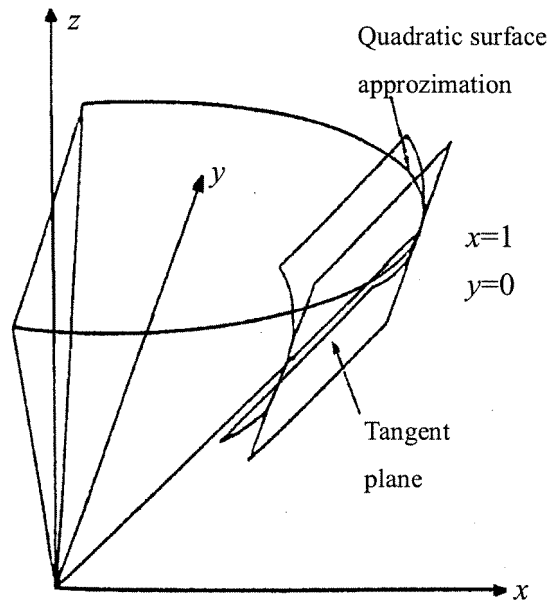
$$g(x, y, z) = \cos(x+y+z) - \cos x \cos y \cos z = -\cos x \sin y \sin z - \sin x \sin y \cos z - \sin x \cos y \sin z$$

$$\begin{aligned}&= -\left(1 - \frac{x^2}{2} + \dots\right) \left[1 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \dots\right] \left[1 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + \dots\right] \\ &\quad - (x - \dots) \left[1 - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \dots\right] \left[-\left(z - \frac{\pi}{2}\right) - \dots\right] - (x - \dots) \left[-\left(y - \frac{\pi}{2}\right) - \dots\right] \left[1 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + \dots\right] \\ &= -\left[1 - \frac{x^2}{2} - \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 - \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2\right] - \left[-x\left(z - \frac{\pi}{2}\right)\right] - \left[-x\left(y - \frac{\pi}{2}\right)\right] \\ &= -1 + \frac{x^2}{2} + \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + \frac{1}{2}\left(z - \frac{\pi}{2}\right)^2 + x\left(z - \frac{\pi}{2}\right) + x\left(y - \frac{\pi}{2}\right) + \dots\end{aligned}$$

□

A bound for the error involved in approximating a function near a point (a, b) may be obtained by

summing over estimates of the moduli of the terms in $R_N(x, y)$.



Linear and quadratic approximations to the cone $z = \sqrt{x^2 + y^2}$, $(x, y) = (0, 0)$

Example (Estimation of Small Changes)

For (x, y) near (a, b) we may expect

$$f(x, y) \approx f(a, b) + (x - a)f_x(a, b) + (y - b)f_y(a, b)$$

Writing $f(x, y) - f(a, b) = \delta f$, $x - a = \delta x$, $y - b = \delta y$

So from $f(x, y) - f(a, b) \approx (x - a)f_x(a, b) + (y - b)f_y(a, b)$ we have

$\delta f \approx \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$, which gives the approximate change in f due to small changes δx and δy in x and y respectively.

□

Example

- (a) Use Taylor's formula to find a linear approximation of $f(x, y) = \cos x \cos y$ near the origin.
- (b) Estimate the error in the approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.

Solution:

(a)

By Taylor's Theorem, we have

$$f(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2}[x^2 f_{xx}(\theta x, \theta y) + 2xyf_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)] , \text{ where } 0 < \theta < 1.$$

Then the linear approximation of $f(x, y) = \cos x \cos y$ near the origin is

$$p_1(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)].$$

$$\text{And } f(0, 0) = \cos x \cos y \Big|_{\substack{x=0 \\ y=0}} = 1, f_x(0, 0) = -\sin x \cos y \Big|_{\substack{x=0 \\ y=0}} = 0, f_y(0, 0) = -\cos x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0.$$

We have $p_1(x, y) = 1$. If (x, y) is close to $(0, 0)$, $f(x, y) = \cos x \cos y \approx p_1(x, y) = 1$.

(b)

The error in the approximation is

$$\begin{aligned} E(x, y) &= f(x, y) - f(0, 0) - [xf_x(0, 0) + yf_y(0, 0)] \\ &= \frac{1}{2}[x^2 f_{xx}(\theta x, \theta y) + 2xyf_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)] \end{aligned}$$

Observe that $f_{xx}(x, y) = -\cos x \cos y$, $f_{yy}(x, y) = -\cos x \cos y$, $f_{xy}(x, y) = \sin x \sin y$.

$$\begin{aligned} |E(x, y)| &= |f(x, y) - f(0, 0) - [xf_x(0, 0) + yf_y(0, 0)]| \\ &= \left| \frac{1}{2}[x^2 f_{xx}(\theta x, \theta y) + 2xyf_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)] \right| \\ &= \frac{1}{2} \left| [x^2 f_{xx}(\theta x, \theta y) + 2xyf_{xy}(\theta x, \theta y) + y^2 f_{yy}(\theta x, \theta y)] \right| \end{aligned}$$

$$\begin{aligned}
|E(x, y)| &\leq \frac{1}{2} \left(|x^2 f_{xx}(\theta x, \theta y)| + |2xy f_{xy}(\theta x, \theta y)| + |y^2 f_{yy}(\theta x, \theta y)| \right) \\
&= \frac{1}{2} \left(|x|^2 |f_{xx}(\theta x, \theta y)| + 2|x||y| |f_{xy}(\theta x, \theta y)| + |y|^2 |f_{yy}(\theta x, \theta y)| \right) \\
&= \frac{1}{2} \left(|x|^2 |-\cos \theta x \cos \theta y| + 2|x||y| |\cos \theta x \cos \theta y| + |y|^2 |-\cos \theta x \cos \theta y| \right) \\
&= \frac{1}{2} \left(|x|^2 |\cos \theta x| |\cos \theta y| + 2|x||y| |\cos \theta x| |\cos \theta y| + |y|^2 |\cos \theta x| |\cos \theta y| \right) \\
&\stackrel{\substack{|\cos \theta x| \leq 1, \\ |\cos \theta y| \leq 1}}{\leq} \frac{1}{2} \left(|x|^2 + 2|x||y| + |y|^2 \right) \stackrel{\substack{|x| \leq 0.1, \\ |y| \leq 0.1}}{\leq} 2(0.1)^2 = 0.02
\end{aligned}$$

□

Example

- (a) Use Taylor's formula to find a quadratic approximation of $f(x, y) = e^x \sin y$ at the origin.
- (b) Show that in the approximation the error $|E(x, y)| < 1.474 \times 10^{-3}$ if $|x| \leq 0.1$, $|y| \leq 0.1$ and $e^{0.1} \approx 1.1052$.
- (c) Find an approximation of $e^{-0.09} \sin 0.06$ with an error less than 1.474×10^{-3} .

Solution:

(a)

By Taylor's Theorem, we have

$$\begin{aligned}
f(x, y) &= f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] \\
&+ \frac{1}{6} [x^3 f_{xxx}(\theta x, \theta y) + 3x^2 y f_{xxy}(\theta x, \theta y) + 3xy^2 f_{xyy}(\theta x, \theta y) + y^3 f_{yyy}(\theta x, \theta y)] \\
&, \text{ where } 0 < \theta < 1.
\end{aligned}$$

Then the quadratic approximation of $f(x, y) = e^x \sin y$ at the origin is:

$$p_2(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2} [x^2 f_{xx}(0, 0) + 2xy f_{xy}(0, 0) + y^2 f_{yy}(0, 0)].$$

And

$$\begin{aligned}
f(0, 0) &= e^x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0, \quad f_x(0, 0) = e^x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0, \quad f_y(0, 0) = e^x \cos y \Big|_{\substack{x=0 \\ y=0}} = 1 \\
f_{xx}(0, 0) &= e^x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0, \quad f_{xy}(0, 0) = e^x \cos y \Big|_{\substack{x=0 \\ y=0}} = 1, \quad f_{yy}(0, 0) = -e^x \sin y \Big|_{\substack{x=0 \\ y=0}} = 0
\end{aligned}$$

$$\text{So } p_2(x, y) = f(0, 0) + [xf_x(0, 0) + yf_y(0, 0)] + \frac{1}{2}[x^2 f_{xx}(0, 0) + 2xyf_{xy}(0, 0) + y^2 f_{yy}(0, 0)] = y + xy.$$

If (x, y) is close to $(0, 0)$, $f(x, y) = e^x \sin y \approx p_2(x, y) = y + xy$.

(b)

The error in the approximation is

$$E(x, y) = f(x, y) - p_2(x, y) = \frac{1}{6}[x^3 f_{xxx}(\theta x, \theta y) + 3x^2 y f_{xxy}(\theta x, \theta y) + 3xy^2 f_{xyy}(\theta x, \theta y) + y^3 f_{yyy}(\theta x, \theta y)].$$

Observe that

$$\begin{aligned} f_{xx}(x, y) &= e^x \sin y, f_{xy}(x, y) = e^x \cos y, f_{yy}(x, y) = -e^x \sin y \\ \Rightarrow f_{xxx}(x, y) &= e^x \sin y, f_{xxy}(x, y) = e^x \cos y, f_{xyy}(x, y) = -e^x \sin y, f_{yyy}(x, y) = -e^x \cos y \end{aligned}$$

$$\begin{aligned} |E(x, y)| &= |f(x, y) - p_2(x, y)| = \frac{1}{6} |[x^3 f_{xxx}(\theta x, \theta y) + 3x^2 y f_{xxy}(\theta x, \theta y) + 3xy^2 f_{xyy}(\theta x, \theta y) + y^3 f_{yyy}(\theta x, \theta y)]| \\ &\leq \frac{1}{6} [|x|^3 |f_{xxx}(\theta x, \theta y)| + 3|x|^2 |y| |f_{xxy}(\theta x, \theta y)| + 3|x| |y|^2 |f_{xyy}(\theta x, \theta y)| + |y|^3 |f_{yyy}(\theta x, \theta y)|] \\ &= \frac{1}{6} [|x|^3 |e^{\theta x} \sin \theta y| + 3|x|^2 |y| |e^{\theta x} \cos \theta y| + 3|x| |y|^2 |-e^{\theta x} \sin \theta y| + |y|^3 |-e^{\theta x} \cos \theta y|] \\ &= \frac{1}{6} [|x|^3 |e^{\theta x}| |\sin \theta y| + 3|x|^2 |y| |e^{\theta x}| |\cos \theta y| + 3|x| |y|^2 |e^{\theta x}| |\sin \theta y| + |y|^3 |e^{\theta x}| |\cos \theta y|] \end{aligned}$$

$$0 < \theta < 1, -0.1 \leq x \leq 0.1 \Rightarrow -0.1 < \theta x < 0.1 \Rightarrow |\theta x| < 0.1.$$

$$\text{Observe that } |\sin \theta y|, |\cos \theta y| \leq 1, |e^{\theta x}| = e^{\theta x} \leq e^{|\theta x|} < e^{0.1}.$$

Thus,

$$\begin{aligned} |E(x, y)| &\leq \frac{1}{6} [|x|^3 |e^{\theta x}| |\sin \theta y| + 3|x|^2 |y| |e^{\theta x}| |\cos \theta y| + 3|x| |y|^2 |e^{\theta x}| |\sin \theta y| + |y|^3 |e^{\theta x}| |\cos \theta y|] \\ &\stackrel{\substack{|\sin \theta y|, |\cos \theta y| \leq 1 \\ |x|, |y| \leq 0.1 \\ |e^{\theta x}| < e^{0.1}}}{<} \frac{1}{6} [(0.1)^3 e^{0.1} + 3(0.1)^2 (0.1) e^{0.1} + 3(0.1)(0.1)^2 e^{0.1} + (0.1)^3 e^{0.1}] = \frac{0.008e^{0.1}}{6} \approx 1.474 \times 10^{-3} \end{aligned}$$

(c)

$$\begin{aligned} f(-0.09, 0.06) &= e^{-0.09} \sin 0.06 \approx p_2(-0.09, 0.06) = (y + xy) \Big|_{\substack{x=-0.09 \\ y=0.06}} = 0.06 - 0.09 \times 0.06 \\ &= 0.06 \times 0.91 = 0.0546 \end{aligned}$$

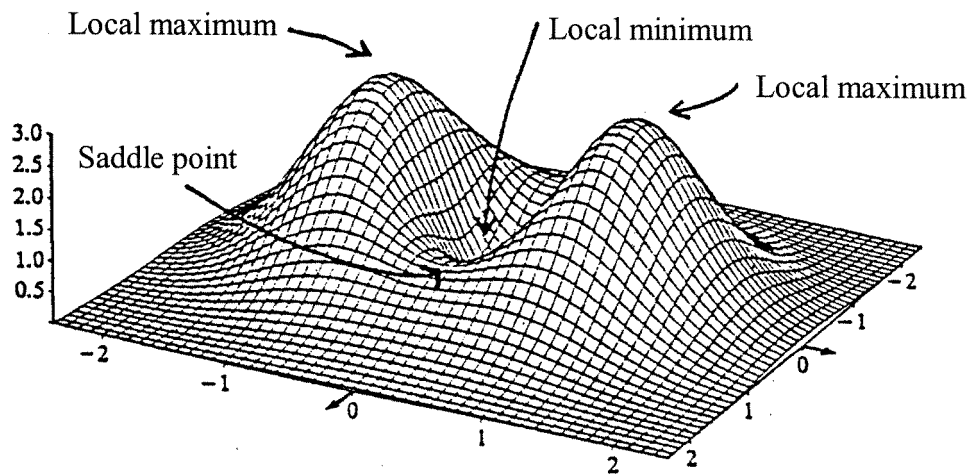
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7 Maxima ad Minima

A function $f(x, y)$ has a local maximum at a point (a, b) if for all points (x, y) close to (a, b) , we have $f(x, y) \leq f(a, b)$ and the point (a, b) is called a local maximum point.

A function $f(x, y)$ has a local minimum at a point (a, b) if for all points (x, y) close to (a, b) , we have $f(x, y) \geq f(a, b)$ and the point (a, b) is called a local minimum point.

In a region the global (absolute) maximum (minimum) of a function is either the greatest (least) of the local maxima (minima) or occurs on the boundary of the region.



$$z = f(x, y) = (x^2 + 3y^2)e^{1-x^2-y^2}$$

For a local maximum or minimum at (a, b) the curves lying in the two planes $x = a$ and $y = b$ must also have a maximum or minimum at (a, b) . Hence we must have $\frac{\partial f}{\partial x}(a, b) = \frac{\partial f}{\partial y}(a, b) = 0$

If this condition holds we say (a, b) is a stationary point.

It is possible to have a stationary point which is neither a local maximum nor a local minimum. If along one path through (a, b) the point is a local minimum, whilst along a different path it is a local maximum, we say that the point (a, b) is a saddle point. There is no equivalent point for functions of one variable.

Theorem

Assume $f(x, y)$ has the partial derivative up to the order we need and all partial derivatives are continuous, also $f_x(a, b) = f_y(a, b) = 0$. If $f_{xx}(a, b) > 0$, $f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) > 0$ for $f(x, y)$ at (a, b) then $f(x, y)$ has a local minimum there. If $f_{xx}(a, b) < 0$, $f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) > 0$ for $f(x, y)$ at (a, b) then $f(x, y)$ has a local maximum there.

If $f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b) < 0$, then $f(x, y)$ has a saddle point at (a, b) .

Examples

(a) Determine the nature of the stationary points of $f(x, y) = x^2 + 2y^2$.

Solution:

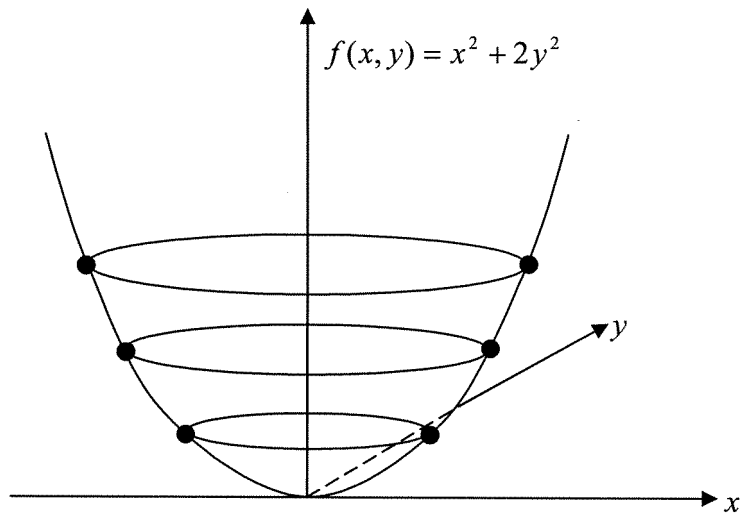
$$f(x, y) = x^2 + 2y^2 \Rightarrow \begin{cases} f_x(x, y) = 2x \\ f_y(x, y) = 4y \end{cases}$$

So $f_x(x, y) = 2x = 0$ & $f_y(x, y) = 4y = 0 \Rightarrow (x, y) = (0, 0)$ is the only stationary point.

$$f_{xx}(x, y) = 2, f_{yy}(x, y) = 4, f_{xy}(x, y) = 0, f_{xx}(0, 0) = 2 > 0 \text{ and } f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = 8 > 0.$$

$f(0, 0) = 0$ is a local minimum and $(0, 0)$ is a local minimum point and clearly $f(x, y) \geq 0$ hence

$f(0, 0) = 0$ is a global minimum.



(b) Determine the nature of the stationary points of $f(x, y) = 2x - x^2 - 2y^2$.

Solution:

$$f_x(x, y) = 2 - 2x, f_y(x, y) = -4y$$

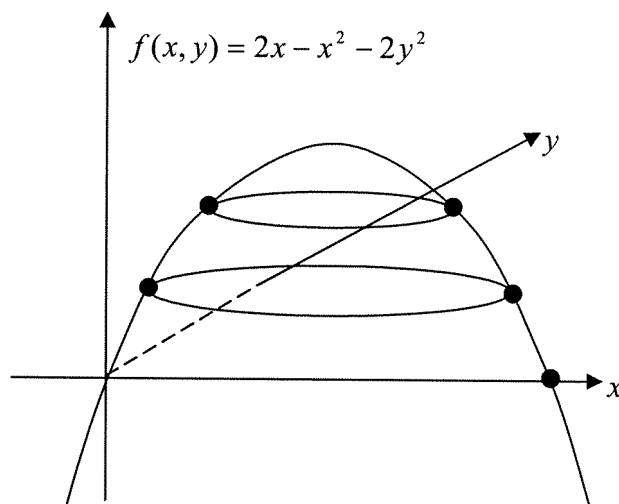
$$f_x(x, y) = 2 - 2x = 0 \text{ \& } f_y(x, y) = -4y = 0 \Rightarrow (x, y) = (1, 0), \text{ which is the only stationary point.}$$

$$f_{xx}(x, y) = -2, f_{yy}(x, y) = -4, f_{xy}(x, y) = 0, f_{xx}(1, 0) = -2 < 0 \text{ and } f_{xx}(1, 0) f_{yy}(1, 0) - f_{xy}^2(1, 0) = 8 > 0.$$

$f(1, 0) = 1$ is a local maximum. In addition,

$$f(x, y) - f(1, 0) = 2x - x^2 - 2y^2 - 1 = -(x - 1)^2 - 2y^2 \leq 0$$

Thus $f(x, y) \leq f(1, 0)$ and $(1, 0)$ is a global maximum point.



(c) Determine the nature of the stationary points of $f(x, y) = x^2 - 2y^2$.

Solution:

$$f_x(x, y) = 2x, f_y(x, y) = -4y.$$

$$f_x(x, y) = 2x = 0, f_y(x, y) = -4y = 0 \Rightarrow (x, y) = (0, 0), \text{ which is the only stationary point.}$$

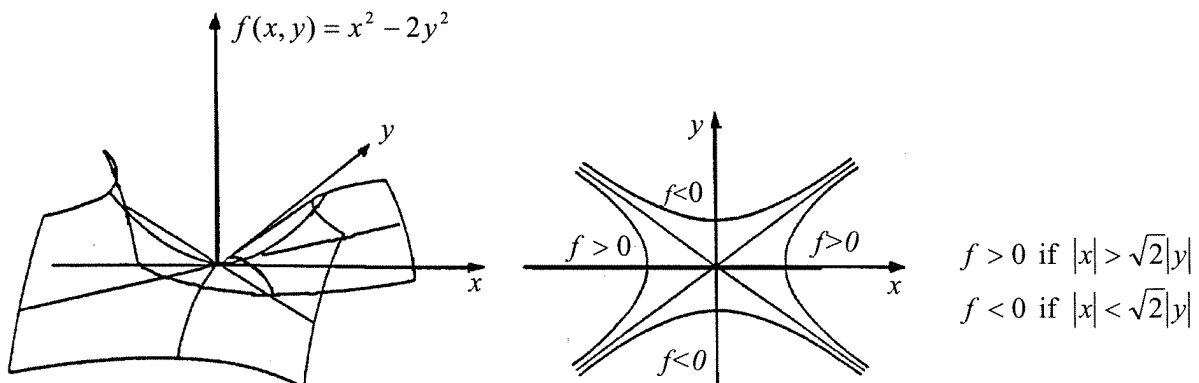
$$f_{xx}(x, y) = 2, f_{yy}(x, y) = -4, f_{xy}(x, y) = 0$$

Since $f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = -8$, $(0, 0)$ is a saddle point.

$(0, 0)$ being a saddle point can also be observed by noting that $f(x, y)$ may be either positive or negative depending on the relative magnitudes of $|x|$ and $|y|$.

$$\text{If } y = 0, x \neq 0, f(x, 0) - f(0, 0) = x^2 > 0. \text{ If } x = 0, y \neq 0, f(0, y) - f(0, 0) = -2y^2 \leq 0.$$

Thus $(0, 0)$ is a saddle point



□

Example

Determine the nature of the stationary points of $f(x, y) = x^3 + y^3 - 3(x + y)$.

Solution:

$$f(x, y) = x^3 + y^3 - 3(x + y) \Rightarrow \frac{\partial f}{\partial x} = 3x^2 - 3, \quad \frac{\partial f}{\partial y} = 3y^2 - 3.$$

$$\begin{cases} \frac{\partial f}{\partial x} = 3x^2 - 3 = 0 \\ \frac{\partial f}{\partial y} = 3y^2 - 3 = 0 \end{cases} \Rightarrow \begin{cases} x = \pm 1 \\ y = \pm 1 \end{cases}$$

Hence there are 4 stationary points $(1, 1), (1, -1), (-1, 1), (-1, -1)$. In addition,

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial y^2} = 6y, \quad \frac{\partial^2 f}{\partial x \partial y} = 0$$

Stationary Point	$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial x \partial y}$	$\frac{\partial^2 f}{\partial y^2}$	$\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$	Conclusion
$(1, 1)$	6	0	6	36	Local Minimum Point
$(-1, -1)$	-6	0	-6	36	Local Maximum Point
$(1, -1)$	6	0	-6	-36	Saddle Point
$(-1, 1)$	-6	0	6	-36	Saddle Point

□

Example

Determine the nature of the stationary points of $f(x, y) = x^4 + y^4$.

Solution:

$$f(x, y) = x^4 + y^4 \Rightarrow \begin{cases} \frac{\partial f}{\partial x} = 4x^3 \\ \frac{\partial f}{\partial y} = 4y^3 \end{cases}. \text{ We observe } (0, 0) \text{ is the only stationary point.}$$

And $\Delta(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}^2(0, 0) = 0$ the test inconclusive, but clearly $(0, 0)$ is a local minimum Point.

□

Example

Consider a point $P(x_0, y_0)$ and a straight line $y = ax + b$. The value $[y_0 - (ax_0 + b)]^2$ is called the square of the vertical displacement of the data point $P(x_0, y_0)$ from the line $y = ax + b$.

Determine the straight line $y = ax + b$ such that the sum S of the squares of the vertical displacements of the data points $P_1(1, 3.1), P_2(2, 6.1), P_3(3, 9.2)$ from the line $y = ax + b$ is the smallest.

In other words, determine the values of a, b such that S is the smallest where

$$S = (3.1 - a - b)^2 + (6.1 - 2a - b)^2 + (9.2 - 3a - b)^2$$

Solution:

$$S(a, b) = (3.1 - a - b)^2 + (6.1 - 2a - b)^2 + (9.2 - 3a - b)^2$$

$$\Rightarrow \begin{cases} 0 = \frac{\partial S}{\partial a} = -2[(3.1 - a - b) + 2(6.1 - 2a - b) + 3(9.2 - 3a - b)] \\ 0 = \frac{\partial S}{\partial b} = -2[(3.1 - a - b) + (6.1 - 2a - b) + (9.2 - 3a - b)] \end{cases}$$

$$\Leftrightarrow \begin{cases} (1 + 2^2 + 3^2)a + (1 + 2 + 3)b = 1 \times 1.3 + 2 \times 6.1 + 3 \times 9.2 \\ (1 + 2 + 3)a + 3b = 3.1 + 6.1 + 9.2 \end{cases}$$

$$\Leftrightarrow \begin{cases} 14a + 6b = 42.9 \\ 6a + 3b = 18.4 \end{cases} \Leftrightarrow \begin{cases} 14a + 6b = 42.9 \\ 12a + 6b = 36.8 \end{cases} \Leftrightarrow \begin{cases} a = \frac{6.1}{2} = 3.05 \\ b = \frac{18.4 - 18.3}{3} = \frac{0.1}{3} = \frac{1}{30} \end{cases}$$

Then $S(a, b) = (3.1 - a - b)^2 + (6.1 - 2a - b)^2 + (9.2 - 3a - b)^2 \Rightarrow \frac{\partial^2 S}{\partial a^2} = 28, \frac{\partial^2 S}{\partial b^2} = 6, \frac{\partial^2 S}{\partial b \partial a} = 12$

Observe that $\frac{\partial^2 S}{\partial a^2} = 28 > 0$ and

$$\Delta\left(3.05, \frac{1}{30}\right) = S_{aa}\left(3.05, \frac{1}{30}\right)S_{bb}\left(3.05, \frac{1}{30}\right) - S_{ab}^2\left(3.05, \frac{1}{30}\right) = 28 \times 6 - 144 = 168 - 144 = 24 > 0.$$

It follows that $S(a, b)$ has a local minimum value at $a = 3.05, b = \frac{1}{30}$.

Observe that $S(a, b) = (3.1 - a - b)^2 + (6.1 - 2a - b)^2 + (9.2 - 3a - b)^2$ is a degree 2 polynomial in a, b

and $\left(3.05, \frac{1}{30}\right)$ is the only local minimum point, $S(a, b)$ actually has the global minimum value

$$S\left(3.05, \frac{1}{30}\right) = \frac{1}{600} \text{ at } \left(3.05, \frac{1}{30}\right) \text{ and this fact can be shown as follows:}$$

Due to Taylor's expansion theorem and noticing that $S(a, b)$ is a degree 2 polynomial in a, b , we have

$$\begin{aligned} & S(a, b) - S\left(3.05, \frac{1}{30}\right) \\ &= \frac{1}{2} \left[S_{aa}\left(3.05, \frac{1}{30}\right)(a - 3.05)^2 + 2S_{ab}\left(3.05, \frac{1}{30}\right)(a - 3.05)\left(b - \frac{1}{30}\right) + S_{bb}\left(3.05, \frac{1}{30}\right)\left(b - \frac{1}{30}\right)^2 \right] \\ &= \frac{1}{2} \left[28(a - 3.05)^2 + 24(a - 3.05)\left(b - \frac{1}{30}\right) + 6\left(b - \frac{1}{30}\right)^2 \right] \\ &= \frac{1}{2} \left[\begin{pmatrix} a - 3.05 & b - \frac{1}{30} \end{pmatrix} \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix} \right] \end{aligned}$$

See that $\begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix}$ is a 2×2 real symmetric matrix which eigenvalues are $17 \pm \sqrt{265} (> 0)$.

Then there exist a 2×2 orthogonal matrix P and the diagonal matrix $D = \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix}$

$$\text{such that } \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} = P \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix} P^{-1} \stackrel{P^{-1} = P^T}{=} P \begin{pmatrix} 17 + \sqrt{265} & 0 \\ 0 & 17 - \sqrt{265} \end{pmatrix} P^T$$

Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P^T \begin{pmatrix} a - 3.05 \\ b - \frac{1}{30} \end{pmatrix}$. Then

$$\begin{aligned}
& \begin{pmatrix} a-3.05 & b-\frac{1}{30} \end{pmatrix} \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix} = \begin{pmatrix} a-3.05 & b-\frac{1}{30} \end{pmatrix} P \begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix} P^T \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix} \\
& = \left[P^T \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix} \right]^T \begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix} P^T \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix} \\
& = (y_1 \ y_2) \begin{pmatrix} 17+\sqrt{265} & 0 \\ 0 & 17-\sqrt{265} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = (17+\sqrt{265})y_1^2 + (17-\sqrt{265})y_2^2 \geq 0
\end{aligned}$$

It follows that $S(a,b) - S\left(3.05, \frac{1}{30}\right) = \frac{1}{2} \left[\begin{pmatrix} a-3.05 & b-\frac{1}{30} \end{pmatrix} \begin{pmatrix} 28 & 12 \\ 12 & 6 \end{pmatrix} \begin{pmatrix} a-3.05 \\ b-\frac{1}{30} \end{pmatrix} \right] \geq 0$.

Therefore, $S(a,b)$ has the global minimum value $S\left(3.05, \frac{1}{30}\right) = \frac{1}{600}$ at $\left(3.05, \frac{1}{30}\right)$.

□

Example

Find the global (absolute) maximum and global minimum values of the function

$$f(x, y) = x^2 + y^2 - x - y + 1 \text{ in the disk defined by } x^2 + y^2 \leq 1.$$

Solution:

Remark: Since $f(x, y) = x^2 + y^2 - x - y + 1$ is a continuous function defined on a bounded closed disk

$x^2 + y^2 \leq 1$, according to some theorems not mentioned $f(x, y)$ can attain global maximum and global minimum at stationary points or at those points where $f(x, y)$ do not have partial derivatives or at the points on the boundary.

Consider $f(x, y) = x^2 + y^2 - x - y + 1$ on $x^2 + y^2 = 1$, where $x^2 + y^2 = 1$ is the boundary of $x^2 + y^2 \leq 1$.

Let $x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$, consider

$$g(t) \equiv f(\cos t, \sin t) = \cos^2 t + \sin^2 t - \cos t - \sin t + 1 = 2 - \cos t - \sin t, \text{ where } 0 \leq t \leq 2\pi.$$

$$\text{Then } g'(t) = \sin t - \cos t = 0 \Rightarrow \sin t = \cos t \Rightarrow t = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$

$$g''(t) = \cos t + \sin t \text{ and } g''\left(\frac{\pi}{4}\right) > 0, g''\left(\frac{5\pi}{4}\right) < 0. \quad g(0) = g(2\pi) = 1, \quad g\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}, \quad g\left(\frac{5\pi}{4}\right) = 2 + \sqrt{2}.$$

It is obvious that $g(t)$ attains the greatest value $g\left(\frac{5\pi}{4}\right) = 2 + \sqrt{2}$ at $t = \frac{5\pi}{4}$ and the smallest value

$$g\left(\frac{\pi}{4}\right) = 2 - \sqrt{2} \text{ at } t = \frac{\pi}{4}$$

When $t = \frac{5\pi}{4}, x = -\frac{\sqrt{2}}{2}, y = -\frac{\sqrt{2}}{2}$.

When $t = \frac{\pi}{4}, x = \frac{\sqrt{2}}{2}, y = \frac{\sqrt{2}}{2}$

Consider $f(x, y) = x^2 + y^2 - x - y + 1$ inside $x^2 + y^2 = 1$.

$$\begin{cases} \frac{\partial f}{\partial x} = 2x - 1 = 0 \\ \frac{\partial f}{\partial y} = 2y - 1 = 0 \end{cases} \Rightarrow \begin{cases} x = 1/2 \\ y = 1/2 \end{cases} \quad \text{and} \quad f_{xx} = 2, f_{yy} = 2, f_{xy} = 0$$

$$\Delta\left(\frac{1}{2}, \frac{1}{2}\right) = f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right)f_{yy}\left(\frac{1}{2}, \frac{1}{2}\right) - f_{xy}^2\left(\frac{1}{2}, \frac{1}{2}\right) = 4 \text{ and } f_{xx}\left(\frac{1}{2}, \frac{1}{2}\right) = 2.$$

$f(x, y) = x^2 + y^2 - x - y + 1$ has a local minimum at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$

Compare $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$ and $f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$.

We conclude that $f(x, y) = x^2 + y^2 - x - y + 1$ has a absolute minimum value at $\left(\frac{1}{2}, \frac{1}{2}\right)$ and

$$f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}.$$

Also $f(x, y) = x^2 + y^2 - x - y + 1$ has a absolute maximum value at $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ and

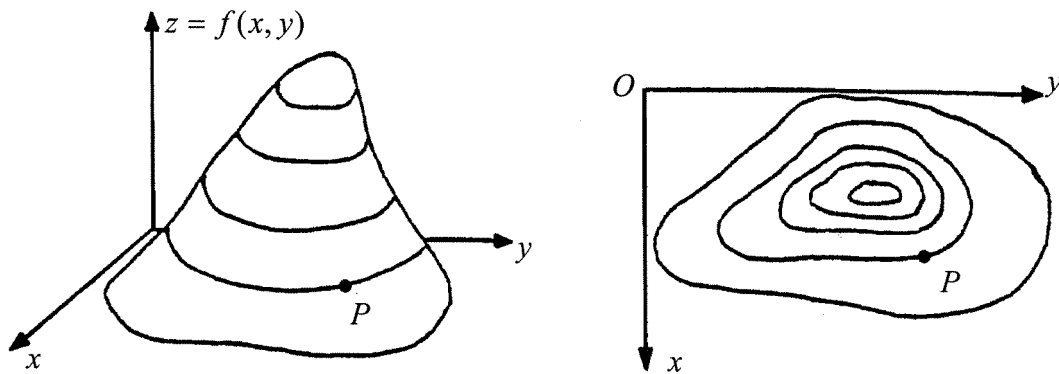
$$f\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2}.$$

□

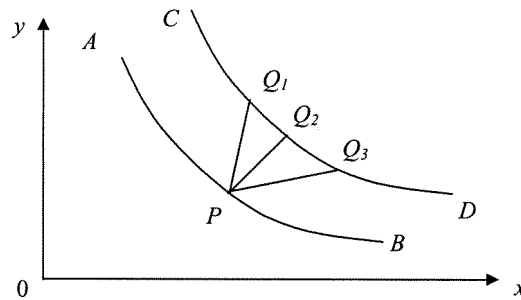
□

8 Directional Derivative and Gradient

For the scalar function $z = f(x, y)$ the partial derivatives $f_x(x, y)$ and $f_y(x, y)$ measure the rates of change of $z = f(x, y)$ in the x and y directions respectively. We would like to know the rate of change in any direction.



Consider a point P which lies on some level curve AB and a (variable) point Q on a nearby level curve CD .

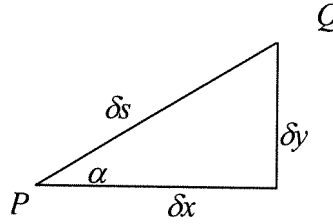


The rate of change of f in the direction PQ is known as the directional derivative of $f(x, y)$ in the direction

$$PQ \text{ and is } \frac{df}{ds} = \lim_{\delta s \rightarrow 0} \frac{\delta f}{\delta s} \text{ where } \frac{\delta f}{\delta s} = \frac{f(P) - f(Q)}{PQ}, PQ = \delta s$$

Geometrically it is clear that $\frac{df}{ds}$ will be a maximum in the direction normal to the level curve at P (since $f(P) - f(Q)$ is constant and PQ is least in that direction). $\frac{df}{ds}$ is also denoted as $D_{\vec{v}}f$ where $\vec{v}$ is the unit vector in the direction PQ .

If the direction PQ is inclined an angle α to the x -axis we have



$$\delta x = \delta s \cos \alpha, \quad \delta y = \delta s \sin \alpha$$

From Taylor's Theorem $\delta f \approx \frac{\partial f}{\partial x} \delta x + \frac{\partial f}{\partial y} \delta y$, thus $\frac{\delta f}{\delta s} \approx \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha$

And as $\delta s \rightarrow 0$, $\frac{df}{ds} = D_{\vec{v}}f = \frac{\partial f}{\partial x} \cos \alpha + \frac{\partial f}{\partial y} \sin \alpha \dots (*)$, where $\vec{v} = \vec{i} \cos \alpha + \vec{j} \sin \alpha$.

By considering $\frac{df}{ds}$ as a function of α it is easy to show that $\frac{df}{ds}$ has a maximum value of

$$\left[\left(\frac{\partial f}{\partial x} \right)^2 + \left(\frac{\partial f}{\partial y} \right)^2 \right]^{\frac{1}{2}} \text{ when } \alpha = \tan^{-1} \left(\frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial x}} \right). \text{ If we define a vector } \text{grad} f = \nabla f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} \text{ known as the}$$

gradient of f , its magnitude is that of the maximum value of $\frac{df}{ds}$ and its direction is that in which $\frac{df}{ds}$ has its maximum value.

Hence ∇f gives us the magnitude and direction (normal to the level curve at P) of the greatest rate of change of f .

Also, from (*), if $\vec{v}$ is a unit vector in any other direction PQ , then $\vec{v} = (\cos \alpha) \vec{i} + (\sin \alpha) \vec{j}$ and $\frac{df}{ds} = D_{\vec{v}}f = \text{grad} f \cdot \vec{v} = \nabla f \cdot \vec{v}$

The above discussion can be generalized to n independent variables.

For example, if $\vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then $D_{\vec{v}} f(a_1, a_2, a_3) = \begin{bmatrix} f_x(a_1, a_2, a_3) \\ f_y(a_1, a_2, a_3) \\ f_z(a_1, a_2, a_3) \end{bmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = f_x(a_1, a_2, a_3)$

Now, we would to show that the gradient $\nabla f(x_1, \dots, x_n) = \begin{bmatrix} f_{x_1}(x_1, \dots, x_n) \\ \vdots \\ f_{x_n}(x_1, \dots, x_n) \end{bmatrix} \neq \vec{0}$ of f at $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ points in the

directions in which f increases most rapidly.

Let $\vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$ be a unit vector i.e. $|\vec{v}| = 1$. Then by Cauchy-Schawz inequality, we have

$$|D_{\vec{v}} f(a_1, a_2, a_3)| = |\nabla f(a_1, a_2, a_3) \cdot \vec{v}| \leq |\nabla f(a_1, a_2, a_3)| |\vec{v}| = |\nabla f(a_1, a_2, a_3)|$$

In addition, let $\vec{w} = \frac{\nabla f(a_1, \dots, a_n)}{|\nabla f(a_1, \dots, a_n)|}$.

Then,

$$\begin{aligned} D_{\vec{w}} f(a_1, \dots, a_n) &= \nabla f(a_1, \dots, a_n) \cdot \vec{w} = \nabla f(a_1, \dots, a_n) \cdot \frac{\nabla f(a_1, \dots, a_n)}{|\nabla f(a_1, \dots, a_n)|} \\ &= \frac{|\nabla f(a_1, \dots, a_n)|^2}{|\nabla f(a_1, \dots, a_n)|} = |\nabla f(a_1, \dots, a_n)| \end{aligned}$$

Example

Let $f(x, y, z) = 10 + xy + xz + yz$ and $\vec{v} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T$. Find $D_{\vec{v}} f(1, 1, 1)$.

Solution:

Method 1:

$$\begin{aligned} D_{\vec{v}} f(1, 1, 1) &= \lim_{t \rightarrow 0} \frac{f\left(1+t\frac{1}{\sqrt{2}}, 1+t\frac{1}{\sqrt{2}}, 1\right) - f(1, 1, 1)}{t} \\ &= \lim_{t \rightarrow 0} \frac{10 + \left(1+\frac{t}{\sqrt{2}}\right)\left(1+\frac{t}{\sqrt{2}}\right) + \left(1+\frac{t}{\sqrt{2}}\right) + \left(1+\frac{t}{\sqrt{2}}\right) - (10+1+1+1)}{t} \\ &= \lim_{t \rightarrow 0} \frac{\left(1+\frac{t}{\sqrt{2}}\right)^2 + 2\left(1+\frac{t}{\sqrt{2}}\right) - 3}{t-0} \end{aligned}$$

Let $g(t) = \left(1+\frac{t}{\sqrt{2}}\right)^2 + 2\left(1+\frac{t}{\sqrt{2}}\right)$. Then

$$\begin{aligned} D_{\vec{v}} f(1, 1, 1) &= \lim_{t \rightarrow 0} \frac{\left(1+\frac{t}{\sqrt{2}}\right)^2 + 2\left(1+\frac{t}{\sqrt{2}}\right) - 3}{t-0} = \lim_{t \rightarrow 0} \frac{g(t) - g(0)}{t-0} \\ &= \frac{dg}{dt}\bigg|_{t=0} = \frac{d\left[\left(1+\frac{t}{\sqrt{2}}\right)^2 + 2\left(1+\frac{t}{\sqrt{2}}\right)\right]}{dt}\bigg|_{t=0} = \left[2\left(1+\frac{t}{\sqrt{2}}\right)\frac{1}{\sqrt{2}} + \frac{2}{\sqrt{2}}\right]\bigg|_{t=0} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \end{aligned}$$

Method 2:

$g(t) = f\left(1+t\frac{1}{\sqrt{2}}, 1+t\frac{1}{\sqrt{2}}, 1\right)$. Then using chain rule, we have

$$\begin{aligned} \frac{dg}{dt}\bigg|_{t=0} &= \frac{\partial f}{\partial x}(1, 1, 1) \frac{dx}{dt}(0) + \frac{\partial f}{\partial y}(1, 1, 1) \frac{dy}{dt}(0) + \frac{\partial f}{\partial z}(1, 1, 1) \frac{dz}{dt}(0) \\ &= \begin{bmatrix} \frac{\partial f}{\partial x}(1, 1, 1) \\ \frac{\partial f}{\partial y}(1, 1, 1) \\ \frac{\partial f}{\partial z}(1, 1, 1) \end{bmatrix} \cdot \begin{bmatrix} \frac{dx}{dt}(0) \\ \frac{dy}{dt}(0) \\ \frac{dz}{dt}(0) \end{bmatrix} = \begin{pmatrix} y+z \\ x+z \\ x+y \end{pmatrix} \bigg|_{\substack{x=1 \\ y=1 \\ z=1}} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \frac{4}{\sqrt{2}} = 2\sqrt{2} \end{aligned}$$

□

Example

Suppose that the temperature (in degree Celsius) in the point (x, y, z) is

$$w = f(x, y, z) = \frac{1}{180}(7400 - 4x - 9y - 0.03xy) - 2z. \text{ Suppose a hawk hovering at the point}$$

$P(200, 200, 5)$ above the airport suddenly dives at a speed of 3 km/min in the direction specified by the vector $(3, 4, -12)^T$. What instantaneous rate of change of temperature does the bird experience?

Solution:

The unit vector in the direction of the given vector $3\vec{i} + 4\vec{j} - 12\vec{k}$ is

$$\vec{u} = \frac{3\vec{i} + 4\vec{j} - 12\vec{k}}{\sqrt{3^2 + 4^2 + (-12)^2}} = \frac{3}{13}\vec{i} + \frac{4}{13}\vec{j} - \frac{12}{13}\vec{k}$$

The temperature gradient vector $\nabla f(x, y, z) = -\frac{4 + 0.03y}{180}\vec{i} - \frac{9 + 0.03x}{180}\vec{j} - 2\vec{k}$ has the value

$$\nabla f(200, 200, 5) = -\frac{4 + 0.03 \times 200}{180}\vec{i} - \frac{9 + 0.03 \times 200}{180}\vec{j} - 2\vec{k} = -\frac{10}{180}\vec{i} - \frac{15}{180}\vec{j} - 2\vec{k} \text{ at the initial position}$$

$P(200, 200, 5)$ of the hawk. Therefore the hawk's initial rate of change of temperature with respect to distance is:

$$D_{\vec{u}} f(200, 200, 5) = \nabla f(200, 200, 5) \cdot \vec{u} = \left(-\frac{10}{180}\vec{i} - \frac{15}{180}\vec{j} - 2\vec{k}\right) \cdot \left(\frac{3}{13}\vec{i} + \frac{4}{13}\vec{j} - \frac{12}{13}\vec{k}\right) = 1.808 \frac{^{\circ}\text{C}}{\text{km}}$$

Its speed is $\frac{ds}{dt} = 3 \frac{\text{km}}{\text{min}}$, so the time rate of change of temperature experienced by the hawk is

$$\frac{dw}{dt} = \frac{dw}{ds} \frac{ds}{dt} = D_{\vec{u}} w \frac{ds}{dt} = \left(1.808 \frac{^{\circ}\text{C}}{\text{km}}\right) \left(3 \frac{\text{km}}{\text{min}}\right) = 5.424 \frac{^{\circ}\text{C}}{\text{min}}.$$

Thus the hawk initially gets warmer by almost 5.5 degrees per minute as it dives toward the ground.

□

Example (for reference only)

Consider $f(x, y, z) = axy^2 + byz + cz^2x^3$.

- (a) Determine the value of a, b, c such that the value of the partial derivative $\frac{\partial f}{\partial z}$ at $P(1, 2, -1)$ is the greatest among all directional derivatives of f at $P(1, 2, -1)$.
- (b) Let a, b, c be the values found in (i), find the directional derivative $D_{\vec{v}} f(1, 2, -1)$ with $\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k}$.

Solution:

(a)

$$f(x, y, z) = axy^2 + byz + cz^2x^3 \Rightarrow \nabla f(x, y, z) = (ay^2 + 3cz^2x^2)\vec{i} + (2axy + bz)\vec{j} + (by + 2czx^3)\vec{k}$$

$$\text{And } \nabla f(1, 2, -1) = (4a + 3c)\vec{i} + (4a - b)\vec{j} + (2b - 2c)\vec{k}.$$

In order the value of the partial derivative $\frac{\partial f}{\partial z}$ at $P(1, 2, -1)$ is the greatest among all directional

$$\text{derivatives of } f \text{ at } P(1, 2, -1) \text{ we have } \begin{cases} 4a + 3c = 0 \\ 4a - b = 0 \\ 2b - 2c > 0 \end{cases} \Rightarrow \begin{cases} c = -\frac{4a}{3} \\ b = 4a \\ b - c > 0 \end{cases} \Rightarrow 4a + \frac{4a}{3} > 0 \Rightarrow \frac{16a}{3} > 0 \Rightarrow a > 0$$

$$\text{Then } \nabla f(1, 2, -1) = \left(8a + \frac{8a}{3}\right)\vec{k} = \frac{32a}{3}\vec{k}, \quad a > 0.$$

(b)

$$D_{\vec{i}+2\vec{j}+3\vec{k}} f(1, 2, -1) = \nabla f(1, 2, -1) \cdot \frac{\vec{i}+2\vec{j}+3\vec{k}}{|\vec{i}+2\vec{j}+3\vec{k}|} = \frac{32a}{3}\vec{k} \cdot \frac{\vec{i}+2\vec{j}+3\vec{k}}{\sqrt{14}} = \frac{32a}{\sqrt{14}}, \quad a > 0.$$

□

Example (for reference only)

Consider the function $f(x, y) = \begin{cases} \frac{xy^2}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$.

Recall that $f(x, y)$ is not differentiable at $(0, 0)$.

(a) Given $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ where $a^2 + b^2 = 1$ and $a \neq 0, b \neq 0$, the straight line passing through the origin with direction $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$ is $\begin{pmatrix} x \\ y \end{pmatrix} = t \begin{pmatrix} a \\ b \end{pmatrix}$, where t is any real number. Find the directional derivative $f_v(0, 0)$.

(b) Show that the chain rule is not applicable to find $f_v(0, 0)$.

Solution:

(a)

$$f(at, bt) = \begin{cases} \frac{ab^2}{a^2 + b^2} t & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases} \Rightarrow f(at, bt) = \frac{ab^2}{a^2 + b^2} t$$

$$\text{Then } f_v(0, 0) = \lim_{t \rightarrow 0} \frac{\frac{ab^2}{a^2 + b^2} t}{t} = \frac{ab^2}{a^2 + b^2}.$$

$f(x, y)$ has directional derivatives in all directions.

(b)

By (a), $f_v(0, 0)$ exists and $f_v(0, 0) = \frac{ab^2}{a^2 + b^2} \neq 0$ where $a^2 + b^2 = 1$ and $a \neq 0, b \neq 0$.

$$\text{However, } \nabla f(0, 0) \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} f_x(0, 0) \\ f_y(0, 0) \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 0 \neq f_v(0, 0) = \frac{ab^2}{a^2 + b^2}.$$

Remark: This example shows the fact that the chain rule is not applicable if f is not differentiable.

□