

CITY UNIVERSITY OF HONG KONG

Course code and title : MA1200 Calculus and Basic Linear Algebra I

Session : Semester A, 2016/2017

Time allowed : Three hours

This paper has **SIX** pages (including this cover page).

A brief table of derivatives is attached on pages 5 and 6.

Instructions to candidates:

1. This paper has **TEN** questions.
 2. Attempt **ALL** questions.
 3. Each question carries 10 marks.
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*This is a **closed-book** examination.*

Candidates are allowed to use the following materials/aids:

Non-programmable calculators

Materials/aids other than those stated above are not permitted. Candidates will be subject to disciplinary action if any unauthorised materials or aids are found on them.

Question 1

The functions $f(x)$ and $g(x)$ are defined by

$$f(x) = 2x - 3, \text{ for } x \in \mathbb{R},$$

$$g(x) = \frac{1}{x-2}, \text{ for } x \in \mathbb{R} \setminus \{2\}.$$

Find, in a similar form

(a) the inverse function $f^{-1}(x)$, (5 marks)

(b) the composite function $(g \circ f)(x)$. (5 marks)

In each case state the largest possible domain and the range of the function.

Question 2

Evaluate the following limits:

(a) $\lim_{x \rightarrow 0} \frac{\sinh x}{x}$, (3 marks)

(b) $\lim_{x \rightarrow \infty} \cos^{-1}\left(\frac{1-x}{1+x}\right)$, (3 marks)

(c) $\lim_{x \rightarrow 0^+} \frac{2}{3 + e^{\frac{1}{x}}}$ (4 marks)

Question 3

An ellipse has equation $9x^2 + 25y^2 + 36x - 50y - 164 = 0$.

(a) Sketch the ellipse. (2 marks)

(b) Find the coordinates of the foci of the ellipse. (4 marks)

(c) Find the equation of the tangent to the ellipse at the point $P\left(2, \frac{14}{5}\right)$ (4 marks)

Question 4

- (a) In 263 A.D., a Chinese Mathematician, Liu Hui proposed a method to compute an approximation to the value of π .

Given a unit circle, he calculated the area of inscribed and circumscribed regular hexagon as shown in Figure 1, thus obtaining lower and upper bounds for the area of the unit circle ($= \pi$ sq.unit). Find the areas of hexagon ABCDEF and hexagon PQRSTU, correct your answers to 3 decimal places.

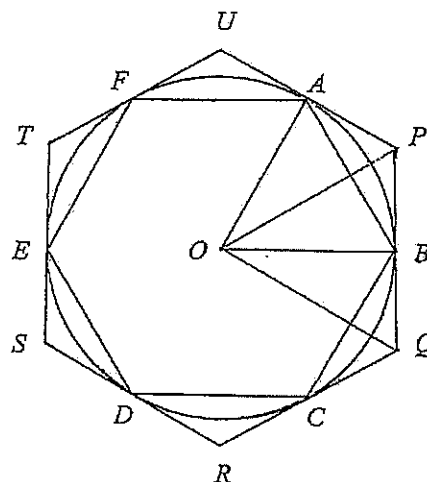


Figure 1

(6 marks)

- (b) Find, in radians, the general solution of the trigonometric equation $\sin(5x) = -\frac{1}{2}$.

(4 marks)

Question 5

- (a) Differentiate with respect to x :

(i) $x^3 e^{-2x}$,

(3 marks)

(ii) $\log_e(\cot x + \operatorname{cosec} x)$.

(3 marks)

- (b) Find the general formula for the n th derivative of the function $F(x) = \frac{3x}{x+3}$ with respect to x .

(4 marks)

Question 6

- (a) A curve has parametric equations $x = t - t^{-1}$, $y = t + t^{-1}$, for $t \in (0, \infty)$.

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t .

(5 marks)

- (b) Differentiate with respect to x :

(i) $(\cos x)^x$,

(3 marks)

(ii) $\sqrt{\cos x} + (\cos x)^x$.

(2 marks)

Question 7

Given that $(x^2 + \sqrt{2}x + 1)$ is a factor of $x^4 + 1$, express $\frac{4x^3 + 8x}{x^4 + 1}$ as partial fractions. (10 marks)

Question 8

(a) If $y = (1 + x^2)^{\frac{1}{2}} \log_e(x + \sqrt{x^2 + 1})$, show that $(1 + x^2) \frac{dy}{dx} + xy = 1$. (3 marks)

(b) Deduce that $(1 + x^2) \frac{d^{n+2}y}{dx^{n+2}} + (2n + 3)x \frac{d^{n+1}y}{dx^{n+1}} + (n + 1)^2 \frac{d^n y}{dx^n} = 0$. (3 marks)

(c) Hence, or otherwise, find the expansion of $(1 + x^2)^{-\frac{1}{2}} \log_e(x + \sqrt{x^2 + 1})$ in ascending powers of x as far as the term in x^5 . (4 marks)

Question 9

The equation of a curve is $y = \frac{x + 4}{x(x + 3)}$, for $x \in \mathbb{R} \setminus \{-3, 0\}$.

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. (3 marks)

(b) Show that $P(-6, -\frac{1}{9})$ and $Q(-2, -1)$ are the stationary points of the curve. (4 marks)

(c) Use the second derivative test, determine the nature of the stationary points P and Q . (3 marks)

Question 10

(a) Prove from first principles that $\frac{d}{dx}(\sin x) = \cos x$. (5 marks)

(Hint: You may use $\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$.)

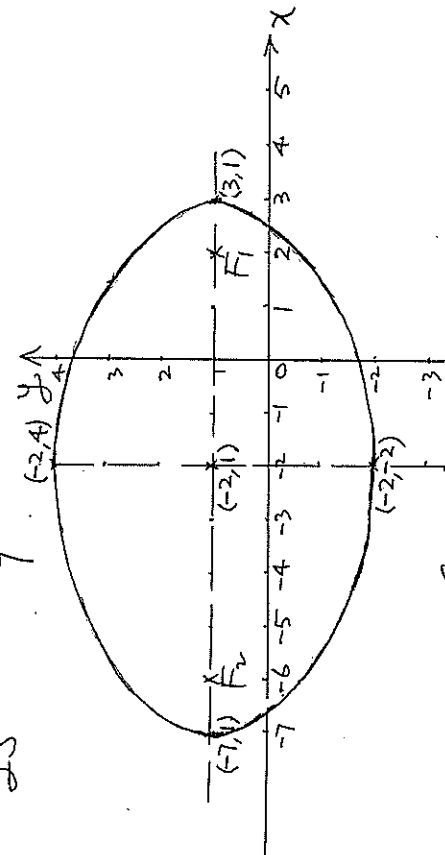
(b) Let $F(x) = |\sin x|$, for $x \in \mathbb{R}$. Determine whether $F(x)$ is differentiable at $x = 0$. Give your reason. (5 marks)

Short Table of Derivatives of $y = f(u)$ with respect to x , where u is a function of x

Functions, $y = f(u)$	Derivative of y with respect to x
$y = c$, where c is a constant.	$\frac{dy}{dx} = 0$
$y = cu$, where c is a constant.	$\frac{dy}{dx} = c \frac{du}{dx}$
$y = u^p$, where p is a constant.	$\frac{dy}{dx} = pu^{p-1} \frac{du}{dx}$
$y = u + v$	$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$
$y = uv$	$\frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$
$y = f(u)$, where u is a function of x .	$\frac{dy}{dx} = \frac{df(u)}{du} \cdot \frac{du}{dx}$, the chain rule
$y = \log_a u$, $a > 0$.	$\frac{dy}{dx} = \frac{1}{u} \log_a e \frac{du}{dx}$
$y = a^u$, $a > 0$.	$\frac{dy}{dx} = a^u \log_e a \frac{du}{dx}$
$y = e^u$	$\frac{dy}{dx} = e^u \frac{du}{dx}$
$y = u^v$	$\frac{dy}{dx} = vu^{v-1} \frac{du}{dx} + u^v \log_e u \frac{dv}{dx}$
$y = \sin u$	$\frac{dy}{dx} = \cos u \frac{du}{dx}$
$y = \cos u$	$\frac{dy}{dx} = -\sin u \frac{du}{dx}$
$y = \tan u$	$\frac{dy}{dx} = \sec^2 u \frac{du}{dx}$
$y = \cot u$	$\frac{dy}{dx} = -\operatorname{cosec}^2 u \frac{du}{dx}$
$y = \sec u$	$\frac{dy}{dx} = \sec u \tan u \frac{du}{dx}$
$y = \operatorname{cosec} u$	$\frac{dy}{dx} = -\operatorname{cosec} u \cot u \frac{du}{dx}$
$y = \sin^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \frac{du}{dx}$
$y = \cos^{-1} u$	$\frac{dy}{dx} = \frac{-1}{\sqrt{1-u^2}} \frac{du}{dx}$
$y = \tan^{-1} u$	$\frac{dy}{dx} = \frac{1}{1+u^2} \frac{du}{dx}$

Functions, $y = f(u)$	Derivative of y with respect to x
$y = \cot^{-1} u$	$\frac{dy}{dx} = -\frac{1}{1+u^2} \frac{du}{dx}$
$y = \sec^{-1} u$	$\frac{dy}{dx} = \frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}$
$y = \operatorname{cosec}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{ u \sqrt{u^2-1}} \frac{du}{dx}$
$y = \sinh u$	$\frac{dy}{dx} = \cosh u \frac{du}{dx}$
$y = \cosh u$	$\frac{dy}{dx} = \sinh u \frac{du}{dx}$
$y = \tanh u$	$\frac{dy}{dx} = \operatorname{sech}^2 u \frac{du}{dx}$
$y = \coth u$	$\frac{dy}{dx} = -\operatorname{cosech}^2 u \frac{du}{dx}$
$y = \operatorname{sech} u$	$\frac{dy}{dx} = -\operatorname{sech} u \tanh u \frac{du}{dx}$
$y = \operatorname{cosech} u$	$\frac{dy}{dx} = -\operatorname{cosech} u \coth u \frac{du}{dx}$
$y = \sinh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{1+u^2}} \frac{du}{dx}$
$y = \cosh^{-1} u$	$\frac{dy}{dx} = \frac{1}{\sqrt{u^2-1}} \frac{du}{dx}$
$y = \tanh^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$
$y = \coth^{-1} u$	$\frac{dy}{dx} = \frac{1}{1-u^2} \frac{du}{dx}$
$y = \operatorname{sech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{u\sqrt{1-u^2}} \frac{du}{dx}$
$y = \operatorname{cosech}^{-1} u$	$\frac{dy}{dx} = -\frac{1}{ u \sqrt{u^2+1}} \frac{du}{dx}$

Question Number	Marks	Department of Mathematics, CityU Examination Solutions Code : MA 1200 Academic Year: 2016/17, Sem. A. Title: Calculus and Basic Linear Algebra I	Department of Mathematics, CityU Examination Solutions Code : MA Academic Year: _____ Title: _____	Ma
Q.1.		(a) $f(x)$ is one-to-one. Let $y = f(x) = 2x - 3$. Then $x = \frac{1}{2}(y + 3)$ $\therefore f^{-1}(x) = \frac{1}{2}(x + 3)$, for $x \in \mathbb{R}$. $\text{Dom}(f^{-1}) = \mathbb{R}$, $\text{Ran}(f^{-1}) = \mathbb{R}$. (b) $(g \circ f)(x) = g(f(x))$ $= g(2x - 3)$ $= \frac{1}{(2x - 3) - 2}$ $= \frac{1}{2x - 5}$, for $x \in \mathbb{R} \setminus \{\frac{5}{2}\}$ $\text{Dom}(g \circ f) = \mathbb{R} \setminus \{\frac{5}{2}\}$ $\text{Ran}(g \circ f) = \mathbb{R} \setminus \{0\}$	(a) $\lim_{x \rightarrow 0} \frac{\sinh x}{x}$ ($= \frac{0}{0}$ form) $= \lim_{x \rightarrow 0} \frac{\cosh x}{1}$, by De L'Hôpital rule $= 1$ (b) For large value of x, we have $\frac{1-x}{1+x} = \frac{-(1+x)+2}{1+x} = -1 + \frac{2}{1+x} \in (-1, 1] \subset [-1, 1]$ Since $\text{Dom}(\cos^{-1}) = [-1, 1]$ $\lim_{x \rightarrow \infty} \cos^{-1}\left(\frac{1-x}{1+x}\right) = \cos^{-1}\left(\lim_{x \rightarrow \infty} \frac{1-x}{1+x}\right)$ $= \cos^{-1}\left(\lim_{x \rightarrow \infty} \frac{\frac{1}{x} - 1}{\frac{1}{x} + 1}\right)$ $= \cos^{-1}\left(-\frac{1}{1}\right) = \cos^{-1}(-1) = \pi$ (c) $\lim_{x \rightarrow 0^-} \frac{2}{3 + e^{\frac{1}{x}}}$ $= \frac{2}{3 + 0}$, since $\frac{1}{x} \rightarrow -\infty$, $= \frac{2}{3}$, as $x \rightarrow 0^-$	3 1 + 1 = 2 3 1 1 4
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Question Number	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Marks	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Marks
Q.3.	(a) $9x^2 + 36x + 25y^2 - 50y = 164$ $9(x^2 + 4x + 4) + 25(y^2 - 2y + 1) = 164 + 36 + 25$ $9(x+2)^2 + 25(y-1)^2 = 225$ $\frac{(x+2)^2}{25} + \frac{(y-1)^2}{9} = 1$  Graph (b) $b^2 = a^2(1 - e^2)$ $e^2 = 1 - \frac{9}{25} = \frac{16}{25}$, $\therefore e = \frac{4}{5}$ $ae = 5(\frac{4}{5}) = 4$ $\therefore$ The coordinates of the foci of the ellipse are $(2, 1)$ and $(-6, 1)$.	(c) $9x^2 + 25y^2 + 36x - 50y - 164 = 0$ — (1) Differentiating both sides of eq. (1) w.r.t. x, $18x + 50y \frac{dy}{dx} + 36 - 50 \frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{18(x+2)}{50(y-1)}$ Slope of the tangent to the ellipse at $P(2, \frac{14}{5})$ is given by $m = \frac{dy}{dx} \bigg _{x=2, y=\frac{14}{5}} = -\frac{9}{25} \times \frac{4}{\frac{9}{5}} = -\frac{4}{5}$ $\therefore$ The equation of the tangent is $\frac{y - \frac{14}{5}}{x - 2} = -\frac{4}{5}$, that is $4x + 5y - 22 = 0$. //
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Question Number	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Marks	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Marks
		(c) $9x^2 + 25y^2 + 36x - 50y - 164 = 0$ — (1) Differentiating both sides of eq. (1) w.r.t. x, $18x + 50y \frac{dy}{dx} + 36 - 50 \frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{18(x+2)}{50(y-1)}$ Slope of the tangent to the ellipse at $P(2, \frac{14}{5})$ is given by $m = \frac{dy}{dx} \bigg _{x=2, y=\frac{14}{5}} = -\frac{9}{25} \times \frac{4}{\frac{9}{5}} = -\frac{4}{5}$ $\therefore$ The equation of the tangent is $\frac{y - \frac{14}{5}}{x - 2} = -\frac{4}{5}$, that is $4x + 5y - 22 = 0$. //
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Q.4.	(a) Area of hexagon ABCDEF $= 6 \times \text{area of } \triangle OAB$ $= 6 \left(\frac{1}{2} OA \times OB \sin \angle AOB \right)$ $= 6 \left(\frac{1}{2} \times 1 \times 1 \times \frac{\sqrt{3}}{2} \right)$ $= \frac{3\sqrt{3}}{2} \text{ units}^2 = \underline{\underline{2.598 \text{ units}^2 \text{ (to 3 dec. places)}}}$ Area of hexagon PQRSU $= 6 \times \text{area of } \triangle OPQ$ $= 6 \left(\frac{1}{2} OP \times OQ \sin \angle POQ \right)$ $= 6 \left(\frac{1}{2} \times \frac{1}{\cos 30^\circ} \times \frac{1}{\cos 30^\circ} \sin 60^\circ \right)$ $= \underline{\underline{2\sqrt{3} \text{ units}^2 = 3.464 \text{ units}^2 \text{ (to 3 dec. places)}}}$ (b) $\sin(5x) = -\frac{1}{2}$ $5x = n\pi + (-1)^n \alpha, \quad n = 0, \pm 1, \pm 2, \dots$ where $\alpha = \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$ $\therefore$ The general solution of the equation is $x = \frac{1}{5} \left(n\pi + (-1)^{n+1} \frac{\pi}{6} \right), \quad n = 0, \pm 1, \pm 2, \dots$	4
	Signature of Examiner : Total :	

Question Number	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Mar	Marks
Q.5.	(a) (i) $\frac{d}{dx}(x^3 e^{-2x})$ $= x^3 \frac{d}{dx}(e^{-2x}) + (e^{-2x}) \frac{d}{dx}(x^3)$ $= x^3(-2e^{-2x}) + e^{-2x}(3x^2)$ $= x^2 e^{-2x}(3 - 2x)$ (ii) $\frac{d}{dx}(\log_e(\cot x + \operatorname{cosec} x))$ $= \frac{1}{\cot x + \operatorname{cosec} x} (-\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x)$ $= -\operatorname{cosec} x$ (b) $F(x) = \frac{3x}{x+3} = \frac{3(x+3)-9}{x+3} = 3 - \frac{9}{x+3}$ $\frac{dF(x)}{dx} = 0 - 9 \frac{d}{dx}(x+3)^{-1} = 9(-1)^2(x+3)^{-2}$ $\frac{d^2F(x)}{dx^2} = 9(-1)^2 \frac{d}{dx}(x+3)^{-2} = (-1)^3 9(2!)(x+3)^{-3}$ $\therefore \frac{d^n F(x)}{dx^n} = (-1)^{n+1} 9(n!)(x+3)^{-(n+1)}, \quad n=1, 2, 3, \dots$	3 3 4
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Question Number	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Marks
Q.6.	(a) $\frac{dx}{dt} = \frac{d}{dt}(t - t^{-1}) = 1 + t^{-2}$ $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - t^{-2}}{1 + t^{-2}} = \frac{t^2 - 1}{t^2 + 1}$ " (2) $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{t^2 - 1}{t^2 + 1}\right) = \frac{d}{dt}\left(\frac{t^2 - 1}{t^2 + 1}\right) \cdot \frac{dt}{dx}$ by the chain rule $= \frac{(t^2 + 1)(2t) - (t^2 - 1)(2t)}{(t^2 + 1)^2} \cdot \frac{1}{1 + t^{-2}}$ $= \frac{4t}{(t^2 + 1)^2} \cdot \frac{t^2}{t^2 + 1} = \frac{4t^3}{(t^2 + 1)^3}$ " (3) (b) (i) Let $y = (\cos x)^x$ Then $\log_e y = \log_e (\cos x)^x = x \log_e (\cos x)$ — (1) Differentiating both sides of (1) w.r.t. x, $\frac{1}{y} \frac{dy}{dx} = x \cdot \frac{1}{\cos x} (-\sin x) + \log_e (\cos x)$ — (2)
	Signature of Examiner : Total :

Question Number	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Marks
Q.6.	Multiplying both sides of (2) by y, we have $\frac{dy}{dx} = (\cos x)^x [-x \tan x + \log_e (\cos x)]$ " (3) (ii) $\frac{d}{dx} [\sqrt{\cos x} + (\cos x)^x]$ $= \frac{d}{dx} (\cos x)^{\frac{1}{2}} + \frac{d}{dx} (\cos x)^x$ $= \frac{1}{2} (\cos x)^{-\frac{1}{2}} (-\sin x) + (\cos x)^x [-x \tan x + \log_e (\cos x)]$ $= -\frac{\sin x}{2 \sqrt{\cos x}} + (\cos x)^x [-x \tan x + \log_e (\cos x)]$ (2)
	Signature of Examiner : Total :

Question Number	Department of Mathematics, CityU Examination Solutions Code: MA Title: _____ Academic Year: _____	Marks
Q.7.	$x^4 + 1 = (x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$ Let $\frac{4x^3 + 8x}{x^4 + 1} = \frac{Ax + B}{x^2 + \sqrt{2}x + 1} + \frac{Cx + D}{x^2 - \sqrt{2}x + 1}$ Then $4x^3 + 8x = (Ax + B)(x^2 - \sqrt{2}x + 1) + (Cx + D)(x^2 + \sqrt{2}x + 1)$ $4x^3 + 8x = (A+C)x^3 + (B+D - \sqrt{2}A + \sqrt{2}C)x^2 + (A+C - \sqrt{2}B + \sqrt{2}D)x + B+D$ Equating the corresponding coefficients, we have $A + C = 4 \quad \text{--- (1)}$ $-\sqrt{2}A + B + \sqrt{2}C + D = 0 \quad \text{--- (2)}$ $A - \sqrt{2}B + C + \sqrt{2}D = 8 \quad \text{--- (3)}$ $B + D = 0 \quad \text{--- (4)}$ ④ → ② : $\sqrt{2}(-A + C) = 0 \Rightarrow -A + C = 0 \quad \text{--- (5)}$ From ① and ⑤, we obtain $A = C = 2$ Substituting $A = C = 2$ into ③, we have $\sqrt{2}(-B + D) = 4 \Rightarrow -B + D = 2\sqrt{2} \quad \text{--- (6)}$ From ④ and ⑥, we obtain $B = -D = -\sqrt{2}$ $\therefore \frac{4x^3 + 8x}{x^4 + 1} = \frac{2x - \sqrt{2}}{x^2 + \sqrt{2}x + 1} + \frac{2x + \sqrt{2}}{x^2 - \sqrt{2}x + 1} //$	① ③ ⑥ ② ②

Question Number	Department of Mathematics, CityU Examination Solutions Code: MA Title: _____ Academic Year: _____	Marks
Q.8	(a) $\frac{dy}{dx} = \frac{d}{dx} \left[(1+x^2)^{-\frac{1}{2}} \log_e(x + (x^2+1)^{\frac{1}{2}}) \right]$ $= (1+x^2)^{-\frac{1}{2}} \cdot \frac{1}{x + (x^2+1)^{\frac{1}{2}}} \cdot \left[1 + \frac{1}{2}(x^2+1)^{-\frac{1}{2}} \cdot (2x) \right]$ $+ \log_e(x + (x^2+1)^{\frac{1}{2}}) \cdot \left[-\frac{1}{2}(1+x^2)^{-\frac{3}{2}} \cdot (2x) \right]$ Multiplying both sides by $(1+x^2)$, we have $(1+x^2) \frac{dy}{dx} = 1 - x(1+x^2)^{-\frac{1}{2}} \log_e(x + (x^2+1)^{\frac{1}{2}})$ $\Rightarrow (1+x^2) \frac{dy}{dx} + xy = 1 \quad \text{--- (1)}$ (b) Differentiating both sides of ① $\frac{dy}{dx}$ times with respect to x, and using Leibnitz' rule $\frac{d^{n+1}}{dx^{n+1}} \left[\left(\frac{dy}{dx} \right) \cdot (1+x^2) \right] + \frac{d^{n+1}}{dx^{n+1}} [y \cdot x] = \frac{d^{n+1}}{dx^{n+1}} (1)$ $\frac{d^{n+2}y}{dx^{n+2}} \cdot (1+x^2) + \frac{d^{n+2}y}{dx^{n+2}} \cdot (2x) + \frac{d^{n+1}y}{dx^{n+1}} \cdot (2x) + \frac{d^{n+1}y}{dx^{n+1}} \cdot (1) = 0$ $\Rightarrow (1+x^2) \frac{d^{n+2}y}{dx^{n+2}} + (2n+3)x \frac{d^{n+1}y}{dx^{n+1}} + (n+1)^2 \frac{d^{n+1}y}{dx^{n+1}} = 0$	③ ② ②
Signature of Examiner : _____		Total : _____

Question Number	Department of Mathematics, CityU Examination Solutions	Academic Year : _____	Code : MA _____	Title : _____	Marks
Q.8.	(c) Putting $x=0$, we have the recurrence relation $\frac{d^{n+2}y}{dx^{n+2}}(0) = -(n+1)^2 \frac{d^ny}{dx^n}(0) \text{ --- (3)}$ where $n=1, 2, 3, \dots$ $y = (1+x^2)^{-\frac{1}{2}} \log_e(x + \sqrt{x^2+1})$ $y(0)=0$, $(1+x^2) \frac{dy}{dx} + xy = 1$ Differentiating both sides with respect to x, $(1+x^2) \frac{d^2y}{dx^2} + 3x \frac{dy}{dx} + y = 0$ Put: $x=0$, we have $\frac{dy}{dx}(0)=1$, $\frac{d^2y}{dx^2}(0)=0$. From (3), we have $\frac{d^3y}{dx^3}(0) = -(1+1)^2 \frac{dy}{dx}(0) = -4$, $\frac{d^4y}{dx^4}(0) = -(2+1)^2 \frac{d^2y}{dx^2}(0) = 0$, $\frac{d^5y}{dx^5}(0) = -(3+1)^2 \frac{d^3y}{dx^3}(0) = 64$.				
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Question Number	Department of Mathematics, CityU Examination Solutions	Academic Year : _____	Code : MA _____	Title : _____	Marks
	Using the Maclaurin series for $y(x)$, $y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \dots + \frac{y^{(r)}(0)}{r!}x^r + \dots$ We have $(1+x^2)^{-\frac{1}{2}} \log_e(x + \sqrt{x^2+1})$ $= x - \frac{2}{3}x^3 + \frac{8}{15}x^5 + \dots$				
	Signature of Examiner :				Total :

Question Number	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Signature of Examiner : Total :	Marks	Department of Mathematics, CityU Examination Solutions Code : MA Title : Academic Year : Signature of Examiner : Total :
Q.1.	(a) $\frac{dy}{dx} = \frac{d}{dx} \left(\frac{x+4}{x(x+3)} \right) = \frac{x(x+3) - (x+4)(2x+3)}{x^2(x+3)^2}$ $= \frac{-x^2 - 8x - 12}{x^2(x+3)^2}$ $\frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{-x^2 - 8x - 12}{x^2(x+3)^2} \right)$ $= \frac{x^2(x+3)^2(-2x-8) + (x+2)(x+6)(4x^3+18x^2+18x)}{x^4(x+3)^4}$ $= \frac{1}{x^2(x+3)^2} \left[-2(x+4) + \frac{2(x+2)(x+6)(2x+3)}{x(x+3)} \right]$ $= \frac{2(x^3+12x^2+36x+36)}{x^3(x+3)^3}$	2	(a) Using the second derivative test for the stationary points of the curve, $\frac{d^2y}{dx^2} \Big _{x=-6} = \frac{1}{(+ve)} [-2(-6+4) + 0] > 0,$ $\therefore P(-6, -\frac{1}{9})$ is a local minimum point of the curve. " $\frac{d^2y}{dx^2} \Big _{x=-2} = \frac{1}{(+ve)} [-2(-2+4) + 0] < 0,$ $\therefore Q(-2, -1)$ is a local maximum point of the curve.
Q.1.	(b) If $\frac{dy}{dx} = 0$, then $x^2+8x+12=0$ $(x+2)(x+6)=0$ $\therefore x = -2, -6$ When $x = -2$, $y = \frac{-2+4}{-2(-2+3)} = -1$, When $x = -6$, $y = \frac{-6+4}{-6(-6+3)} = -\frac{1}{9}$. $\therefore P(-6, -\frac{1}{9})$ and $Q(-2, -1)$ are the stationary points of the curve.	2	

