

MA1201 Calculus and Basic Algebra II (Section: CE1)

Quiz 1: Integration

Name:

SID:

Score:

Time allowed: 25 minutes

Instruction: Write all your answer in the spaces provided. Show all your working steps (including evaluation of integrals) clearly. The brief table of integrals, sum-to-product formula and product-to-sum formula are included in the last page.

Appendix A: Brief Table for Integration

$$1. \int x^a dx = \begin{cases} \frac{x^{a+1}}{a+1} + C & \text{if } a \neq -1 \\ \ln|x| & \text{if } a = -1 \end{cases}$$

$$2. \int e^x dx = e^x + C, \quad \int a^x dx = \frac{a^x}{\ln a} + C, a > 0.$$

$$3. \int \sin x dx = -\cos x + C$$

$$4. \int \cos x dx = \sin x + C$$

$$5. \int \sec^2 x dx = \tan x + C$$

$$6. \int \sec x dx = \ln|\sec x + \tan x| + C$$

$$7. \int \csc x dx = -\ln|\csc x + \cot x| + C$$

$$8. \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$9. \int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

Appendix B: Product-to-Sum Formula

$$1. \sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)] \quad 2. \cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$3. \sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

Appendix C: Sum-to-product formula

$$1. \sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$2. \sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$$

$$3. \cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$$

$$4. \cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$$

Problem 1 (50 marks)

Compute the following integrals

(a) $\int \frac{26x^2 - 14x - 1}{(x-2)(2x+1)(3x-1)} dx$

(b) $\int_1^{\sqrt{3}} \frac{1}{(3+x^2)^{\frac{1}{2}}} dx$

Solution

(a) Using method of partial fraction, we decompose the integrand as $\frac{26x^2 - 14x - 1}{(x-2)(2x+1)(3x-1)} = \frac{A}{x-2} + \frac{B}{2x+1} + \frac{C}{3x-1} \Rightarrow \begin{matrix} A(2x+1)(3x-1) + B(x-2)(3x-1) \\ + C(x-2)(2x+1) = 26x^2 - 14x - 1. \end{matrix}$

• Put $x = -\frac{1}{2}$, $\frac{25}{4}B = \frac{50}{4} \Rightarrow B = 2$.

• Put $x = \frac{1}{3}$, $-\frac{25}{9}C = -\frac{25}{9} \Rightarrow C = 1$

• Put $x = 2$, $25A = 75 \Rightarrow A = 3$

$$\int \frac{26x^2 - 14x - 1}{(x-2)(2x+1)(3x-1)} dx = 3 \int \frac{1}{x-2} dx + 2 \int \frac{1}{2x+1} dx + \int \frac{1}{3x-1} dx$$

$$= 3 \ln|x-2| + \ln|2x+1| + \frac{1}{3} \ln|3x-1| + C$$

(b) Let $x = \sqrt{3} \tan \theta$ (so that $3+x^2 = 3+3\tan^2 \theta = 3\sec^2 \theta$),

then $\frac{dx}{d\theta} = \sqrt{3} \sec^2 \theta \Rightarrow dx = \sqrt{3} \sec^2 \theta d\theta$.

When $x = \sqrt{3}$, $\sqrt{3} = \sqrt{3} \tan \theta \Rightarrow \tan \theta = 1 \Rightarrow \theta = \frac{\pi}{4}$

$x = 1$, $1 = \sqrt{3} \tan \theta \Rightarrow \tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6}$.

$$\int_1^{\sqrt{3}} \frac{1}{(3+x^2)^{\frac{1}{2}}} dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{(3\sec^2 \theta)^{\frac{1}{2}}} \sqrt{3} \sec^2 \theta d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sec \theta d\theta$$

$$= \ln|\sec \theta + \tan \theta| \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = \ln|\sqrt{2}+1| - \ln\sqrt{3}.$$

Problem 2 (25 marks)

Compute the integral using suitable method

$$\int_0^{\pi} x \cos 2x \cos 3x \, dx$$

(Hint: Do something on $\cos 2x \cos 3x$ first!)Solution

$$\begin{aligned} \int_0^{\pi} x \cos 2x \cos 3x \, dx &= \int_0^{\pi} x \left[\frac{1}{2} (\cos(2x+3x) + \cos(2x-3x)) \right] dx \\ &= \frac{1}{2} \int_0^{\pi} x \cos 5x \, dx + \frac{1}{2} \int_0^{\pi} x \cos x \, dx \end{aligned}$$

Using integration by parts with $u=x$ and $dv = \begin{cases} \cos 5x \, dx & \text{1st integral} \\ \cos x \, dx & \text{2nd integral} \end{cases}$

$\Rightarrow v = \begin{cases} \frac{1}{5} \sin 5x & \text{for 1st integral} \\ \sin x & \text{for 2nd integral} \end{cases}$, we have

$$\begin{aligned} &\frac{1}{2} \int_0^{\pi} \underbrace{x}_u \underbrace{\cos 5x \, dx}_{dv} + \frac{1}{2} \int_0^{\pi} \underbrace{x}_u \underbrace{\cos x \, dx}_{dv} \\ &= \frac{1}{2} \left(\frac{1}{5} x \sin 5x \Big|_0^{\pi} - \int_0^{\pi} \frac{1}{5} \sin 5x \, dx \right) + \frac{1}{2} \left(x \sin x \Big|_0^{\pi} - \int_0^{\pi} \sin x \, dx \right) \\ &= \frac{1}{50} \cos 5x \Big|_0^{\pi} + \frac{1}{2} \cos x \Big|_0^{\pi} \\ &= -\frac{16}{25}. \end{aligned}$$

Problem 3 (25 points)

Compute the following integral using suitable method

$$\int_0^4 x e^{x^2 + |x^2 - 4|} dx$$

☺Hint:

- Be careful, $|??|$ is the *absolute value function*.

$$\text{Recall that } |y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$$

- The graph of $y = x^2 - 4$ is shown on the right.

Solution

Note that

$$|x^2 - 4| = \begin{cases} x^2 - 4 & \text{if } x \geq 2 \\ 4 - x^2 & \text{if } 0 \leq x < 2 \end{cases}$$

Then the integral becomes

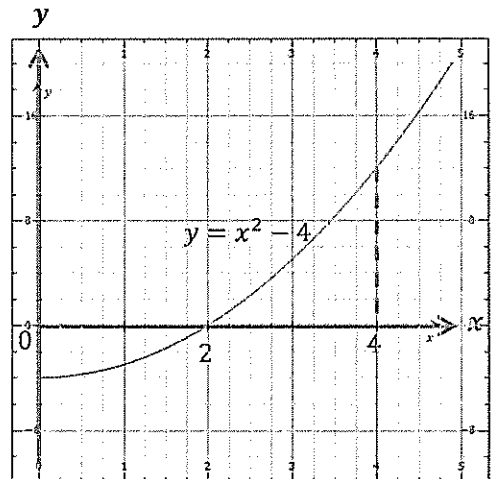
$$\begin{aligned} \int_0^4 x e^{x^2 + |x^2 - 4|} dx &= \int_2^4 x e^{x^2 + x^2 - 4} dx + \int_0^2 x e^{x^2 + (4 - x^2)} dx \\ &= \int_2^4 x e^{2x^2 - 4} dx + \int_0^2 x e^4 dx \end{aligned}$$

$$y = 2x^2 - 4 \Rightarrow \int_4^{28} x e^{2x^2 - 4} \left(\frac{1}{4x} dy \right) + e^4 \int_0^2 x dx$$

$$\frac{dy}{dx} = 4x$$

$$\Rightarrow dx = \frac{1}{4x} dy = \frac{1}{4} \int_4^{28} e^y dy + e^4 \frac{x^2}{2} \Big|_0^2$$

$$\begin{aligned} \text{When } x=2, y=28 \\ x=0, y=4 \end{aligned} \quad \begin{aligned} &= \frac{1}{4} e^y \Big|_4^{28} + 2e^4 \\ &= \frac{e^{28}}{4} + \frac{7}{4} e^4 \end{aligned}$$



☺End of the Quiz☺