

EE3109 Applied Electromagnetics
2022/23 Semester B
Assignment 1 Solution

1. Write Maxwell's Equations in differential form. Write within brackets the unit for each variable (eg. $\mathbf{E}$ [V/m]).

Solution: See the lecture slides

2. Show that the plane wave with magnetic field $\mathbf{H}^-(z, t) = H_0 \cos(\omega t + kz + \phi) \hat{\mathbf{x}}$ [mA/m] travels in the $-z$ -direction in free space with the speed of $v_p = \omega/k$. What is the electric field at the origin?

Solution:

Given $\mathbf{H}^-(z, t) = H_0 \cos(\omega t + kz + \phi) \hat{\mathbf{x}}$ [mA/m], the phase can be written as $\theta = \omega t + kz + \phi$. Phase velocity is the rate at which the constant phase point travels i.e $v_p = dz/dt$ when θ is constant. Re-evaluate the phase expression to get:

$$z = -\frac{\omega}{k}t + \frac{\theta - \phi}{k}$$

$$v_p = dz/dt = -\frac{\omega}{k} \quad [\text{m/s}]$$

Conclusion: The given wave travels in the negative z -direction with the speed of ω/k .

We find the corresponding electric field using phasor and plane wave relationships:

$$\mathbf{H} = H_0 e^{jkz} e^{j\phi} \hat{\mathbf{x}};$$

$$\mathbf{E} = \eta \mathbf{H} \times \hat{\mathbf{k}} = \eta H_0 e^{jkz} e^{j\phi} (\hat{\mathbf{x}} \times (-\hat{\mathbf{z}})) = \eta H_0 e^{jkz} e^{j\phi} \hat{\mathbf{y}};$$

$$\mathbf{E}(z, t) = \eta H_0 \cos(\omega t + kz + \phi) \hat{\mathbf{y}} \quad [\text{mV/m}]$$

@ $y = 0$ (the origin): $\mathbf{E}(\mathbf{0}, t) = \eta H_0 \cos(\omega t + \phi) \hat{\mathbf{y}}$

3. Given the time-domain electric field $\mathbf{E}(\mathbf{r}, t) = [A_1 \cos(\omega t - \beta_1 x + \phi_1) - A_2 \cos(\omega t - \beta_2 x + \phi_2)](\hat{\mathbf{z}} + \hat{\mathbf{y}})$ in free space, find the corresponding magnetic flux density
- In phasor form, and
 - In time-varying form.

Solution:

- (a) Electric field phasor: $\mathbf{E} = (\overline{A_1} e^{-j\beta_1 x} - \overline{A_2} e^{-j\beta_2 x})(\hat{\mathbf{z}} + \hat{\mathbf{y}}) = \mathbf{E}_1 - \mathbf{E}_2$, where $\mathbf{E}_n = \overline{A_n} e^{-j\beta_n x}(\hat{\mathbf{z}} + \hat{\mathbf{y}})$ and $\overline{A_n} = A_n e^{j\phi_n}$ for $n = 1$ to 2 . Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be magnetic flux densities corresponding to $\mathbf{E}_1$ and $\mathbf{E}_2$.

$$\mathbf{B}_1 = -\frac{\mu}{\eta} \mathbf{E}_1 \times \hat{\mathbf{x}} = \mathbf{E}_n = -\frac{\mu}{\eta} E_1 (\hat{\mathbf{z}} + \hat{\mathbf{y}}) \times \hat{\mathbf{x}} = \frac{\mu}{\eta} E_1 (-\hat{\mathbf{y}} + \hat{\mathbf{z}}); \text{ similarly, } \mathbf{B}_2 = \frac{\mu}{\eta} E_2 (-\hat{\mathbf{y}} + \hat{\mathbf{z}})$$

Hence

$$\mathbf{B} = \mathbf{B}_1 - \mathbf{B}_2 = \frac{\mu}{\eta} (E_1 - E_2)(-\hat{\mathbf{y}} + \hat{\mathbf{z}}) = \sqrt{\mu\epsilon} (\overline{A_1} e^{-j\beta_1 x} - \overline{A_2} e^{-j\beta_2 x})(-\hat{\mathbf{y}} + \hat{\mathbf{z}})$$

- (b) Convert the phasor into time-domain:

$$\mathbf{B}(\mathbf{r}, t) = \sqrt{\mu\epsilon} (A_1 \cos(\omega t - \beta_1 x + \phi_1) - A_2 \cos(\omega t - \beta_2 x + \phi_2))(-\hat{\mathbf{y}} + \hat{\mathbf{z}})$$

4. The magnetic field of a plane wave is $\mathbf{H}(\mathbf{r}, t) = 15 \cos(10^9 \pi t + mx + ny + 80^\circ) (\hat{\mathbf{x}} + 2\hat{\mathbf{y}})$ [A/m]. The wave travels in a lossless, nonmagnetic material with $\epsilon_r = 9$. Find:
- Its frequency
 - Its wavelength
 - Its propagation direction
 - The constants m and n
 - The corresponding $\mathbf{E}(\mathbf{r}, t)$.
 - The amplitude of the $\mathbf{E}$ field at the origin, at time $t = 0$.

Solution:

$$(a) f = \frac{\omega}{2\pi} = \frac{\pi \times 10^9}{2\pi} = 5 \times 10^8 = 500 \text{ [MHz]}$$

$$(b) k = \frac{\omega}{c} n = \frac{\omega \sqrt{\epsilon_r}}{c} = \frac{(10^9 \pi)(\sqrt{9})}{3 \times 10^8} = 10\pi; \quad \lambda = \frac{2\pi}{k} = \frac{2\pi}{10\pi} = 0.2 \text{ [m]}$$

$$(c) \text{ Firstly, } \mathbf{k} \cdot \mathbf{r} = mx + ny \Rightarrow \mathbf{k} = (m, n, 0)$$

$\mathbf{k}$ is perpendicular to $\mathbf{H}$, so $0 = \mathbf{k} \cdot \mathbf{H} = (m, n, 0) \times 15 \cos(10^9 t + mx + ny + 80^\circ) (\hat{\mathbf{x}} + 2\hat{\mathbf{y}}) = 0$.
Simplifying gives $m + 2n = 0 \Rightarrow m = -2n$.

So the propagation direction is either $(2, -1, 0)$ or $(-2, 1, 0)$ (two possible solutions).

$$(d) k = |\mathbf{k}| = \sqrt{m^2 + n^2} = \pm \sqrt{5}n, \text{ so } n = \pm \frac{k}{\sqrt{5}} = \pm 2\sqrt{5}\pi = \pm 14.05 \text{ [m}^{-1}\text{]}; m = -2n = \mp 28.10 \text{ [m}^{-1}\text{]}.$$

Combining gives $\mathbf{k} = (m, n, 0) = \pm(28.10, -14.05, 0) \text{ [m}^{-1}\text{]}$ (again, two possible solutions).

$$(e) \text{ We will choose the solution } \mathbf{k} = (28.10, -14.05, 0) \text{ [m}^{-1}\text{]}.$$

Convert $\mathbf{H}(\mathbf{r}, t)$ into phasor form: $\mathbf{H} = 15 \exp(28.10x - 14.05y + 80^\circ) (\hat{\mathbf{x}} + 2\hat{\mathbf{y}})$ [A/m].

Using the relationship for plane waves,

$$\begin{aligned} \mathbf{E} &= \eta \mathbf{H} \times \hat{\mathbf{k}} = \frac{\eta_0}{\sqrt{\epsilon_r}} 15 \exp(28.10x - 14.05y + 80^\circ) (1, 2, 0) \times \frac{(28.10, -14.05, 0)}{10\pi} \\ &= \frac{(120\pi)}{(3)} (15) \exp(28.10x - 14.05y + 80^\circ) (-0.4472 - 2 \times 0.8945) \hat{\mathbf{z}} \\ &= 4.215 \exp(28.10x - 14.05y + 80^\circ) \hat{\mathbf{z}} \text{ [kV/m]} \end{aligned}$$

Converting into time domain gives $\mathbf{E}(\mathbf{r}, t) = 4.215 \cos(10^9 \pi t + 28.10x - 14.05y + 80^\circ) \hat{\mathbf{z}} \text{ [kV/m]}.$

(If we choose the other $\mathbf{k}$, we would find a corresponding $\mathbf{E}(\mathbf{r}, t)$ with its direction reversed.)

$$(f) \mathbf{E}(x = 0, y = 0, t = 0) = 4.215 \cos(80^\circ) \hat{\mathbf{z}} = 0.7319 \hat{\mathbf{z}} \text{ [kV/m]}$$

5. A 2.5 GHz left-hand circularly polarized plane wave travels in wet soil ($\epsilon_r = 9, \sigma = 5$ [S/m]). At $t = 0$ and at the origin ($z = 0$), the electric field is $\mathbf{E} = 2 \hat{\mathbf{y}}$ [V/m]. The plane wave travels in the +z-direction.
- Is the soil a good dielectric, a good conductor or neither?
 - What is the attenuation constant in [dB/m]?
 - What is the complex permittivity?
 - How fast does the wave travel in the soil?
 - Write the magnetic field phasor.
 - Draw the time evolution of the E-field at $z = 0$.

Solution:

(a)

$$\frac{\sigma}{\omega\epsilon} = \frac{5}{(2\pi \times 2.5 \times 10^9)(8.854 \times 10^{-12})(9)} = 3.995 > 1 \text{ but not } \gg 1$$

So the soil is neither a good dielectric nor a good conductor at this frequency.

- (b) We calculate the necessary constants then put them into the expression for α :
- $$\omega = 2\pi(2.5 \times 10^9) = 5\pi \times 10^9 \text{ [rad/s]; } \epsilon' = \epsilon_0\epsilon_r = (8.854 \times 10^{-12})(9) = 7.969 \times 10^{-11} \text{ [F/m];}$$

$$\epsilon'' = \frac{\sigma}{\omega} = \frac{5}{(2\pi \times 2.5 \times 10^9)} = 3.183 \times 10^{-10} \text{ [F/m]; } \mu = \mu_0 = 4\pi \times 10^{-7} \text{ [H/m]}$$

$$\begin{aligned} \alpha &= \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} - 1 \right] \right\}^{1/2} \\ &= (5\pi \times 10^9) \left\{ \frac{(4\pi \times 10^{-7})(7.969 \times 10^{-11})}{2} \left[\sqrt{1 + \left(\frac{3.183 \times 10^{-10}}{7.969 \times 10^{-11}}\right)^2} - 1 \right] \right\}^{1/2} \\ &= 196.3 \text{ [Np/m]} \end{aligned}$$

$$\alpha_{dB} = 8.686\alpha = 1704 \text{ [dB/m]}$$

We also calculate β for use in a later sub-question

$$\begin{aligned} \beta &= \omega \left\{ \frac{\mu\epsilon'}{2} \left[\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'}\right)^2} + 1 \right] \right\}^{1/2} \\ &= (5\pi \times 10^9) \left\{ \frac{(4\pi \times 10^{-7})(7.969 \times 10^{-11})}{2} \left[\sqrt{1 + \left(\frac{3.183 \times 10^{-10}}{7.969 \times 10^{-11}}\right)^2} + 1 \right] \right\}^{1/2} \\ &= 251.4 \text{ [rad/s]} \end{aligned}$$

(c)

$$\epsilon_c = \epsilon' - j\epsilon'' = (7.969 - j31.83) \times 10^{-11} \text{ [F/m]}$$

(d)

$$v_p = \frac{\omega}{\beta} = \frac{(5\pi \times 10^9)}{251.4} = 6.247 \times 10^7 \text{ [m/s]}$$

(e) The impedance is

$$\eta_c = \sqrt{\frac{\mu}{\epsilon'}} \left(1 - \frac{j\epsilon''}{\epsilon'} \right)^{-\frac{1}{2}} = \sqrt{\frac{4\pi \times 10^{-7}}{7.969 \times 10^{-11}}} \left(1 - \frac{j3.183 \times 10^{-10}}{7.969 \times 10^{-11}} \right)^{-\frac{1}{2}} = 61.89e^{j0.6627} \text{ [\Omega]}$$

Let the E-field (time domain) be

$$\mathbf{E}(z, t) = E_0 e^{-\alpha z} \left(\cos(\omega t - \beta z + \phi) \hat{\mathbf{x}} + \cos\left(\omega t - \beta z + \phi + \frac{\pi}{2}\right) \hat{\mathbf{y}} \right)$$

@ $t = 0$ and at the origin, $\mathbf{E}(0, 0) = E_0 \left(\cos(\phi) \hat{\mathbf{x}} + \cos\left(\phi + \frac{\pi}{2}\right) \hat{\mathbf{y}} \right) = 2 \hat{\mathbf{y}}$ [V/m]. Comparing the two equations gives $E_0 = 2$ [V/m] and $\phi = -\frac{\pi}{2}$.

The E-field phasor is

$$\mathbf{E} = E_0 e^{j\phi} e^{-(\alpha + j\beta)z} (\hat{\mathbf{x}} + j\hat{\mathbf{y}})$$

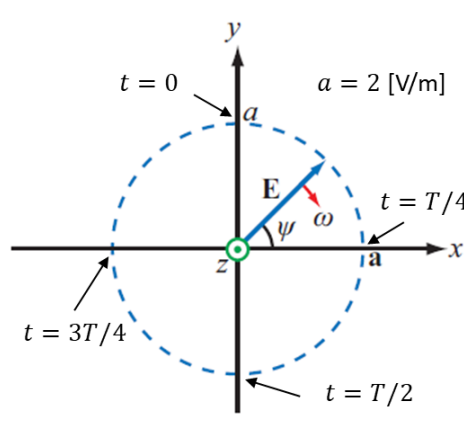
Substituting in α, β, E_0 and ϕ , we have

$$\mathbf{E} = 2e^{-j\frac{\pi}{2}} e^{-196.3z} e^{-j251.4z} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \text{ [V/m]}$$

The H-field phasor is

$$\begin{aligned} \mathbf{H} &= \frac{1}{\eta_c} \hat{\mathbf{k}} \times \mathbf{E} = \frac{2e^{-j\frac{\pi}{2}} e^{-196.3z} e^{-j251.4z}}{61.89e^{j0.6627}} (\hat{\mathbf{z}} \times (\hat{\mathbf{x}} + j\hat{\mathbf{y}})) \\ &= 32.32e^{-j2.233} e^{-196.3z} e^{-j251.4z} (\hat{\mathbf{y}} - j\hat{\mathbf{x}}) \text{ [mA/m]} \end{aligned}$$

- (f) This is an LHC polarized wave with an amplitude of $E_0 = 2$ [V/m], pointing in the y-direction at $t = 0$.



6. A 20 GHz right-hand circular polarized plane wave, travelling in air in the $-z$ -direction, has an electric field phasor $\mathbf{E} = 15 \text{ [V/m]} \hat{\mathbf{x}}$ at time $t = 5 \text{ ns}$ and $z = 4.56 \text{ m}$. Write its magnetic field as a function of time and space.

Solution:

The electric field phasor has the form $\mathbf{E} = E_0 e^{j\phi_0} e^{jkz} (\hat{\mathbf{x}} + j\hat{\mathbf{y}}) \text{ [V/m]}$, where $k = \frac{\omega}{c} = \frac{2\pi(20 \times 10^9)}{3 \times 10^8} = \frac{400\pi}{3} = 418.8 \text{ [m}^{-1}\text{]}$. The corresponding time domain electric field is

$$\mathbf{E}(z, t) = E_0 \left[\cos(\omega t + kz + \phi_0) \hat{\mathbf{x}} + \cos\left(\omega t + kz + \phi_0 + \frac{\pi}{2}\right) \hat{\mathbf{y}} \right]$$

We now find E_0 and ϕ_0 . At $t = 5 \text{ ns}$ $\mathbf{E}$ points completely in the $+x$ direction. At that time $E_y = 0 \Rightarrow$

$\cos\left(\omega t + kz + \phi_0 + \frac{\pi}{2}\right) = 0 \Rightarrow \omega t + kz + \phi_0 = n\pi$ (n is an integer). Substituting this into the x -component gives $E_x = E_0 \cos(n\pi + \frac{\pi}{2}) = E_0 \cos(\pm \frac{\pi}{2}) = \pm E_0$. We note that an even (odd) n gives a positive (negative) E_x . Since $\mathbf{E} = 15 \text{ [V/m]} \hat{\mathbf{x}}$ (ie. positive E_x), an even n should be chosen, which means $\omega t + kz + \phi_0 = 0 \text{ [rad]}$. Rearranging gives

$$\phi_0 = -\omega t - kz = -(2\pi)(20 \times 10^9)(5 \times 10^{-9}) - \frac{400\pi}{3}(4.56) = -(2\pi)(100) - (2\pi)(304) = 0 \text{ [rad]}$$

Note: one needs to keep the exact value of k , or account for enough significant figures, to make an accurate calculation for ϕ_0 . The previous version of this answer had rounding errors.

From the electric field phasor, we find the corresponding magnetic field using the plane wave relationships:

$$\mathbf{H} = \frac{E_0 e^{j\phi_0}}{\eta_0} \hat{\mathbf{k}} \times \hat{\mathbf{E}} = \frac{15 e^{j\phi_0}}{120\pi} (-\hat{\mathbf{z}} \times (\hat{\mathbf{x}} + j\hat{\mathbf{y}})) = \frac{1}{8\pi} (-\hat{\mathbf{y}} + j\hat{\mathbf{x}}) \text{ [A/m]}$$

7. A plane wave propagates in free-space, in the z-direction, and has a magnetic field $\mathbf{H}(\mathbf{r}, t) = 7.0 \cos(\omega t - kz + 30^\circ) \hat{\mathbf{x}} + 3.5 \sin(\omega t - kz - 120^\circ) \hat{\mathbf{y}}$ [mA/m].
- Find its polarization state.
 - Trace the time evolution of the magnetic field.
 - Trace the time evolution of the electric field.
 - Write the electric field as a summation of two circularly polarized plane waves (one LHC and one RHC). Which component is larger?

Solution:

We rewrite $\mathbf{H}(\mathbf{r}, t)$ into cosine form: $\mathbf{H}(\mathbf{r}, t) = 7.0 \cos(\omega t - kz + 30^\circ) \hat{\mathbf{x}} + 3.5 \cos(\omega t - kz + 150^\circ) \hat{\mathbf{y}}$
 The phasor is $\mathbf{H} = e^{j\frac{\pi}{6}} e^{-jkz} (7.0 \hat{\mathbf{x}} + 3.5 \hat{\mathbf{y}} e^{j\frac{2\pi}{3}})$ [mA/m].

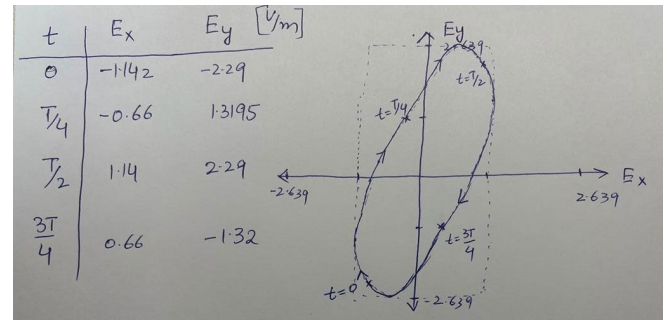
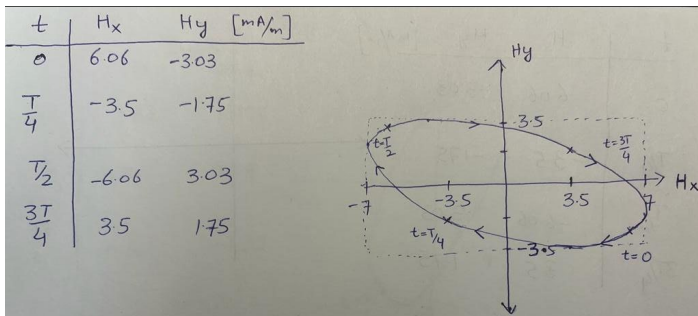
- (a) Elliptical (because $\delta = 120^\circ = \frac{2\pi}{3}$)

(b) and (c): Electric field phasor is

$$\begin{aligned} \mathbf{E} &= \eta_0 e^{j\frac{\pi}{6}} e^{-jkz} \left((7.0 \hat{\mathbf{x}} + 3.5 e^{j\frac{2\pi}{3}} \hat{\mathbf{y}}) \times \hat{\mathbf{z}} \right) = 120\pi e^{j\frac{\pi}{6}} e^{-jkz} (-7.0 \hat{\mathbf{y}} + 3.5 e^{j\frac{2\pi}{3}} \hat{\mathbf{x}}) \\ &= 1319 e^{j\frac{\pi}{6}} e^{-jkz} (-2 \hat{\mathbf{y}} + e^{j\frac{2\pi}{3}} \hat{\mathbf{x}}) \text{ [mV/m]} \\ &= 1.319 e^{j\frac{\pi}{6}} e^{-jkz} (e^{j\frac{2\pi}{3}} \hat{\mathbf{x}} - 2 \hat{\mathbf{y}}) \text{ [V/m]} \end{aligned}$$

The time domain electric field is

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= 1.319 \cos(\omega t - kz + 150^\circ) \hat{\mathbf{x}} - 2.639 \cos(\omega t - kz + 30^\circ) \hat{\mathbf{y}} \\ &= 1.319 \cos(\omega t - kz + 150^\circ) \hat{\mathbf{x}} + 2.639 \cos(\omega t - kz + 210^\circ) \hat{\mathbf{y}} \text{ [V/m]} \end{aligned}$$



Time Evolution of the H-field

Time Evolution of the E-field

- (d) We let $\mathbf{L} = \hat{\mathbf{x}} + j\hat{\mathbf{y}}$; $\mathbf{R} = \hat{\mathbf{x}} - j\hat{\mathbf{y}}$ represent the LHC and RHC components of the electric field. Rearranging gives $\hat{\mathbf{x}} = \frac{1}{2}(\mathbf{L} + \mathbf{R})$; $\hat{\mathbf{y}} = \frac{1}{2j}(\mathbf{L} - \mathbf{R})$. Substitution into $\mathbf{E}$ gives

$$\begin{aligned} \mathbf{E} &= 1.319 e^{j\frac{\pi}{6}} e^{-jkz} \left(e^{j\frac{2\pi}{3}} \left(\frac{1}{2}(\mathbf{L} + \mathbf{R}) \right) - 2 \left(\frac{1}{2j}(\mathbf{L} - \mathbf{R}) \right) \right) \\ &= 1319 e^{j\frac{\pi}{6}} e^{-jkz} \left(0.5 e^{j\frac{2\pi}{3}} (\mathbf{L} + \mathbf{R}) + j(\mathbf{L} - \mathbf{R}) \right) \end{aligned}$$

$$\begin{aligned} &= 1319 e^{j\frac{\pi}{6}} e^{-jkz} \left[\left(0.5 e^{j\frac{2\pi}{3}} + j \right) \mathbf{L} + \left(0.5 e^{j\frac{2\pi}{3}} - j \right) \mathbf{R} \right] \\ &= 1319 e^{j\frac{\pi}{6}} e^{-jkz} [(-0.250 + j1.433) \mathbf{L} + (-0.250 - j0.567) \mathbf{R}] \text{ [mV/m]} \end{aligned}$$

Therefore LHC component is larger.