

MA1201 Calculus and Basic Algebra II (Section: CC1)

Quiz 1: Integration

Name:

SID:

Score:

Time allowed: 25 minutes

Instruction: Write all your answer in the spaces provided. Show all your working steps (including evaluation of integrals) clearly. The brief table of integrals, sum-to-product formula and product-to-sum formula are included in the last page.

Appendix A: Brief Table for Integration

1. $\int x^a dx = \begin{cases} \frac{x^{a+1}}{a+1} + C & \text{if } a \neq -1 \\ \ln|x| + C & \text{if } a = -1 \end{cases}$
2. $\int e^x dx = e^x + C, \int a^x dx = \frac{a^x}{\ln a} + C, a > 0.$
3. $\int \sin x dx = -\cos x + C$
4. $\int \cos x dx = \sin x + C$
5. $\int \sec^2 x dx = \tan x + C$
6. $\int \sec x dx = \ln|\sec x + \tan x| + C$
7. $\int \csc x dx = -\ln|\csc x + \cot x| + C$
8. $\int \frac{1}{1+x^2} dx = \tan^{-1} x + C$
9. $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$

Appendix B: Product-to-Sum Formula

1. $\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$
2. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
3. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$

Appendix C: Sum-to-product formula

1. $\sin A + \sin B = 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}$
2. $\sin A - \sin B = 2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}$
3. $\cos A + \cos B = 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2}$
4. $\cos A - \cos B = -2 \sin \frac{A+B}{2} \sin \frac{A-B}{2}$

Problem 1 (50 marks)

Compute the following integrals

(a) $\int_2^{2\sqrt{2}} \frac{1}{\sqrt{x^2-2}} dx$

(b) $\int \frac{-4x^2+7x+9}{(2x+1)(x-1)^2} dx$

Solution

$$1 + \tan^2 \theta = \sec^2 \theta$$

(a) Let $x = \sqrt{2} \sec \theta$ (so that $x^2 - 2 = 2 \sec^2 \theta - 2 = 2 \tan^2 \theta$), then
 $\frac{dx}{d\theta} = \sqrt{2} \sec \theta \tan \theta \Rightarrow dx = \sqrt{2} \sec \theta \tan \theta d\theta$.

When $x = 2\sqrt{2} \Rightarrow 2\sqrt{2} = \sqrt{2} \sec \theta \Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \pi/3$

$x = 2 \Rightarrow 2 = \sqrt{2} \sec \theta \Rightarrow \cos \theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \pi/4$.

$$\int_2^{2\sqrt{2}} \frac{1}{\sqrt{x^2-2}} dx = \int_{\pi/4}^{\pi/3} \frac{1}{\sqrt{2 \tan^2 \theta}} \sqrt{2} \sec \theta \tan \theta d\theta = \int_{\pi/4}^{\pi/3} \sec \theta d\theta$$

$$= \ln |\sec \theta + \tan \theta| \Big|_{\pi/4}^{\pi/3} = \ln |2 + \sqrt{3}| - \ln |\sqrt{2} + 1|.$$

(b) Using method of partial fraction, we decompose the integrand as

$$\frac{-4x^2+7x+9}{(2x+1)(x-1)^2} = \frac{A}{2x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \Rightarrow \frac{A(x-1)^2 + B(2x+1)(x-1) + C(2x+1)}{(2x+1)(x-1)^2} = \frac{-4x^2+7x+9}{(2x+1)(x-1)^2}$$

• Put $x=1$, we get $12 = C(3) \Rightarrow C=4$.

• Put $x=-1/2$, we get $9/2 = \frac{9}{4}A \Rightarrow A=2$

• Put $x=0$, we get $9 = A - B + C \Rightarrow B = -3$

Thus

$$\begin{aligned} \int \frac{-4x^2+7x+9}{(2x+1)(x-1)^2} dx &= 2 \int \frac{1}{2x+1} dx - 3 \int \frac{1}{x-1} dx + 4 \int \frac{1}{(x-1)^2} dx \\ &= 2 \left(\frac{1}{2} \ln |2x+1| \right) - 3 \ln |x-1| + 4 \frac{(x-1)^{-2+1}}{-2+1} + C \\ &= \ln |2x+1| - 3 \ln |x-1| - \frac{4}{x-1} + C. \end{aligned}$$

Problem 2 (25 marks)

Compute the integral using suitable method

$$\int_0^1 x^3 e^{5-2x^2} dx$$

IDEA:

Simplify e^{5-2x^2} by changing it into e^y .

(Hint: Do something on e^{5-2x^2} first!)

Solution

Let $y = 5 - 2x^2$. Then $\frac{dy}{dx} = -4x \Rightarrow dx = -\frac{1}{4x} dy$.

When $x=1$, $y=3$. When $x=0$, $y=5$.

$$\begin{aligned} \int_0^1 x^3 e^{5-2x^2} dx &= \int_5^3 x^3 e^{5-2x^2} \left(-\frac{1}{4x} dy\right) = -\frac{1}{4} \int_5^3 x^2 e^{5-2x^2} dy \\ &= -\frac{1}{4} \int_5^3 \left(\frac{5-y}{2}\right) e^y dy = -\frac{1}{8} \int_5^3 (5-y) e^y dy \end{aligned}$$

Using integration by parts with $u=5-y$ and $dv=e^y dy$.
 $\Rightarrow v = \int e^y dy = e^y$.

We have

$$\begin{aligned} -\frac{1}{8} \int_5^3 \underbrace{(5-y)}_u \underbrace{e^y dy}_{dv} &\stackrel{v=e^y}{=} -\frac{1}{8} \left[\underbrace{(5-y)e^y}_{uv} \Big|_5^3 - \int_5^3 e^y d(5-y) \right] \\ &= -\frac{1}{8} \left[2e^3 + \int_5^3 e^y dy \right] \\ &= -\frac{1}{8} \left[2e^3 + e^y \Big|_5^3 \right] \\ &= -\frac{3}{8} e^3 + \frac{1}{8} e^5. \end{aligned}$$

Problem 3 (25 points)

Compute the following integral using suitable method

$$\int_0^{\pi} \left(\left| \sin x - \frac{1}{2} \right| + \frac{1}{2} \right)^2 dx$$

☺Hint:

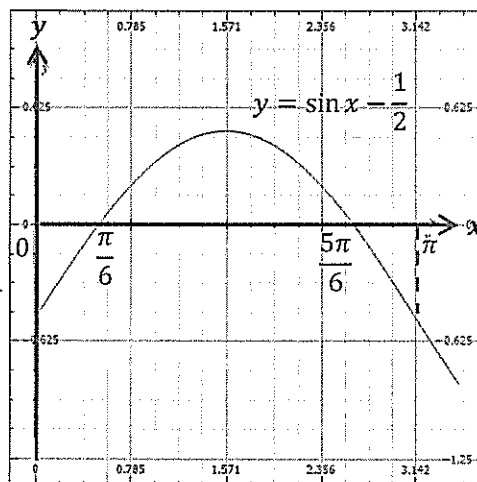
- Be careful, $|??|$ is the *absolute value function*.

Recall that $|y| = \begin{cases} y & \text{if } y \geq 0 \\ -y & \text{if } y < 0 \end{cases}$

- The graph of $y = \sin x - \frac{1}{2}$ is shown on the right.

Solution

$$\left| \sin x - \frac{1}{2} \right| = \begin{cases} \sin x - \frac{1}{2} & \text{if } \frac{\pi}{6} \leq x \leq \frac{5\pi}{6} \\ \frac{1}{2} - \sin x & \text{if } 0 < x < \frac{\pi}{6} \text{ or } \frac{5\pi}{6} < x < \pi \end{cases}$$



Thus

$$\begin{aligned} & \int_0^{\pi} \left(\left| \sin x - \frac{1}{2} \right| + \frac{1}{2} \right)^2 dx \\ &= \int_0^{\pi/6} \left(\frac{1}{2} - \sin x + \frac{1}{2} \right)^2 dx + \int_{\pi/6}^{5\pi/6} \left(\sin x - \frac{1}{2} + \frac{1}{2} \right)^2 dx + \int_{5\pi/6}^{\pi} \left(\frac{1}{2} - \sin x + \frac{1}{2} \right)^2 dx \\ &= \int_0^{\pi/6} (1 - 2\sin x + \sin^2 x) dx + \int_{\pi/6}^{5\pi/6} \sin^2 x dx + \int_{5\pi/6}^{\pi} (1 - 2\sin x + \sin^2 x) dx \\ &= \left(x + 2\cos x - \frac{1}{4}\sin 2x + \frac{x}{2} \right) \Big|_0^{\pi/6} + \left(-\frac{1}{4}\sin 2x + \frac{x}{2} \right) \Big|_{\pi/6}^{5\pi/6} + \left(x + 2\cos x - \frac{1}{4}\sin 2x + \frac{x}{2} \right) \Big|_{5\pi/6}^{\pi} \\ &= \frac{5\pi}{6} - \frac{3\sqrt{3}}{4} - 4. \end{aligned}$$

* Note :

$$\int \sin^2 x dx = \int (\sin x)(\sin x) dx = -\frac{1}{2} \int (\cos(x+x) - \cos(x-x)) dx = -\frac{1}{2} \int (\cos 2x - 1) dx = -\frac{1}{4}\sin 2x + \frac{x}{2} + C.$$

☺End of the Quiz☺