

Practice Problem Set 2 Solutions

Question 1: Find the derivative of the inverse of the following function

$$f(x) = x^3 + 1$$

If $y = x^3 + 1$, then $x = (y - 1)^{1/3}$. In other words, the inverse function $g = f^{-1}$ is given by

$$g(y) = (y - 1)^{1/3}$$

Using the formula, we have

$$g'(y) = \frac{1}{f'(x)}$$

Now, $f'(x) = 3x^2$ and so

$$g'(y) = \frac{1}{3x^2} = \frac{1}{3\left((y - 1)^{1/3}\right)^2} = \frac{1}{3}(y - 1)^{-2/3}$$

Here, we have used the inverse function rule/formula. However, you could have differentiated g directly.

Question 2: Work out the derivatives of the following functions using the standard rules:

(a) $\ln(2x^2 + 5x + 4)$

Applying the chain rule, we get

$$f'(x) = \left(\frac{1}{2x^2 + 5x + 4} \right) \frac{d}{dx}(2x^2 + 5x + 4) = \frac{4x + 5}{2x^2 + 5x + 4}$$

(b) $3x^4 e^x$

Applying the product rule, we get

$$f'(x) = 3 \left(x^4 \frac{d}{dx} e^x + e^x \frac{d}{dx} x^4 \right) = 3(x^4 e^x + 4x^3 e^x) = 3x^3 e^x (x + 4)$$

(c) $\ln\left(\frac{x^4 + 6x^2 + 9}{x^2 + 1}\right)$

We have

$$f(x) = \ln\left(\frac{x^4 + 6x^2 + 9}{x^2 + 1}\right) = \ln(x^4 + 6x^2 + 9) - \ln(x^2 + 1)$$

Then

$$\begin{aligned}
 f'(x) &= \left(\frac{1}{x^4 + 6x^2 + 9} \right) \frac{d}{dx}(x^4 + 6x^2 + 9) - \left(\frac{1}{x^2 + 1} \right) \frac{d}{dx}(x^2 + 1) \\
 &= \frac{4x^3 + 12x}{x^4 + 6x^2 + 9} - \frac{2x}{x^2 + 1} = \frac{4x^3 + 12x}{(x^2)^2 + (2)(3)x^2 + 3^2} - \frac{2x}{x^2 + 1} \\
 &= \frac{4x(x^2 + 3)}{(x^2 + 3)^2} - \frac{2x}{x^2 + 1} = \frac{4x}{x^2 + 3} - \frac{2x}{x^2 + 1} = \frac{4x(x^2 + 1) - 2x(x^2 + 3)}{(x^2 + 3)(x^2 + 1)} \\
 &= \frac{4x^3 + 4x - 2x^3 - 6x}{x^4 + x^2 + 3x^2 + 3} = \frac{2x^3 - 2x}{x^4 + 4x^2 + 3}
 \end{aligned}$$

Question 3: Determine if $f(x) = (x - 6)\ln x$ is increasing or decreasing at the following points

(a) $x = 1$

(b) $x = 20$

The derivative of a function gives the rate of change for the function. Here, we have

$$f'(x) = \ln x + \frac{x - 6}{x}$$

(a) $f'(1) = \ln 1 + \frac{1-6}{1} = -5 < 0$

Since, $f'(1) < 0$, the function $f(x)$ is decreasing at $x = 1$.

(b) $f'(20) = \ln 20 + \frac{20-6}{20} = 3.69 > 0$

Since, $f'(20) > 0$, the function $f(x)$ is increasing at $x = 20$.

Question 4: Find the maximum value of the function

$$f(x) = -2x^3 + 3x^2 + 12x + 9$$

in the interval $[0, \infty)$.

The first-order condition is:

$$f'(x) = -6x^2 + 6x + 12 = 0$$

$$\implies -6(x^2 - x - 2) = 0$$

$$\implies -6(x^2 + x - 2x - 2) = -6(x + 1)(x - 2) = 0$$

$$\implies x = -1 \quad \text{or} \quad 2$$

Then

$$f''(x) = -12x + 6$$

$$f''(-1) = (-12)(-1) + 6 = 18 > 0$$

$$f''(2) = (-12)(2) + 6 = -18 < 0$$

The second-order condition for a maximum is satisfied at $x = 2$, and the maximum value of the function is

$$f(2) = -2(2)^3 + 3(2)^2 + 12(2) + 9 = 29$$

Question 5: Given that a market's demand equation is

$$q^2(4 + p^3) = 31,$$

find the elasticity of demand and show that it never exceeds $3/2$.

As $q \geq 0$, the demand equation gives

$$q = \sqrt{31}(4 + p^3)^{-1/2}$$

Differentiating this (using the chain rule), we have

$$q' = \sqrt{31} \left(-\frac{1}{2} \right) (4 + p^3)^{-3/2} (3p^2) = -\frac{3\sqrt{31}p^2}{2(4 + p^3)^{3/2}}$$

The elasticity of demand is

$$\varepsilon(p) = -\frac{q'p}{q} = \frac{3\sqrt{31}p^3/[2(4 + p^3)^{3/2}]}{\sqrt{31}/(4 + p^3)^{1/2}} = \frac{3}{2} \left(\frac{p^3}{4 + p^3} \right)$$

As $p \geq 0$, we have

$$\begin{aligned} \frac{p^3}{4 + p^3} &< 1 \\ \implies \varepsilon(p) &< 3/2 \end{aligned}$$