

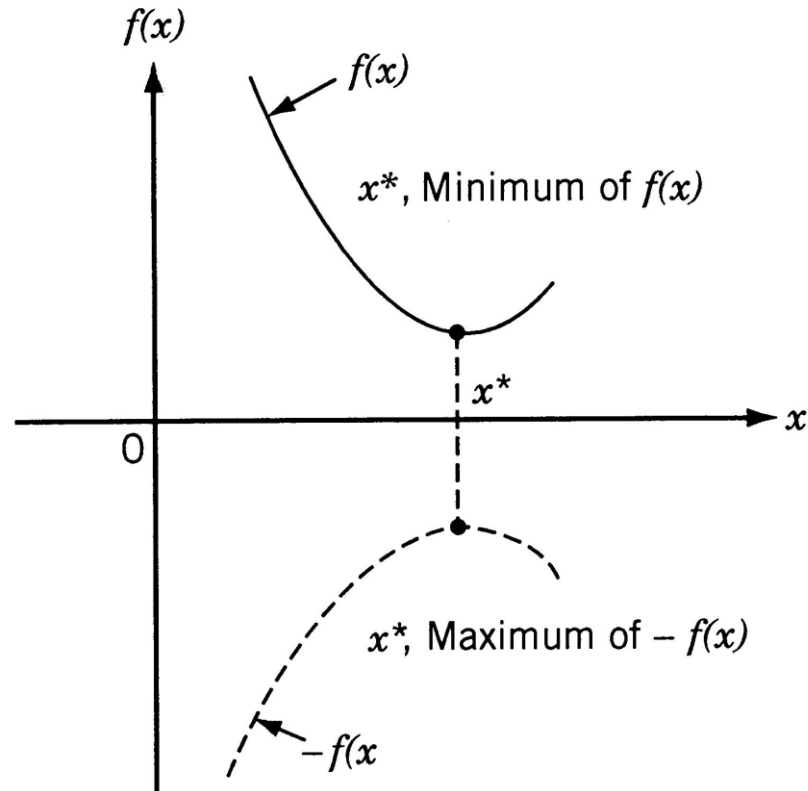
EE3114 Systems and Control

Tutorial - 8

Semester B, 2017-18

Optimization

Optimization: minimization or maximization

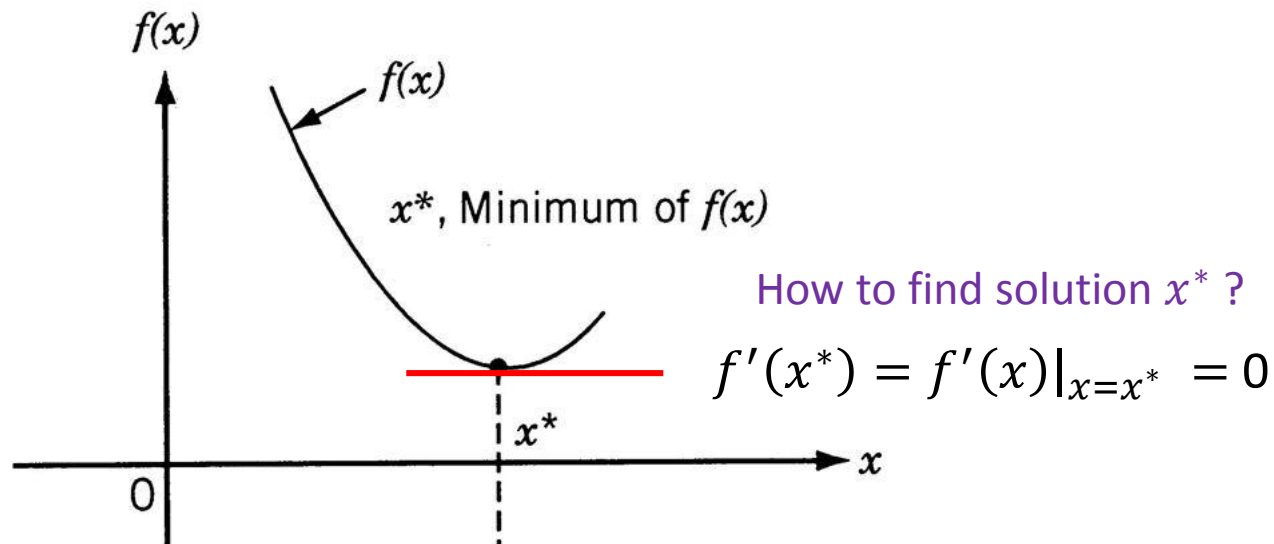


Minimum of $f(x)$ is same as maximum of $-f(x)$.

→ We will only discuss “minimization” below

Example:

$$\min_x f(x)$$



Exercise:

$$\min_x (x - 1)^2$$

(5 minutes)

Exercise:

$$\min_x (x - 1)^2$$

Solution:

$$f(x) = (x - 1)^2$$

$$\rightarrow f'(x) = 2(x - 1) = 0$$

$$\rightarrow x^* = 1$$

Exercise:

$$\min_x \mathbf{x}^T(t) \mathbf{x}(t) \quad (\mathbf{x} \text{ is a vector function}) \quad (5 \text{ minutes})$$

and

$$\min_t \mathbf{x}^T(t) \mathbf{x}(t)$$

Exercise:

$$\min_x \mathbf{x}^T(t)\mathbf{x}(t) \quad (\mathbf{x}(t) \text{ is a vector function})$$

and

$$\min_t \mathbf{x}^T(t)\mathbf{x}(t)$$

Solution:

$$\frac{d}{dx} \mathbf{x}^T(t)\mathbf{x}(t) = 2\mathbf{x}(t) = 0 \rightarrow \mathbf{x}(t) \equiv 0 \quad \forall t$$

and

$$\frac{d}{dt} \mathbf{x}^T(t)\mathbf{x}(t) = \dot{\mathbf{x}}^T(t)\mathbf{x}(t) + \mathbf{x}^T(t)\dot{\mathbf{x}}(t) = 0 \quad \forall t$$

$$\rightarrow \mathbf{x} = \mathbf{0} \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{0}$$

$$\rightarrow \mathbf{x} = \mathbf{0} \quad \text{or} \quad \mathbf{x} = (\text{any}) \text{ constant vector}$$

Note: $\mathbf{x}^T \mathbf{x} \geq 0$; therefore, any $\mathbf{x} = \text{nonzero}$ will yield $\mathbf{x}^T > 0$

$$\rightarrow \mathbf{x}^* = \mathbf{0} \text{ is the only solution, } \forall t$$

Exercise:

$$\min_t \mathbf{x}^T(t) \mathbf{x}(t) \quad (\mathbf{x}(t) \text{ is a vector function})$$

Solution: $\mathbf{x}^* = \mathbf{0}$

Extension:

$$\min_t \mathbf{x}^T(t) Q \mathbf{x}(t) \quad \begin{array}{l} (Q \text{ is a positive-definite and symmetrical matrix} \\ \mathbf{x} \text{ is a vector function}) \end{array}$$

Solution:

$$\frac{d}{dt} \mathbf{x}^T Q \mathbf{x} = \dot{\mathbf{x}}^T Q \mathbf{x} + \mathbf{x}^T Q \dot{\mathbf{x}} = 0$$

$$\rightarrow \mathbf{x} = \mathbf{0} \quad \text{or} \quad \dot{\mathbf{x}} = \mathbf{0}$$

$$\rightarrow \mathbf{x} = \mathbf{0} \quad \text{or} \quad \mathbf{x} = (\text{any}) \text{ constant vector}$$

Note: $\mathbf{x}^T Q \mathbf{x} \geq 0$ because Q is positive definite

$$\rightarrow \mathbf{x}^* = \mathbf{0} \text{ is the solution}$$

Exercise:

$$\begin{aligned} & \min_t \mathbf{x}^T(t) Q \mathbf{x}(t) && (Q \text{ is a positive-definite and symmetrical matrix} \\ & \text{subject to} && \mathbf{x} \text{ is a vector function}) \\ & \dot{\mathbf{x}} = A \mathbf{x} \end{aligned}$$

Find: Optimal solution $\mathbf{x}^* = ?$ (5 minutes)

Exercise:

$\min_t \mathbf{x}^T(t) Q \mathbf{x}(t)$ (Q is a positive-definite and symmetrical matrix
subject to $\mathbf{x}$ is a vector function)
 $\dot{\mathbf{x}} = A\mathbf{x}$

Solution:

$$\begin{aligned} \frac{d}{dt} \mathbf{x}^T Q \mathbf{x} &= \dot{\mathbf{x}}^T Q \mathbf{x} + \mathbf{x}^T Q \dot{\mathbf{x}} \\ &= [A\mathbf{x}]^T Q \mathbf{x} + \mathbf{x}^T Q A \mathbf{x} \\ &= \mathbf{x}^T [A^T Q + Q A] \mathbf{x} \\ &= 0 \end{aligned}$$

→ $\mathbf{x}^* = 0$

But this may also have non-zero optimal solutions **if**

$$A^T Q + Q A = 0$$

Exercise:

$\min_t \mathbf{x}^T(t) Q \mathbf{x}(t)$ (Q is a positive-definite and symmetrical matrix)
subject to $\mathbf{x}$ is a vector function)

Under what condition can this problem have non-zero solutions?

(10 minutes)

Exercise:

$$\min_t \mathbf{x}^T(t) Q \mathbf{x}(t)$$

subject to

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{u} = -K\mathbf{x}$$

Solution:

$$\frac{d}{dt} \mathbf{x}^T Q \mathbf{x} = \dot{\mathbf{x}}^T Q \mathbf{x} + \mathbf{x}^T Q \dot{\mathbf{x}}$$

$$= [A\mathbf{x} + B\mathbf{u}]^T Q \mathbf{x} + \mathbf{x}^T Q [A\mathbf{x} + B\mathbf{u}]$$

$$= \mathbf{x}^T [A - BK]^T Q \mathbf{x} + \mathbf{x}^T Q [A - BK] \mathbf{x}$$

$$= \mathbf{x}^T [[A - BK]^T Q + Q [A - BK]] \mathbf{x}$$

$$= \mathbf{0}$$

$$\rightarrow \mathbf{x}^* = \mathbf{0}$$

Note: It can have non-zero solutions (used for other tasks), **if**

$$[A - BK]^T Q + Q [A - BK] = 0$$

Example:

$$\min_K \int_0^{\infty} [\mathbf{x}^T(\tau)Q\mathbf{x}(\tau) + \mathbf{u}^T(\tau)R\mathbf{u}(\tau)]d\tau$$

Subject to

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{u} &= -K\mathbf{x}\end{aligned}\quad (Q \text{ and } R \text{ are positive-definite and symmetric matrices})$$

Solution:

$$\dot{\mathbf{x}} = (A - BK)\mathbf{x}$$

$$\min_K \int_0^{\infty} [\mathbf{x}^T(\tau)Q\mathbf{x}(\tau) + \mathbf{x}^T(\tau)K^TRK\mathbf{x}(\tau)]d\tau$$

$$= \min_K \int_0^{\infty} \mathbf{x}^T(\tau)[Q + K^TRK]\mathbf{x}(\tau)d\tau$$

Set

$$\mathbf{x}^T[Q + K^TRK]\mathbf{x} = -\frac{d}{dt}\mathbf{x}^TP\mathbf{x} \quad (P \text{ is a positive definite and symmetric matrix})$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$$

$$\mathbf{u}(t) = -K\mathbf{x}(t)$$

$$\mathbf{x}^T(\tau)[Q + K^T R K]\mathbf{x}(\tau) = -\frac{d}{dt}\mathbf{x}^T(t)P\mathbf{x}(t)$$



$$\begin{aligned}\mathbf{x}^T[Q + K^T R K]\mathbf{x} &= -\frac{d}{dt}\mathbf{x}^T P \mathbf{x} = -\dot{\mathbf{x}}^T P \mathbf{x} - \mathbf{x}^T P \dot{\mathbf{x}} \\ &= -\mathbf{x}^T[(A - BK)^T P + P(A - BK)]\mathbf{x}\end{aligned}$$



$$(A - BK)^T P + P(A - BK) = -(Q + K^T R K)$$



$$K = R^{-1}B^T P$$

where P satisfies

$$A^T P + PA - PBR^{-1}B^T P + Q = 0 \quad (\text{Riccati equation})$$

→ $\mathbf{u} = -K\mathbf{x} = -R^{-1}B^T P\mathbf{x}$ (optimal controller) solves

$$\min_K \int_0^{\infty} [\mathbf{x}^T(\tau)Q\mathbf{x}(\tau) + \mathbf{u}^T(\tau)R\mathbf{u}(\tau)]d\tau$$

Example:

$$\begin{aligned}\dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ \mathbf{u}(t) &= -K\mathbf{x}(t)\end{aligned}\quad \min_K \int_0^{\infty} [\mathbf{x}^T(\tau)Q\mathbf{x}(\tau) + \mathbf{u}^T(\tau)R\mathbf{u}(\tau)]d\tau$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix}, \quad R = [1]$$

(Riccati equation) $A^T P + PA - PBR^{-1}B^T P + Q = 0$ $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$



$$\begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{12} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \mu \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$



$$\begin{aligned}1 - p_{12}^2 &= 0 \\ p_{11} - p_{12}p_{22} &= 0 \\ \mu + 2p_{12} - p_{22}^2 &= 0\end{aligned}\quad \Rightarrow \quad \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{\mu+2} & 1 \\ 1 & \sqrt{\mu+2} \end{bmatrix}$$
$$K = R^{-1}B^T P = 1 \cdot [0 \ 1] \cdot P = [1 \ \sqrt{\mu+2}]$$

%MatLab code (with μ = number)

K=lqr(A,B,Q,R)

END