

Solution to Problem Set 6:

Partial Differential Equations

$$1. (a) \quad \frac{\partial^2 u}{\partial x \partial y} = 4xy + e^x \Leftrightarrow \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y} \right) = 4xy + e^x \Rightarrow \frac{\partial u}{\partial y} = \int (4xy + e^x) dx = 2x^2 y + e^x + A(y)$$

$$\text{Put } x = 0, \text{ we have } \frac{\partial u}{\partial y}(0, y) = y \Rightarrow 1 + A(y) = y \Rightarrow A(y) = y - 1.$$

$$\text{Thus, } \frac{\partial u}{\partial y} = 2x^2 y + e^x + y - 1.$$

$$\text{It follows that } u(x, y) = \int (2x^2 y + e^x + y - 1) dy = x^2 y^2 + ye^x + \frac{y^2}{2} - y + B(x).$$

$$\text{Put } y = 0, \text{ we have } 2 = u(x, 0) = B(x).$$

$$\text{Hence, the particular solution is } u(x, y) = x^2 y^2 + ye^x + \frac{y^2}{2} - y + 2.$$

(b)

$$\frac{\partial^2 u}{\partial x \partial y} = 3 \frac{\partial u}{\partial y} + 2y \Rightarrow \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial x} - 3u \right) = 2y \Rightarrow \frac{\partial u}{\partial x} - 3u = y^2 + A(x)$$

$$\Rightarrow e^{-3x} \frac{\partial u}{\partial x} - 3e^{-3x} u = e^{-3x} y^2 + e^{-3x} A(x) \Rightarrow \frac{\partial}{\partial x} (e^{-3x} u) = e^{-3x} y^2 + e^{-3x} A(x)$$

$$\Rightarrow e^{-3x} u = -\frac{e^{-3x} y^2}{3} + \int e^{-3x} A(x) dx + B(y)$$

$$\Rightarrow u(x, y) = -\frac{y^2}{3} + e^{3x} \int e^{-3x} A(x) dx + e^{3x} B(y)$$

$$\Rightarrow \frac{\partial u}{\partial y} = -\frac{2y}{3} + e^{3x} B'(y)$$

Consider $x = 0$, then

$$\frac{\partial u}{\partial y}(0, y) = -\frac{2y}{3} + B'(y) = y^2 - 2y \Rightarrow B(y) = \int \left(y^2 - \frac{4y}{3} \right) dy = \frac{y^3}{3} - \frac{2y^2}{3} + C$$

Consider the equation $u(x, y) = -\frac{y^2}{3} + e^{3x} \int e^{-3x} A(x) dx + e^{3x} B(y)$ again, let $y = 0$, we have

$$u(x, 0) = e^{3x} \int e^{-3x} A(x) dx + e^{3x} B(0) \underset{B(0)=C}{=} e^{3x} \int e^{-3x} A(x) dx + Ce^{3x} = x + 3e^{-x}$$

$$\Rightarrow e^{3x} \int e^{-3x} A(x) dx = x + 3e^{-x} - Ce^{3x}$$

$$\text{Thus, } u(x, y) = -\frac{y^2}{3} + x + 3e^{-x} - Ce^{3x} + e^{3x} \left(\frac{y^3}{3} - \frac{2y^2}{3} + C \right)$$

$$\text{or } u(x, y) = -\frac{y^2}{3} + x + 3e^{-x} + e^{3x} \left(\frac{y^3}{3} - \frac{2y^2}{3} \right)$$

$$2. \quad \frac{\partial w}{\partial x} = \frac{\partial w}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial x} \underset{\frac{\partial r}{\partial x}=1}{=} \frac{\partial w}{\partial r} - \frac{\partial w}{\partial s}, \quad \frac{\partial w}{\partial y} = \frac{\partial w}{\partial r} \frac{\partial r}{\partial y} + \frac{\partial w}{\partial s} \frac{\partial s}{\partial y} \underset{\frac{\partial r}{\partial y}=1}{=} \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s}$$

$$\begin{aligned} \frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} = w &\Rightarrow \frac{\partial w}{\partial r} - \frac{\partial w}{\partial s} + \frac{\partial w}{\partial r} + \frac{\partial w}{\partial s} = w \Rightarrow \frac{\partial w}{\partial r} - \frac{w}{2} = 0 \Rightarrow e^{-\int \frac{1}{2} dr} \frac{\partial w}{\partial r} - e^{-\int \frac{1}{2} dr} \frac{w}{2} = 0 \\ &\Rightarrow \frac{\partial}{\partial r} \left(e^{-\frac{1}{2}r} w \right) = 0 \Rightarrow e^{-\frac{1}{2}r} w = f(s) \Rightarrow w = f(s) e^{\frac{1}{2}r} \Rightarrow w(x, y) = f(y-x) e^{\frac{1}{2}(x+y)} \end{aligned}$$

$r = x+y$
 $s = -x+y$

3. Assume $u(x, y, z) = X(x)Y(y)Z(z)$, then

$$3 \frac{\partial u}{\partial x} - 2 \frac{\partial u}{\partial y} - 5 \frac{\partial u}{\partial z} = 0 \Rightarrow 3X'YZ - 2XY'Z - 5XYZ' = 0 \Rightarrow 3 \frac{X'}{X} - 2 \frac{Y'}{Y} - 5 \frac{Z'}{Z} = 0.$$

$$\text{Let } 3 \frac{X'}{X} = \lambda, \quad 2 \frac{Y'}{Y} = \mu, \text{ then } 3 \frac{X'}{X} - 2 \frac{Y'}{Y} - 5 \frac{Z'}{Z} = 0 \Rightarrow 5 \frac{Z'}{Z} = \lambda - \mu \Rightarrow \frac{Z'}{Z} = \frac{\lambda - \mu}{5}.$$

$$3 \frac{X'}{X} = \lambda \Rightarrow X' - \frac{\lambda}{3} X = 0 \Rightarrow X(x) = C_1 e^{\frac{\lambda}{3}x}, \quad 2 \frac{Y'}{Y} = \mu \Rightarrow Y' - \frac{\mu}{2} Y = 0 \Rightarrow Y(y) = C_2 e^{\frac{\mu}{2}y} \text{ and}$$

$$\frac{Z'}{Z} = \frac{\lambda - \mu}{5} \Rightarrow Z' - \frac{\lambda - \mu}{5} Z = 0 \Rightarrow Z(z) = C_3 e^{\frac{\lambda - \mu}{5}z}.$$

It follows that

$$u(x, y, z) = C_1 e^{\frac{\lambda}{3}x} C_2 e^{\frac{\mu}{2}y} C_3 e^{\frac{\lambda - \mu}{5}z} = C_1 C_2 C_3 e^{\frac{\lambda}{3}x} e^{\frac{\mu}{2}y} e^{\frac{\lambda - \mu}{5}z} =_{C=C_1 C_2 C_3} C e^{\frac{\lambda}{3}x} e^{\frac{\mu}{2}y} e^{\frac{\lambda - \mu}{5}z} = C e^{\frac{\lambda}{3}x + \frac{\mu}{2}y + \frac{\lambda - \mu}{5}z}$$

4. (a) Here the independent variables are x and t and suppose $\psi(x, t) = X(x)T(t)$ is a nontrivial

$$\text{solution of } \begin{cases} \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c} \frac{\partial \psi}{\partial t}, & 0 < x < \pi, t > 0 \\ \psi(0, t) = \psi(\pi, t) = 0, & t > 0 \end{cases}.$$

$$\frac{\partial \psi}{\partial t}(x, t) = c \frac{\partial^2 \psi}{\partial x^2}(x, t) \Rightarrow XT' = cX''T, \text{ where } X'' = \frac{d^2 X}{dx^2}, T' = \frac{dT}{dt}.$$

Then $\frac{T'}{cT} = \frac{X''}{X} = \lambda$ where λ is a constant, so that $X'' - \lambda X = 0$, that is, $X(x)$ is a solution of the ODE $X'' - \lambda X = 0$.

$$0 = \psi(0, t) = X(0)T(t) \Rightarrow X(0) = 0, \text{ otherwise, we have } T(t) \equiv 0 \text{ then } \psi(x, t) \equiv 0.$$

$$\text{Similarly, } 0 = \psi(\pi, t) = X(\pi)T(t) \Rightarrow X(\pi) = 0.$$

$$\text{So } X(x) \text{ satisfies } \begin{cases} X'' - \lambda X = 0 \\ X(0) = X(\pi) = 0 \end{cases}.$$

$$\text{Since } \psi(x, t) = X(x)T(t) \text{ is a nontrivial solution of } \begin{cases} \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c} \frac{\partial \psi}{\partial t}, & 0 < x < \pi, t > 0 \\ \psi(0, t) = \psi(\pi, t) = 0, & t > 0 \end{cases},$$

$$X(x) \text{ must be a nontrivial solution of } \begin{cases} X'' - \lambda X = 0, & 0 < x < \pi \\ X(0) = X(\pi) = 0 \end{cases}.$$

CASE 1: $\lambda = 0$.

Then $X'' = 0$ and $X(x) = c_1 x + c_2$.

$$0 = X(0) = c_2 \text{ so } X(x) = c_1 x.$$

However, $0 = X(\pi) = \pi c_1 \Rightarrow c_1 = 0 \Rightarrow X(x) \equiv 0$ (not accepted).

Therefore, $\lambda = 0$ can't be accepted.

CASE 2: $\lambda > 0$,

Then $X'' - \lambda X = 0 \Rightarrow X(x) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$.

From the first boundary condition $0 = X(0)$ we have $0 = X(0) = c_1 + c_2$.

And $c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$. So $X(x) = c_1 (e^{\sqrt{\lambda}x} - e^{-\sqrt{\lambda}x})$

From the second boundary condition $0 = X(\pi)$, we have $0 = X(\pi) = c_1 (e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi})$

Notice that $e^{\pi\sqrt{\lambda}} - e^{-\pi\sqrt{\lambda}} \neq 0$, $c_1 (e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}) = 0 \Rightarrow c_1 = 0$.

Therefore, $X(x) \equiv 0$ (not accepted).

This shows that we cannot have $c > 0$.

CASE3: $\lambda < 0$,

$X'' - \lambda X = 0 \Rightarrow X(x) = c_1 \cos \sqrt{-\lambda}x + c_2 \sin \sqrt{-\lambda}x$.

From the first boundary condition, $X(0) = 0 \Rightarrow c_1 = 0 \Rightarrow X = c_2 \sin \sqrt{-\lambda}x$.

From the second boundary condition, we have $X(\pi) = 0$, therefore, $0 = c_2 \sin \sqrt{-\lambda}\pi$

Observe that $c_2 \neq 0$, otherwise we have $X(x) \equiv 0$.

So we must have $\sin \pi \sqrt{-\lambda} = 0$, that is, $\pi \sqrt{-\lambda} \stackrel{\sqrt{-\lambda} > 0}{=} n\pi$, $n = 1, 2, \dots$.

$\pi \sqrt{-\lambda} \stackrel{\sqrt{-\lambda} > 0}{=} n\pi \Rightarrow \sqrt{-\lambda} = n$ ($\lambda = -n^2$) $\Rightarrow X(x) = c_2 \sin nx$, where c_2 is a constant $\neq 0$.

In particular, let $c_2 = 1$, $\sin nx$, $n = 1, 2, \dots$ is a nontrivial solution of

$$\begin{cases} X'' + n^2 X = 0, & 0 < x < \pi \\ X(0) = X(\pi) = 0 \end{cases}.$$

Consider the corresponding ODE for T , $T' - (-cn^2)T = 0$, $n = 1, 2, \dots$, we have

$T_n(t) = d_n e^{-cn^2 t}$ ($d_n \neq 0$) as a nontrivial solution of $T' + cn^2 T = 0$, in particular,

$T_n(t) = e^{-cn^2 t}$ is a nontrivial solution of $T' + cn^2 T = 0$.

It follows that $\psi_n(x, t) = X_n(x)T_n(t) = e^{-cn^2 t} \sin nx$, $n = 1, 2, \dots$, is a nontrivial solution of

$$\begin{cases} \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c} \frac{\partial \psi}{\partial t}, & 0 < x < \pi, t > 0 \\ \psi(0, t) = \psi(\pi, t) = 0, & t > 0 \end{cases}.$$

By superposition principle, $\psi(x, t) = \sum_{n=1}^{\infty} c_n \psi_n(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) T_n(t) = \sum_{n=1}^{\infty} c_n e^{-cn^2 t} \sin nx$

formally satisfies

$$\begin{cases} \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c} \frac{\partial \psi}{\partial t}, & 0 < x < \pi, t > 0 \\ \psi(0, t) = \psi(\pi, t) = 0, & t > 0 \end{cases}, \text{ where } c_n \text{ is an arbitrary constant.}$$

Observe that since $c > 0$, $\psi(x, t) = \sum_{n=1}^{\infty} c_n e^{-cn^2 t} \sin nx$ also satisfies the third condition that $\psi(x, t) \rightarrow 0$ as $t \rightarrow \infty$.

Hence the required solution is $\psi(x, t) = \sum_{n=1}^{\infty} c_n e^{-cn^2 t} \sin nx$, where c_n is an arbitrary constant.

(b) With $\frac{\partial \psi}{\partial x}(0, t) = 0$, we have $X'(0)T(t) = 0$, $t > 0 \Rightarrow X'(0) = 0$

With $\frac{\partial \psi}{\partial x}(\pi, t) = 0$, we have $X'(\pi)T(t) = 0$, $t > 0 \Rightarrow X'(\pi) = 0$.

$X(x)$ is a nontrivial solution of the differential eigenvalue problem

$$\begin{cases} X'' - \lambda X = 0, & 0 < x < \pi \\ X'(0) = X'(\pi) = 0 \end{cases}$$

For $\lambda = 0$, the general solution to $X'' = 0$, $0 < x < \pi$ is $X(x) = A_1 x + A_2$, $X' = A_1$. With $X'(0) = 0$, we have $A_1 = 0$ so $X(x) = A_2$.

And $X = A_2$ satisfies the second condition, $X'(\pi) = 0$.

Thus $\lambda = 0$ is an eigenvalue of the problem $\begin{cases} X'' - \lambda X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases}$

In particular, $X(x) = 1$ is a nontrivial solution of $\begin{cases} X'' - \lambda X = 0, & 0 < x < \pi \\ X'(0) = X'(\pi) = 0 \end{cases}$

For $\lambda > 0$, the general solution to $X'' - \lambda X = 0$, $0 < x < \pi$ is $X = A_3 e^{\sqrt{\lambda}x} + A_4 e^{-\sqrt{\lambda}x}$.

$$X(x) = A_3 e^{\sqrt{\lambda}x} + A_4 e^{-\sqrt{\lambda}x} \Rightarrow X'(x) = A_3 \sqrt{\lambda} e^{\sqrt{\lambda}x} - A_4 \sqrt{\lambda} e^{-\sqrt{\lambda}x}$$

$$0 = X'(0) = A_3 \sqrt{\lambda} - A_4 \sqrt{\lambda} \Rightarrow A_3 - A_4 = 0 \Rightarrow A_3 = A_4. \text{ So } X(x) = A_3 (e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x}).$$

$$0 = X'(\pi) = A_3 \sqrt{\lambda} (e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}) \Rightarrow_{\sqrt{\lambda}(e^{\sqrt{\lambda}\pi} - e^{-\sqrt{\lambda}\pi}) \neq 0} A_3 = 0.$$

Thus, $\begin{cases} X'' - \lambda X = 0, & 0 < x < \pi \\ X'(0) = X'(\pi) = 0 \end{cases}$ has no nontrivial solutions. This shows that we cannot

have $\lambda > 0$. It follows problem $\begin{cases} X'' - \lambda X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases}$ has no positive eigenvalue.

For $\lambda < 0$, the general solution to $X'' - \lambda X = 0$, $0 < x < \pi$ is

$$X(x) = A_5 \cos \sqrt{-\lambda}x + A_6 \sin \sqrt{-\lambda}x \text{ and}$$

$$X'(x) = -A_5 \sqrt{-\lambda} \sin \sqrt{-\lambda}x + A_6 \sqrt{-\lambda} \cos \sqrt{-\lambda}x.$$

With $X'(0) = 0$, we have $A_6 \sqrt{-\lambda} = 0 \Rightarrow A_6 = 0$. So $X(x) = A_5 \cos \sqrt{-\lambda}x$ and $X'(x) = -A_5 \sqrt{-\lambda} \sin \sqrt{-\lambda}x$.

With $X'(\pi) = 0$, we have $-A_5 \sqrt{-\lambda} \sin \sqrt{-\lambda} \pi = 0$.

To have a nontrivial solution we need $A_5 \neq 0$ and choose $\sqrt{-\lambda} (\neq 0)$ so that $\sin \sqrt{-\lambda} \pi = 0 \Rightarrow \sqrt{-\lambda} = 1, 2, \dots$.

It leads to $\lambda = -n^2$, $n = 1, 2, \dots$ with corresponding eigenfunctions $X_n(x) = A_n \cos nx$, in

particular, $X_n(x) = \cos nx$ is a nontrivial solution to $\begin{cases} X'' + n^2 X = 0 \\ X'(0) = X'(\pi) = 0 \end{cases}$.

We may combine the case $\lambda = 0$ and the case $\lambda < 0$ by writing the eigenvalues as $\lambda_n = -n^2$, $n = 0, 1, 2, \dots$ and the corresponding eigenfunctions $X_n(x) = \cos nx$.

The corresponding equation for T is $T' + n^2 c T = 0$.

When $n = 0$, we have $T' = 0 \Rightarrow T_0(t) = 1$.

When $n = 1, 2, \dots$, we have $T_n(t) = e^{-n^2 ct}$.

Let $\psi_n(x, t) = X_n(x) T_n(t) = e^{-n^2 ct} \cos nx$, $n = 0, 1, 2, \dots$.

Observe that each of $\psi_n(x, t) = X_n(x) T_n(t) = e^{-n^2 ct} \cos nx$, $n = 0, 1, 2, \dots$ is a solution of

$$\begin{cases} \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c} \frac{\partial \psi}{\partial t}, & 0 < x < \pi, t > 0 \\ \frac{\partial \psi}{\partial x}(0, t) = \frac{\partial \psi}{\partial x}(\pi, t) = 0, & t > 0 \end{cases}$$

By the principle of superposition,

$$\psi(x, t) = \sum_{n=0}^{\infty} c_n \psi_n(x, t) = c_0 + \sum_{n=1}^{\infty} c_n \psi_n(x, t) = c_0 + \sum_{n=1}^{\infty} c_n e^{-n^2 ct} \cos nx$$

$$\text{also formally satisfies } \begin{cases} \frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c} \frac{\partial \psi}{\partial t}, & 0 < x < \pi, t > 0 \\ \frac{\partial \psi}{\partial x}(0, t) = \frac{\partial \psi}{\partial x}(\pi, t) = 0, & t > 0 \end{cases}.$$

With the third condition $\psi(x, t) \rightarrow 0$ as $t \rightarrow \infty$, we have $c_0 = 0$. Finally, the required

solution is $\psi(x, t) = \sum_{n=1}^{\infty} c_n e^{-n^2 ct} \cos nx$.

5. (a) Compare the coefficients of the diffusion equation with the general second-order linear PDE, we have $A = 1$, $B = C = 0 \Rightarrow B^2 - 4AC = 0$. The diffusion equation is a parabolic PDE.

(b) Let $u(x, t) = X(x)T(t)$, then $X(x) \frac{dT}{dt} = \frac{d^2 X}{dx^2} T(t) \Rightarrow \frac{1}{T} \frac{dT}{dt} = \frac{1}{X} \frac{d^2 X}{dx^2} = \lambda$.

$$\text{We have } \frac{d^2 X}{dx^2} - \lambda X = 0, \quad \frac{dT}{dt} - \lambda T = 0.$$

We solved these problems in Q4, obtaining

$$u_n(x, t) = X_n(x) T_n(t) = E_n e^{-n^2 t} \sin nx, \quad n = 1, 2, \dots$$

Using the principle of superposition, $u(x, t) = \sum_{n=1}^{\infty} E_n e^{-n^2 t} \sin nx$

(c) With the initial condition $u(x, 0) = f(x)$, $f(x) = \sum_{n=1}^{\infty} E_n \sin nx$, which is the Fourier sin

series expansion of $f(x)$ on $[0, \pi]$. Hence, $E_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$.

If $f(x)$ satisfies certain conditions, this Fourier sin series converges to $f(x)$ on $[0, \pi]$ and the solution of the initial-boundary value problem is

$$u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\int_0^{\pi} f(x) \sin nx dx \right) e^{-n^2 t} \sin nx.$$

(d) If $f(x) = 3\sin x - 2\sin 2x + 10\sin 5x$, $0 \leq x \leq \pi$, then by equating the coefficients of the

equation $f(x) = \sum_{n=1}^{\infty} E_n \sin nx$ we obtain $E_1 = 3$, $E_2 = -2$, $E_5 = 10$, $E_m = 0$, $m \neq 1, 2, 5$.

Hence the solution is $u(x, t) = 3e^{-t} \sin x - 2e^{-4t} \sin 2x + 10e^{-25t} \sin 5x$

6. Let $V(x, y) = X(x)Y(y)$, then $X''Y + XY'' = 0$ or $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$.

We have $X'' - \lambda X = 0$, $Y'' + \lambda Y = 0$.

Considering the boundary conditions,

$$0 = V(0, y) = X(0)Y(y) \Rightarrow X(0) = 0, \quad 0 = V(3, y) = X(3)Y(y) \Rightarrow X(3) = 0$$

$$0 = V(x, 0) = X(x)Y(0) \Rightarrow Y(0) = 0$$

Let's solve for X first.

For $\lambda = 0$, $\lambda = 0 \Rightarrow X'' = 0 \Rightarrow X = c_1 x + c_2$.

$$X(0) = 0 \Rightarrow c_2 = 0 \Rightarrow X(x) = c_1 x, \quad X(3) = 0 \Rightarrow 3c_1 = 0 \Rightarrow c_1 = 0.$$

Hence $\lambda = 0$ is not an eigenvalue of the problem.

For $\lambda > 0$, the general solution of $X'' - \lambda X = 0$ is $X = A_3 e^{\sqrt{\lambda} x} + A_4 e^{-\sqrt{\lambda} x}$.

$$X(0) = 0 \Rightarrow A_3 + A_4 = 0 \Rightarrow A_3 = -A_4. \text{ And so } X = A_3 (e^{\sqrt{\lambda} x} - e^{-\sqrt{\lambda} x}).$$

$$X(3) = 0 \Rightarrow A_3 (e^{3\sqrt{\lambda}} - e^{-3\sqrt{\lambda}}) = A_3 \left(\frac{e^{6\sqrt{\lambda}} - 1}{e^{3\sqrt{\lambda}}} \right) = 0 \Rightarrow A_3 = 0. \quad \frac{e^{6\sqrt{\lambda}} - 1}{e^{3\sqrt{\lambda}}} \neq 0$$

It follows the problem has no positive eigenvalue.

For $\lambda < 0$, the general solution is of $X'' - \lambda X = 0$ is $X(x) = A_5 \cos \sqrt{-\lambda} x + A_6 \sin \sqrt{-\lambda} x$.

$$X(0) = 0 \Rightarrow A_5 = 0, \text{ so } X(x) = A_6 \sin \sqrt{-\lambda} x. \quad 0 = X(3) \Rightarrow X(x) = A_6 \sin 3\sqrt{-\lambda}.$$

To have a nontrivial solution we need $A_6 \neq 0$, hence $\sin 3\sqrt{-\lambda} = 0$.

$$\sin 3\sqrt{-\lambda} = 0 \Rightarrow \sqrt{-\lambda} = \frac{n\pi}{3} \quad (\lambda = -\frac{n^2 \pi^2}{9}), \quad n = 1, 2, \dots$$

For each such n , let $X_n(x) = \sin \frac{n\pi}{3} x$, which is an eigenfunction of the problem corresponding

to the eigenvalue $\lambda_n = -\frac{n^2\pi^2}{9}$, $n = 1, 2, \dots$.

Now consider $Y_n'' - \frac{n^2\pi^2}{9}Y_n = 0$ and its general solution is $Y_n(y) = B_{n1}e^{\frac{n\pi}{3}y} + B_{n2}e^{-\frac{n\pi}{3}y}$.

$$Y_n(0) = 0 \Rightarrow B_{n1} + B_{n2} = 0 \Rightarrow B_{n1} = -B_{n2},$$

so $Y_n(y) = B_{n1}\left(e^{\frac{n\pi}{3}y} - e^{-\frac{n\pi}{3}y}\right)$ or $Y_n(y) = B_n \sinh \frac{n\pi}{3}y$, where $B_n = 2B_{n1}$, in particular, let

$$B_n = 1, Y_n(y) = \sinh \frac{n\pi}{3}y \text{ is a solution of } Y_n'' - \frac{n^2\pi^2}{9}Y_n = 0, Y_n(0) = 0.$$

It follows that $V_n(x, y) = X_n(x)Y_n(y) = \left(\sin \frac{n\pi x}{3}\right)\left(\sinh \frac{n\pi y}{3}\right)$ satisfies

$$\begin{cases} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \\ V(0, y) = V(3, y) = V(x, 0) = 0 \end{cases}$$

By the principle of superposition, $V(x, y) = \sum_{n=1}^{\infty} A_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{3}\right) \sinh\left(\frac{n\pi y}{3}\right)$

also formally satisfies $\begin{cases} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \\ V(0, y) = V(3, y) = V(x, 0) = 0 \end{cases}$.

Determine the values of A_n such that

$$V(x, y) = \sum_{n=1}^{\infty} A_n X_n(x) Y_n(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{3}\right) \sinh\left(\frac{n\pi y}{3}\right)$$

satisfies the boundary condition $V(x, 2) = V_0$.

We have $V_0 = V(x, 2) = \sum_{n=1}^{\infty} A_n X_n(x) Y_n(2) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{3}\right) \sinh\left(\frac{2n\pi}{3}\right)$, which is the Fourier sine series expansion of V_0 on $[0, 3]$. As a consequence,

$$A_n \sinh\left(\frac{2n\pi}{3}\right) = \frac{2}{3} \int_0^3 V_0 \sin \frac{n\pi x}{3} dx = \frac{2V_0}{3} \left[-\frac{\cos \frac{n\pi x}{3}}{\frac{n\pi}{3}} \right]_0^3 = -\frac{2V_0}{n\pi} (\cos n\pi - 1) = \frac{2V_0}{n\pi} [1 - (-1)^n]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4V_0}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$\Rightarrow A_{2n-1} = \frac{4V_0}{(2n-1)\pi \sinh \frac{2(2n-1)\pi}{3}}, \quad n = 1, 2, \dots$$

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1) \sinh \frac{2(2n-1)\pi}{3}} \sin \left[\frac{(2n-1)\pi x}{3} \right] \sinh \left[\frac{(2n-1)\pi y}{3} \right] \text{ is the solution to}$$

$$\begin{cases} \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \\ V(0, y) = V(3, y) = 0 \\ V(x, 0) = 0, V(x, 2) = V_0 \end{cases}$$

7. (a) Let $V(x, t) = X(x)T(t)$ be a nontrivial solution to
$$\begin{cases} \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 \\ \frac{\partial v}{\partial x} \Big|_{x=0} = 0, \frac{\partial v}{\partial x} \Big|_{x=l} = 0 \end{cases} \dots\dots(I).$$

Then $X(x)T'(t) - X''(x)T(t) = 0$ and $\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda$, where λ is a constant.

$$\frac{X''(x)}{X(x)} = -\lambda \Rightarrow X''(x) + \lambda X(x) = 0.$$

$$\frac{\partial V(x, t)}{\partial x} \Big|_{x=0} = X'(0)T(t) = 0 \quad \Rightarrow \quad X'(0) = 0. \text{ Similarly, } X'(l) = 0.$$

Thus, $X(x)$ is a solution to the boundary-value problem

$$\begin{cases} \frac{d^2 X}{dx^2} + \lambda X = 0 \\ \frac{dX}{dx} \Big|_{x=0} = 0, \frac{dX}{dx} \Big|_{x=l} = 0 \end{cases} \dots\dots(II)$$

(b) For $\lambda < 0$, $\frac{d^2 X}{dx^2} + \lambda X = 0 \Rightarrow X = Ae^{-\sqrt{-\lambda}x} + Be^{\sqrt{-\lambda}x} \Rightarrow X' = -A\sqrt{-\lambda}e^{-\sqrt{-\lambda}x} + B\sqrt{-\lambda}e^{\sqrt{-\lambda}x}.$

$$\begin{cases} X'(0) = -A\sqrt{-\lambda} + B\sqrt{-\lambda} \\ X'(l) = -A\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} + B\sqrt{-\lambda}e^{\sqrt{-\lambda}l} \end{cases} \Rightarrow \begin{pmatrix} X'(0) \\ X'(l) \end{pmatrix} = \begin{pmatrix} -\sqrt{-\lambda} & \sqrt{-\lambda} \\ -\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} & \sqrt{-\lambda}e^{\sqrt{-\lambda}l} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

$$\det \begin{pmatrix} -\sqrt{-\lambda} & \sqrt{-\lambda} \\ -\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} & \sqrt{-\lambda}e^{\sqrt{-\lambda}l} \end{pmatrix} = \lambda (e^{\sqrt{-\lambda}l} - e^{-\sqrt{-\lambda}l}) \neq 0.$$

$$\text{So } \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} X'(0) \\ X'(l) \end{pmatrix} = \begin{pmatrix} -\sqrt{-\lambda} & \sqrt{-\lambda} \\ -\sqrt{-\lambda}e^{-\sqrt{-\lambda}l} & \sqrt{-\lambda}e^{\sqrt{-\lambda}l} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \Rightarrow A = B = 0 \Rightarrow X(x) \text{ is trivial.}$$

For $\lambda > 0$,

$$\frac{d^2 X}{dx^2} + \lambda X = 0 \Rightarrow X = A \cos \sqrt{\lambda}x + B \sin \sqrt{\lambda}x \Rightarrow X' = -A\sqrt{\lambda} \sin \sqrt{\lambda}x + B\sqrt{\lambda} \cos \sqrt{\lambda}x.$$

$$0 = X'(0) = B\sqrt{\lambda} \Rightarrow B = 0. \text{ So } X = A \cos \sqrt{\lambda}x \text{ and } X' = -A\sqrt{\lambda} \sin \sqrt{\lambda}x.$$

$$0 = X'(l) = -A\sqrt{\lambda} \sin \sqrt{\lambda} l \underset{X(x) \text{ is nontrivial}}{\Rightarrow} \sin \sqrt{\lambda} l = 0 \Rightarrow \sqrt{\lambda} l = n\pi, \quad n = 1, 2, \dots \Rightarrow \lambda = \left(\frac{n\pi}{l} \right)^2, \quad n = 1, 2, \dots$$

Therefore, let $A = 1$ and $X(x) = \cos \frac{n\pi}{l} x$, $n = 1, 2, \dots$ are nontrivial solutions to

$$\begin{cases} \frac{d^2 X}{dx^2} + \lambda X = 0 \\ \frac{dX}{dx} \Big|_{x=0} = 0, \quad \frac{dX}{dx} \Big|_{x=l} = 0 \end{cases} \dots\dots(\text{II}) \quad \text{with } \lambda = \left(\frac{n\pi}{l} \right)^2, \quad n = 1, 2, \dots$$

For $\lambda = 0$, $\frac{d^2 X}{dx^2} = 0 \Rightarrow X = Ax + B \Rightarrow X' = A$.

$0 = X'(0) = A$. Thus $X = B$. So $X'(l) = X'(0) = 0$. And choose $B = 1$, so $X = 1 = \cos \frac{n\pi}{l} x$,

$$\text{where } n = 0 \text{ is a nontrivial solution to } \begin{cases} \frac{d^2 X}{dx^2} = 0 \\ \frac{dX}{dx} \Big|_{x=0} = 0, \quad \frac{dX}{dx} \Big|_{x=l} = 0 \end{cases} \dots\dots(\text{II}).$$

-End-