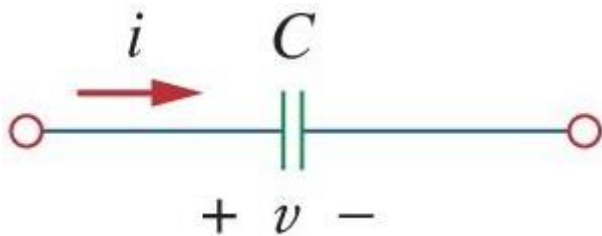


I-V relation in a capacitor

If the voltage across the capacitor is time-varying, then the charge stored on the capacitor must also be time-varying:



$$\frac{dQ(t)}{dt} = \frac{dCV(t)}{dt} \rightarrow C \frac{dV}{dt}$$

$$i(t) = C \frac{dV(t)}{dt}$$

Differentiating with respect to time (t):

Current depends on the **rate of change** of voltage

Case study to consider:

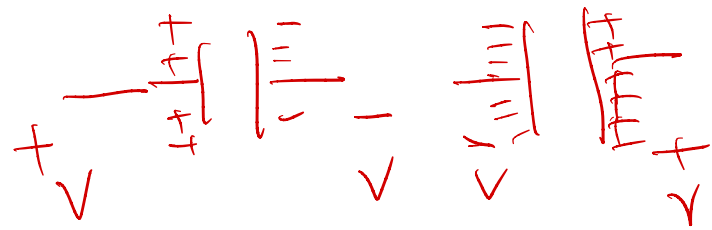
Given that $C = 1 \text{ nF}$

If $V = 1\text{V} \rightarrow Q = \underline{1 \text{ nC}}$ $Q = CV = 1 \text{ nC}$

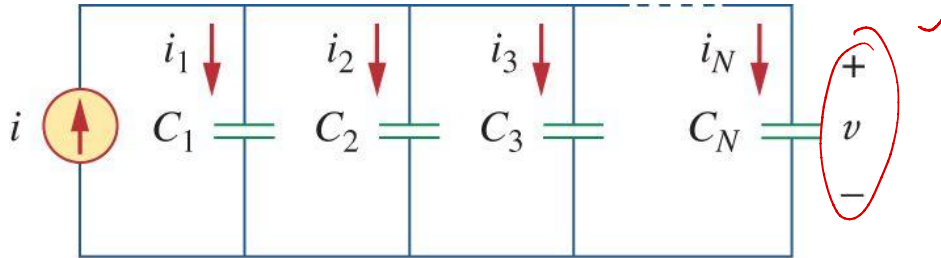
If we reverse V, such that now $V = -1\text{V} \rightarrow Q = \underline{-1 \text{ nC}}$

If the above change was made gradually over 1 ms, what would be the resulting current?

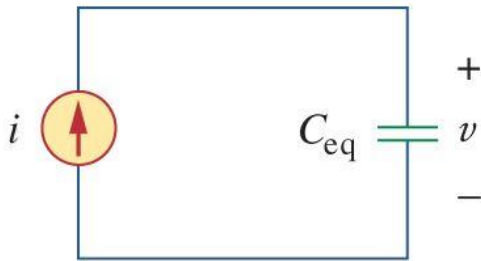
$$I_{\text{avg}} = \frac{Q_f - Q_i}{\Delta t} = \frac{(-1 - 1) \times 10^{-9}}{1 \text{ ms}} = -2 \mu\text{A}$$



Capacitors in parallel



(a)



(b)

$$C_{eq} = C_1 + C_2 + C_3 + \dots$$

$$v C_{eq} = v C_1 + v C_2 + \dots$$

$$\downarrow$$

$$Q_{eq} = Q_1 + Q_2 + \dots$$

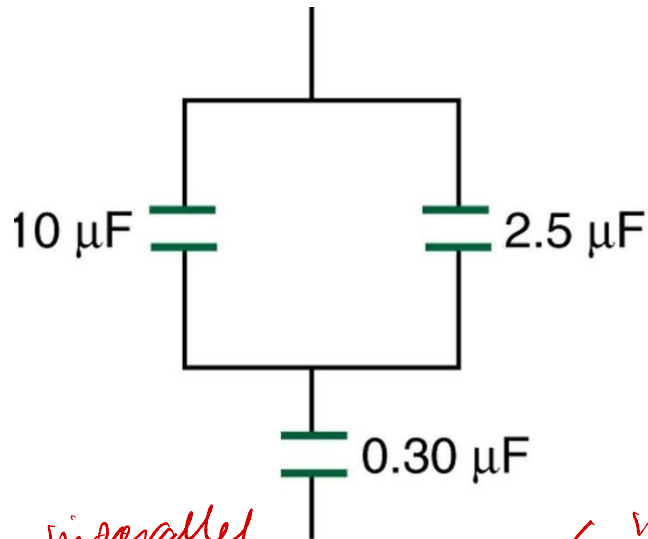
$$C_{eq} = C_1 + C_2 + C_3 + \dots + C_N$$

What is the value of Q_{eq} ?

$$Q_{eq} = Q_1 + Q_2 + \dots$$

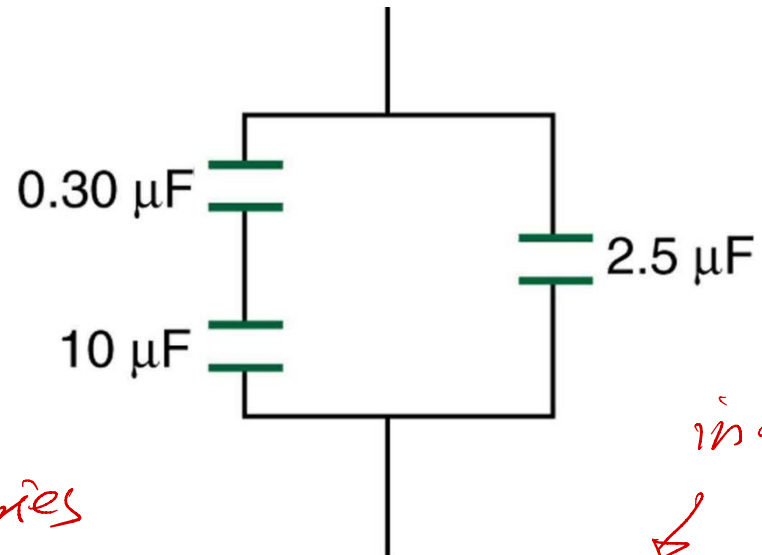
Example on Capacitor in parallel and series

What is the equivalent of the following capacitance circuit?



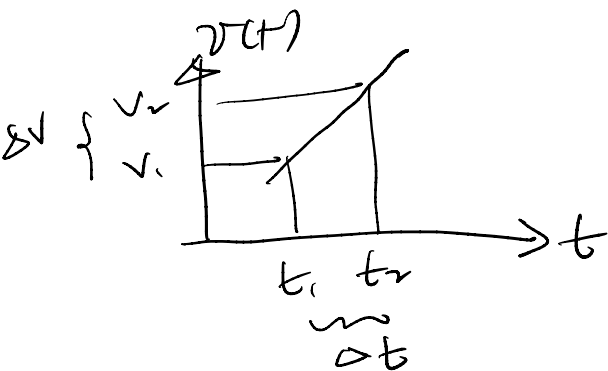
in parallel ↓ *in series* ↙

$$C_{eq} = (10\mu F + 2.5\mu F) // 0.30\mu F$$
$$= 12.5 // 0.30 = 0.293\mu F$$



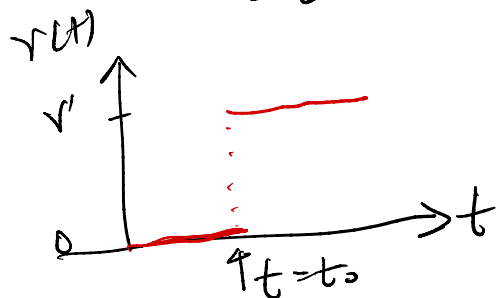
in series ↘ *in parallel* ↓

$$C_{eq} = (0.3\mu F // 10\mu F) + 2.5\mu F$$
$$= 2.781\mu F$$



$$\frac{dv(t)}{dt} = \frac{v_2 - v_1}{t_2 - t_1}$$

$$= \frac{\Delta V}{\Delta t}$$



$$\left. \frac{dv(t)}{dt} \right|_{t=t_0} = \frac{v' - 0}{0} \Rightarrow \frac{v'}{0}$$

$$\Rightarrow \infty$$



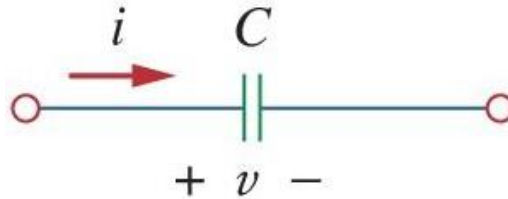
$$i(t) = C \frac{dv(t)}{dt}$$

$\uparrow$ finite $\uparrow$ real $\nwarrow$ finite

$$v(t=t_1) = v(t=t_1 + \Delta t)$$

Examples

$$i(t) = C \frac{dv(t)}{dt}$$

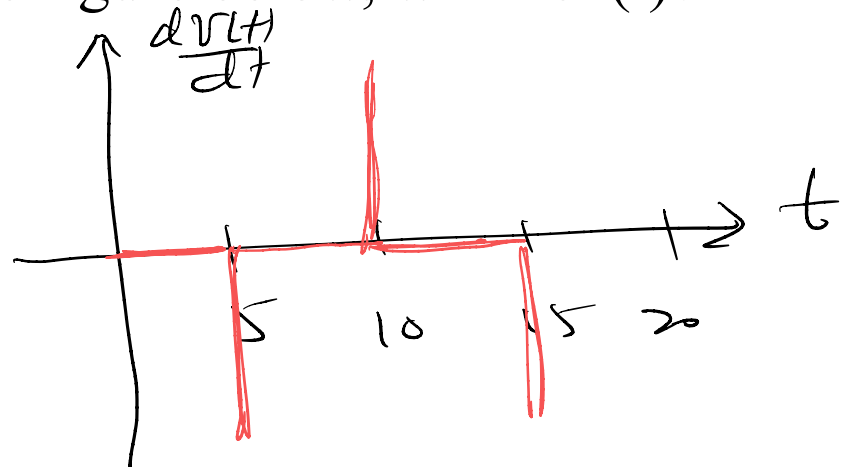
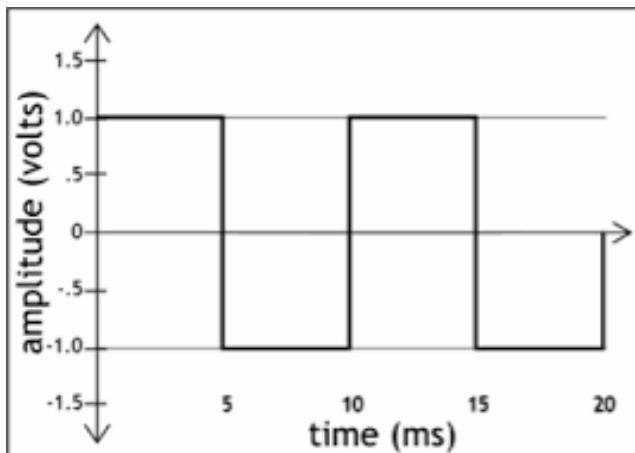


phasor

Case 1: If $v(t) = 10 \sin(2t)$ V, $C = 10 \mu\text{F}$, what is $i(t)$?

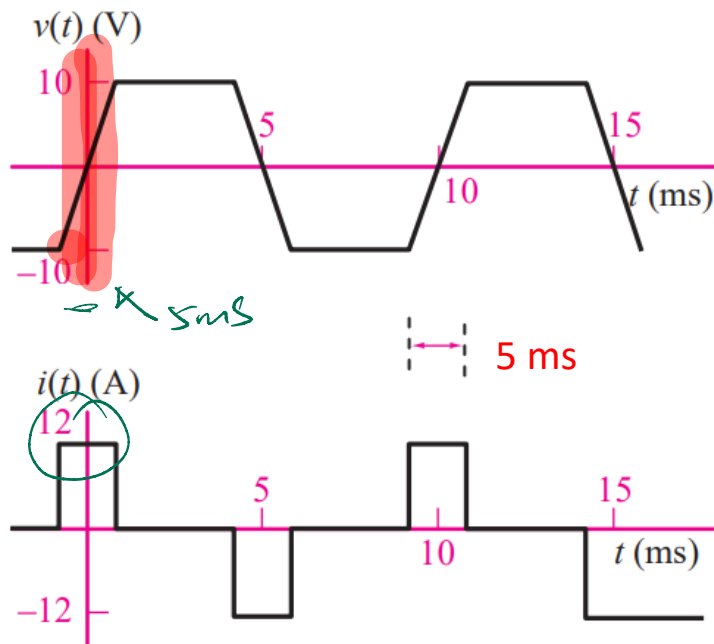
$$i(t) = (10 \mu\text{F}) \frac{d}{dt} [10 \sin(2t)] = (10^{-5}) (20 \cos 2t) = 2 \times 10^{-4} \cos 2t \text{ A}$$

Case 2: If $v(t)$ is shown in the figure below, what is $i(t)$?



Example

If the plots shown in Figure P4.18 are the voltage across and the current through an ideal capacitor, determine the capacitance.



Handwritten notes and equations:

$$i(t) = C \frac{dv(t)}{dt}$$

Consider $0 < t < 5 \text{ ms}$

$$i(t) = 12 \text{ A}$$

$$\frac{dv(t)}{dt} = \frac{10 - (-10)}{5 \text{ ms}} = \frac{20}{5 \times 10^{-3}} = 4000 \text{ V/s}$$

$$12 = C \cdot 4000$$

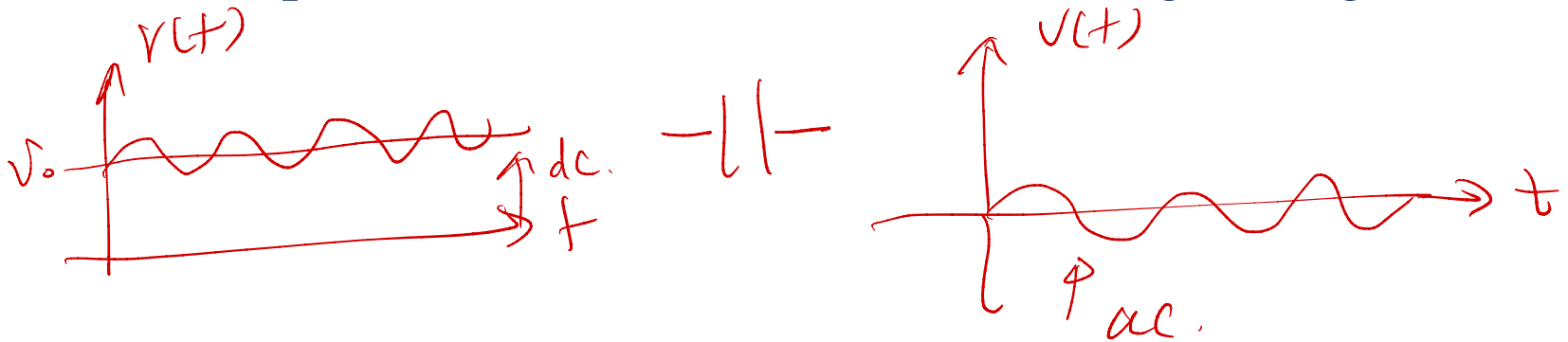
$$C = \frac{12}{4000} = 3 \text{ mF}$$

Summary: Capacitor response to DC

$$\frac{dV(t)}{dt} \rightarrow 0, \quad i(t) \rightarrow 0$$

If the applied voltage is DC

- Insulating dielectric blocks the current from flowing through
- Plates will charge up
- At DC, capacitor blocks current from flowing through



Simple first-order RC circuit

- At $t = 0$, $v_c(0) = 0$
- After $t = 0$, the circuit is closed:

$$\triangleright i(t) = \frac{v_R}{R} = C \frac{dv_c}{dt}, \text{ and}$$

$$\triangleright v_R(t) = V_0 - v_c(t)$$

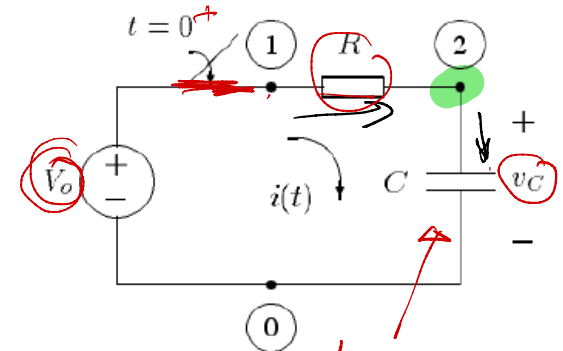
$$\text{Hence, } \frac{V_0 - v_c(t)}{CR} = \frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} + \frac{v_c}{CR} = \frac{V_0}{CR}$$

$$\text{The general solution is } v_c(t) = Ae^{-\frac{t}{CR}} + V_0 \text{ for } t \geq 0 \text{ (A is a constant)}$$

$$\text{As the initial condition is } v_c(0^+) = 0, A + V_0 = 0 \Rightarrow A = -V_0$$

Hence,

$$v_c(t) = V_0 \left(1 - e^{-\frac{t}{CR}} \right)$$



at $t < 0$

$$v_c(0^-) = 0$$

at $t \geq 0$

$$v_c(0^+) = v_c(0^-) = 0$$

$$\frac{V_0 - v_c(t)}{CR} = \frac{dv_c}{dt}$$

$$\int \frac{dx}{x} = \ln x$$

$$\frac{dv_c}{dt} = \frac{-d(V_0 - v_c(t))}{dt} = \frac{V_0 - v_c(t)}{CR}$$

$$\Rightarrow \int \frac{d(V_0 - v_c(t))}{V_0 - v_c(t)} = \int -\frac{dt}{CR} + K$$

$$\Rightarrow \ln(V_0 - v_c(t)) = -\frac{t}{CR} + K$$

$$\text{at } t=0 \quad v_c(0) = 0$$

$$\ln(V_0 - 0) = K$$

$$K = \ln(V_0)$$

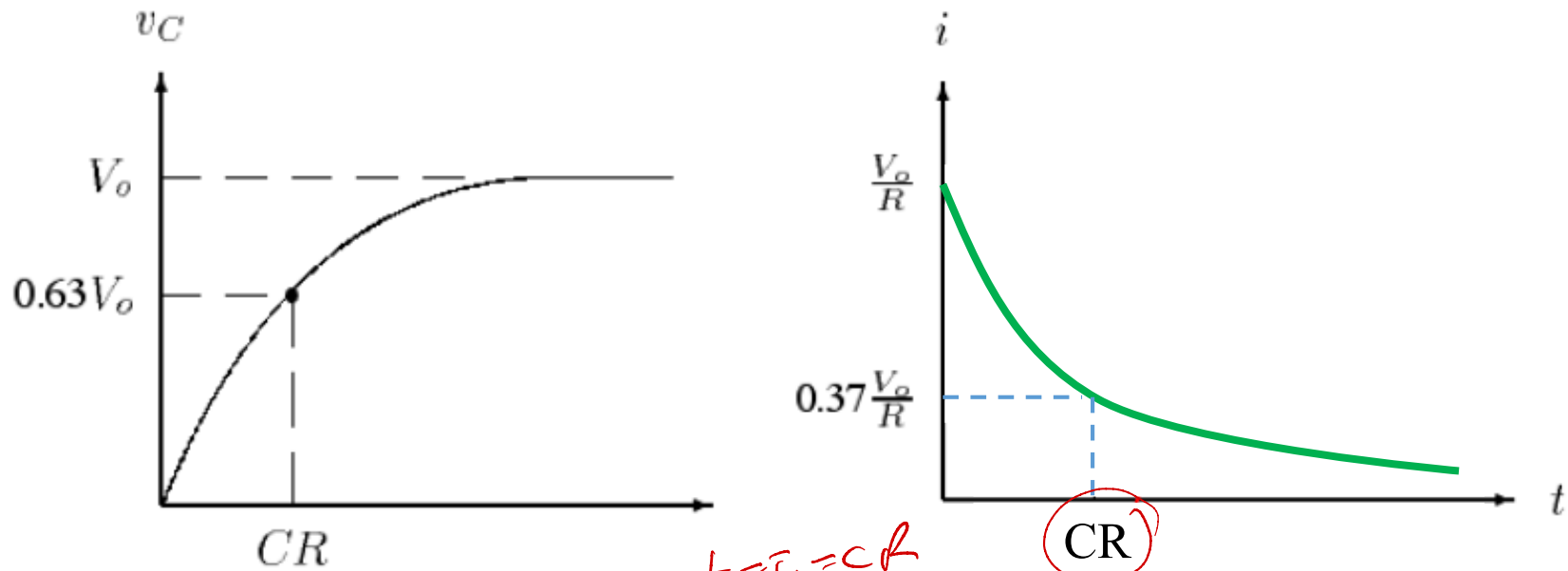
$$\Rightarrow \ln(V_0 - v_c(t)) = -\frac{t}{CR} + \ln(V_0)$$

$$\Rightarrow \ln\left(\frac{V_0 - v_c(t)}{V_0}\right) = -\frac{t}{CR}$$

$$\Rightarrow V_0 - v_c(t) = V_0 e^{-t/CR}$$

$$\Rightarrow v_c(t) = V_0(1 - e^{-t/CR})$$

Transient response of a simple RC circuit



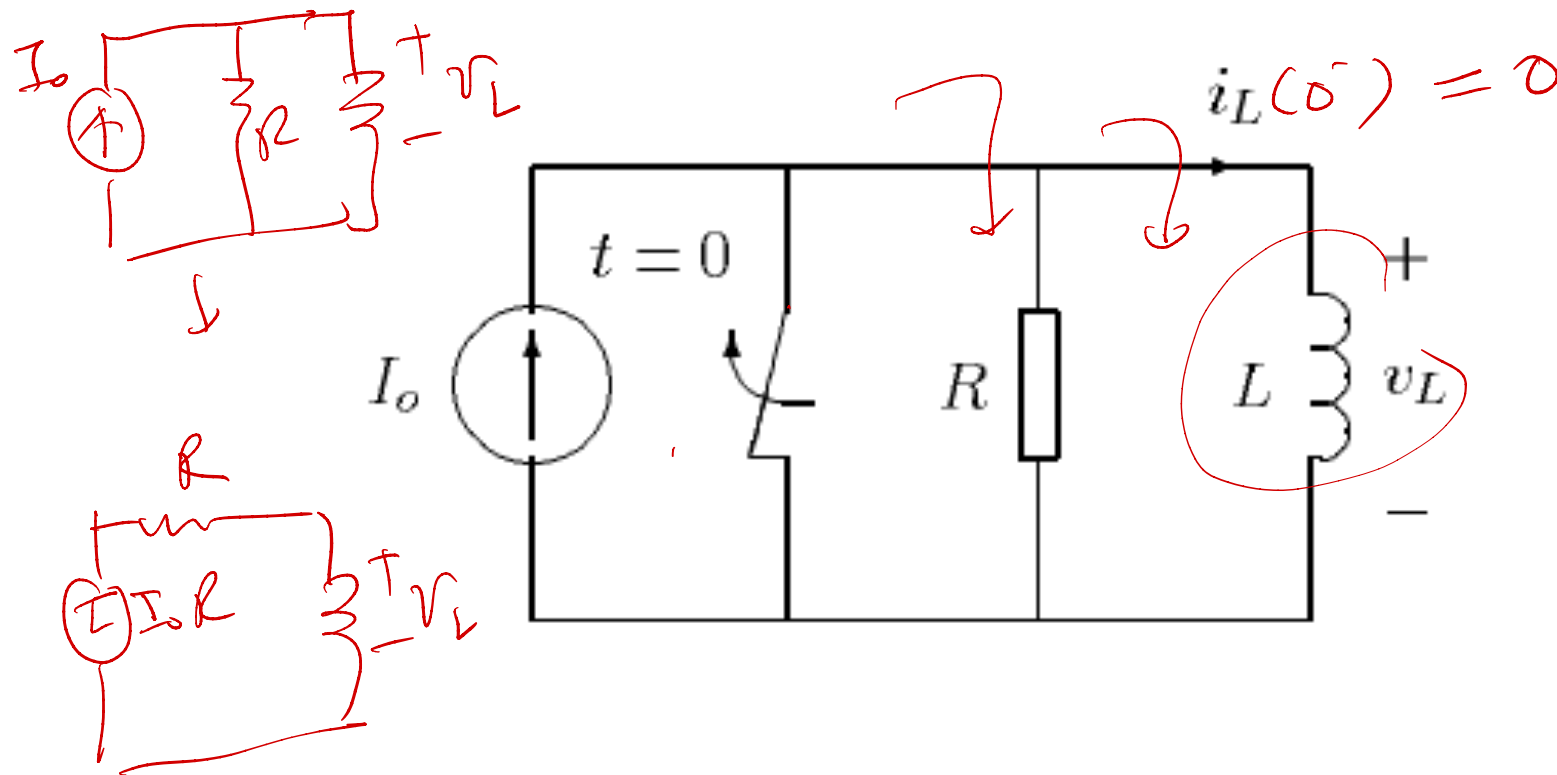
$$v_c(t) = V_0 \left(1 - e^{-\frac{t}{CR}} \right) \quad \rightarrow \quad i(t) = C \frac{dv_c}{dt} = \frac{V_0}{R} e^{-\frac{t}{CR}}$$

When $t = CR$,

$$v_c(CR) = V_0 (1 - e^{-1}) = 0.63V_0 \quad \text{and} \quad i(CR) = \frac{V_0}{R} e^{-1} = 0.37 \frac{V_0}{R}$$

Simple first-order RL circuit

If I_o is a constant current, how the current $i_L(t)$ and the voltage across the inductor $v_L(t)$ will change when the switch is opened at $t = 0$



Simple first-order RL circuit

- When $t < 0$, short-circuit at switch, $i_L = 0$

- For $t \geq 0$,

$\triangleright v_L(t) = L \left(\frac{di_L}{dt} \right)$, and

$\triangleright v_R(t) = (I_0 - i_L(t))R$

$i_R = \frac{V_L}{R} = \frac{L}{R} \frac{di_L}{dt}$

- Hence, $(I_0 - i_L(t))R = L \left(\frac{di_L}{dt} \right) \Rightarrow \frac{di_L}{dt} + \frac{R}{L} i_L(t) = \frac{RI_0}{L}$

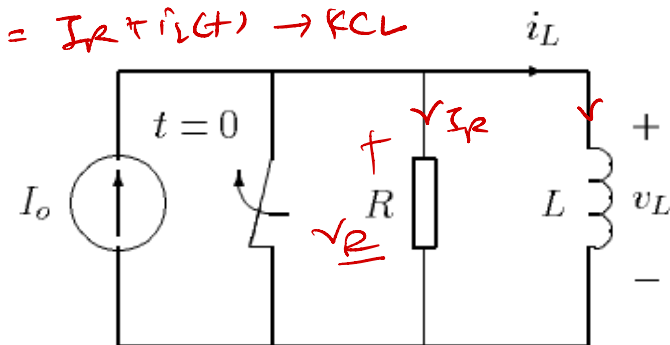
- The general solution is $i_L(t) = Ae^{-\frac{Rt}{L}} + I_0$ where A is a constant

- $i_L(0^+) = 0 = A + I_0 \Rightarrow A = -I_0$

- Hence, for $t \geq 0$, we obtain

$$i_L(t) = I_0 \left(1 - e^{-\frac{Rt}{L}} \right)$$

$I_0 = I_R + i_L(t) \rightarrow KCL$



$V_R = V_L$

Example 1: Initial and final values given

- At $t = 0$, the switch is thrown to the right
- Initial and final values of capacitor voltage are:

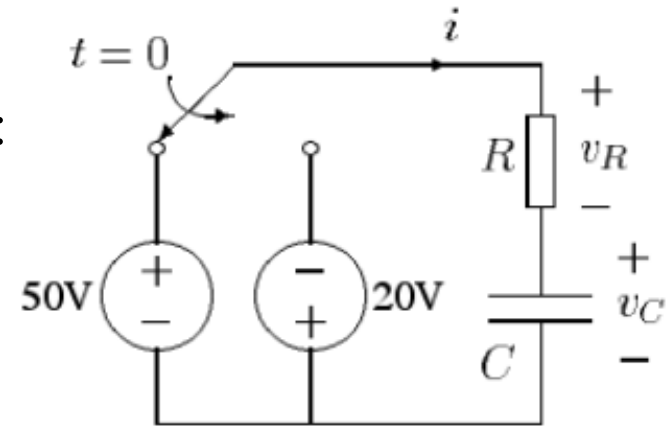
$$v_c(0^+) = 50V \text{ and } v_c(\infty) = -20V$$

- Decrease in $v_c \rightarrow$ Case B ($\tau = CR$)

$$v_c(t) = -20 + 70e^{-\frac{t}{CR}}$$

- $v_R(t) = -20 - v_c(t) = -70e^{-\frac{t}{CR}}$

- $i(t) = \frac{v_R(t)}{R} = -\frac{70}{R}e^{-\frac{t}{CR}}$



$$v_c(t), t \geq 0$$

$$-t/\tau \rightarrow \tau = RC$$

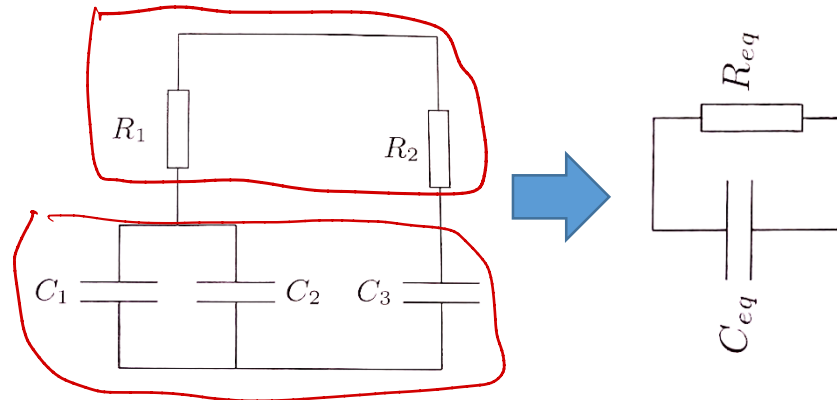
$$v_c(t) = v_c(\infty) + [v_c(0^+) - v_c(\infty)] e^{-t/\tau}$$

$$\begin{matrix} \uparrow & \downarrow \\ -20V & 50V \end{matrix}$$

$$\Rightarrow v_c(t) = -20 + [50 - (-20)] e^{-\frac{t}{RC}}$$

Example 2 (non-trivial boundary conditions)

- At $t = 0$, the switch is thrown to the right



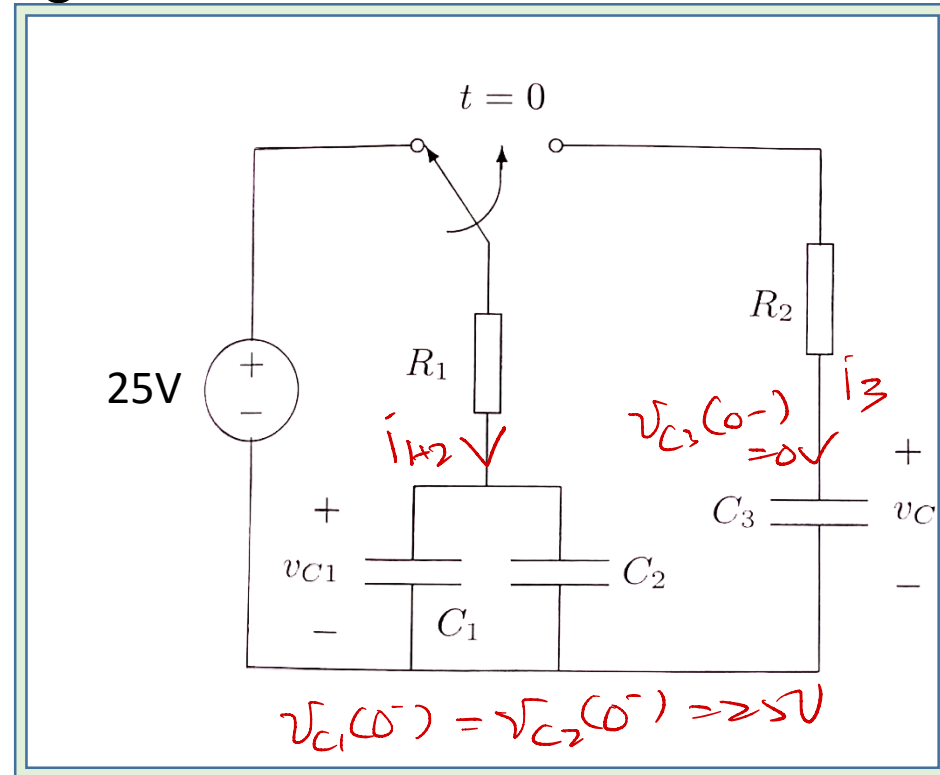
→ Find equivalent resistance:

$$R_{eq} = R_1 + R_2$$

→ Find equivalent capacitance

$$\frac{1}{C_{eq}} = \frac{1}{C_1 + C_2} + \frac{1}{C_3} \rightarrow$$

$$C_{eq} = \frac{C_3(C_1 + C_2)}{(C_1 + C_2 + C_3)}$$



$$\tau = R_{eq} C_{eq}$$

Example 2 (non-trivial boundary conditions)

- Time constant $\tau = \frac{C_3(C_1+C_2)(R_1+R_2)}{C_1+C_2+C_3}$

- Initial values are

$$v_{c_1}(0^+) = 25, \quad v_{c_2}(0^+) = 25, \quad v_{c_3}(0^+) = 0$$

- To find the final values after the switch throw to right

$$(C_1 + C_2) \frac{dv_{c_1}}{dt} + C_3 \frac{dv_{c_3}}{dt} = 0$$

$$\Rightarrow (C_1 + C_2)v_{c_1}(t) + C_3v_{c_3}(t) = \text{constant} \quad \text{for all } t \geq 0$$

- $v_{c_1}(0^+) = 25$ and $v_{c_3}(0^+) = 0 \Rightarrow 25(C_1 + C_2) = \text{constant}$

$$\Rightarrow (C_1 + C_2)v_{c_1}(t) + C_3v_{c_3}(t) = 25(C_1 + C_2)$$

- The final values must satisfy $v_{c_1}(\infty) = v_{c_3}(\infty) \Rightarrow$

$$v_{c_1}(\infty) = v_{c_2}(\infty) = v_{c_3}(\infty) = \frac{25(C_1 + C_2)}{C_1 + C_2 + C_3}$$