

**City University of Hong Kong**  
**EE2104 Introduction to Electromagnetics**  
**Semester A 2022/2023**  
**Final Exam Solution Keys**

Question 1

Wavelength 5 mm / 2.5 mm  
 Propagation towards  $-x$ -direction.

Question 2

Electrons; 2.1/ 4.3;  $2.5 \times 10^{12}$ ; space/position;  
 21 V/ 4.2 V; zero; magnetic; electromagnetic

Question 3

$$\rho = \sqrt{3^2 + 4^2} = 5$$

$$\cos \phi = \frac{3}{5} \Rightarrow \phi = 36.9^\circ / 64.4^\circ$$

$$z = 10 / 20$$

$$\hat{\rho} = (4\hat{x} + 3\hat{y})/5 = \frac{4}{5}\hat{x} + \frac{3}{5}\hat{y}$$

$$\hat{\phi} = -\frac{3}{5}\hat{x} - \frac{4}{5}\hat{y}$$

$$E = \frac{1}{4\pi\epsilon_0} \frac{4 \text{ mC}}{\rho^2 + z^2} = 0.288 \text{ MV/m} \_ / 0.106$$

Question 4

(a) Based on  $\nabla \cdot \mathbf{B} = 0$ ;

$$\nabla \cdot \mathbf{B} = \frac{1}{\rho} \left[ \frac{d}{d\left(\frac{\rho}{b}\right)} \left( \frac{\rho}{b} \frac{2b^3}{\rho^3} \right) + 0 \right] b^2 = -4b^3/\rho^4$$

implies  $b = 0$ .

(b)

$$\nabla \times \mathbf{B} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ \frac{b^3}{\rho^3} & 4 \frac{\rho^{3/2}}{a^{1/2}} & 0 \end{vmatrix} = \frac{4}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\rho^{3/2}}{a^{1/2}} \right) \hat{z} = \frac{6}{\sqrt{a\rho}} \hat{z}$$

at any point.

### Question 5

By symmetry, the field must point along the  $+x$ -direction.

(a) From the diagram, there is an angle  $\alpha$  with  $\tan \alpha = 0.8/1.5$

The magnitude of the field is simply:

$$\frac{3 \mu\text{C}}{4\pi\epsilon_0(1.5 \text{ m})^2} + \frac{4 \mu\text{C} (2\cos\alpha)}{4\pi\epsilon_0(1.5 \text{ m} / \cos \alpha)^2} = \frac{10^{-6}}{4\pi\epsilon_0(1.5 \text{ m})^2} [3 + 8 \cos^3 \alpha]$$

which equals 33.9 kV/m.

### Question 6

Similar to the example in the Chapter 7 (page 12):

The Laplace's equation can be applied.

$$\nabla^2 V = 0$$

Here the voltage depends only on  $x$ :

$$\frac{\partial^2}{\partial x^2} V = 0$$

$$V(x) = Ax + B$$

for some constants  $A$  and  $B$ .

From the boundary conditions:

$$V(0) = 5 \text{ kV} \Rightarrow B = 5 \text{ kV}$$

$$V(L) = 0 \Rightarrow A = -10 \text{ kV/m}$$

$$\therefore V(x) = (-10x + 5) \times 10^3$$

### Question 7

(a) The magnetic field is inversely proportional to the distance:

$$\mathbf{B} = \frac{-\mu_0 I_0}{2\pi x} \hat{\mathbf{z}}$$

where the direction follows the right-hand rule.

The field lines perpendicular to the drawing enters/exits the page for positive/negative  $x$ . Their density reduces as their distance increase from the  $y$ -axis

(b) The magnetic flux is a two-dimensional integration of the  $B$ -field:

$$\begin{aligned} \psi &= \int_{Sq.} \mathbf{B} \cdot d\mathbf{S} \\ \psi &= \frac{2\mu_0 I_0}{2\pi} \left[ \int_b^{b+a/\sqrt{2}} \frac{x-b}{x} dx + \int_{b+a/\sqrt{2}}^{b+\sqrt{2}a} \frac{(b+\sqrt{2}a)-x}{x} dx \right] \\ &= \frac{\mu_0 I_0}{\pi} \left[ \frac{a}{\sqrt{2}} + b \ln \frac{b}{b+\frac{a}{\sqrt{2}}} - \frac{a}{\sqrt{2}} + (b+\sqrt{2}a) \ln \frac{b+\sqrt{2}a}{b+\frac{a}{\sqrt{2}}} \right] \\ &= \frac{\mu_0 I_0}{\pi} \left[ b \ln \frac{b}{b+a/\sqrt{2}} + (b+\sqrt{2}a) \ln \frac{b+\sqrt{2}a}{b+a/\sqrt{2}} \right] \end{aligned}$$

(c) By Faraday's law, the induced voltage is:

$$\begin{aligned} V_0 &= -\frac{d\psi}{dt} = \frac{\mu_0}{\pi} \left[ b \ln \frac{b}{b+\frac{a}{\sqrt{2}}} + (b+\sqrt{2}a) \ln \frac{b+\sqrt{2}a}{b+\frac{a}{\sqrt{2}}} \right] I_{pk} (2\pi f) \sin 2\pi f t \\ V_{pk} &= +2\mu_0 f I_{pk} \left[ b \ln \frac{b}{b+\frac{a}{\sqrt{2}}} + (b+\sqrt{2}a) \ln \frac{b+\sqrt{2}a}{b+\frac{a}{\sqrt{2}}} \right] \\ &= 2.34 \times 10^{-7} \text{ (V)} \end{aligned}$$

### Question 8

(a) By symmetry,  
i-ii)

$\mathbf{D}$  points along the radial direction and

It depends only on  $\rho$ .

Thus,  $\mathbf{D}(\rho, \phi, z) = D(\rho)\hat{\rho}$

iii) For Gaussian surface, consider a cylinder of length  $l$  with radius  $\rho$  centered along the  $z$ -axis. From Gauss's law, we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q_{enc}$$

where  $Q_{enc}$  is the enclosed charge within the surface.

On the curved surface of  $S$ :

$$\int_{\phi=0}^{2\pi} \int_{z=0}^l (D(\rho)\hat{\rho}) \cdot (\rho d\phi dz \hat{\rho}) = Q_0 l / L$$

$$D(\rho)\rho \int_{z=0}^l dz \int_{\phi=0}^{2\pi} d\phi = \frac{Q_0 l}{L}$$

$$D(\rho) = \frac{Q_0}{2\pi L \rho}$$

iv) Therefore, the electric field

$$\mathbf{E} = \begin{cases} \frac{Q_0}{2\pi\epsilon_0\epsilon_r L \rho} \hat{\rho} & \text{in dielectric} \\ \frac{Q_0}{2\pi\epsilon_0 L \rho} \hat{\rho} & \text{in air} \end{cases}$$

For plotting the magnitude of  $\mathbf{E}$ ,

there is a general reduction as inversely proportional to  $\rho$  and

there is a discontinuous increase by  $\epsilon_r$  times at the dielectric-air interface.

v) Since the charge stored is proportional to the voltage, the charge stored would be increased to  $3Q_0$ .

(b) By cylindrical symmetry,  
i-ii)

$\mathbf{H}$  points along  $\hat{\phi}$  and

It depends only on  $\rho$ .

Thus,  $\mathbf{H}(\rho, \phi, z) = H(\rho)\hat{\phi}$

iii) Consider a circular loop  $C$  of radius  $\rho$  centered at the origin on the  $xy$ -plane.  
The Ampere's law

$$\oint_C \mathbf{H} \cdot d\mathbf{L} = I_0$$

$$\oint_C H(\rho)(\hat{\phi} \cdot \hat{\phi})dL = I_0$$

$$H(\rho) \oint_L dL = I_0$$

$$H(\rho) = \frac{I_0}{2\pi\rho}$$

iv) Therefore, the  $\mathbf{B}$ -field is

$$\mathbf{B} = \begin{cases} \frac{\mu_0\mu_r I_0}{2\pi\rho} \hat{\phi} & \text{in the dielectric} \\ \frac{\mu_0 I_0}{2\pi\rho} \hat{\phi} & \text{in air} \end{cases}$$

For plotting the magnitude of  $\mathbf{B}$ ,

there is a general reduction as inversely proportional to  $\rho$  and

there is a discontinuous increase by  $\mu_r$  times at the dielectric-air interface.