

MA2001 Solution to Take-Home Assignment 3

$$1. \quad u = \frac{y}{x^2}, \quad v = \frac{x}{y^2}$$

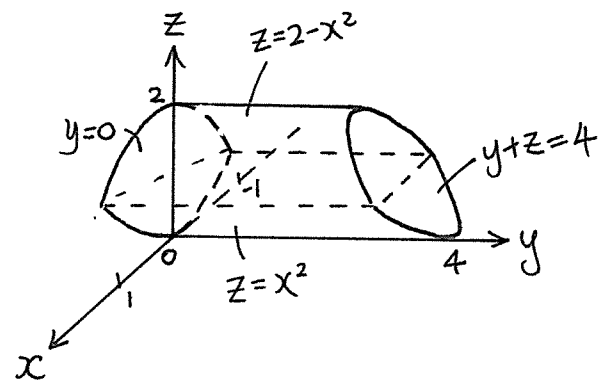
$$R: \begin{cases} y=x^2 \Rightarrow u=1 \\ y=2x^2 \Rightarrow u=2 \\ x=y^2 \Rightarrow v=1 \\ x=4y^2 \Rightarrow v=4 \end{cases}$$

$$\begin{aligned} J &= \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}} = \frac{1}{\begin{vmatrix} \frac{-2y}{x^3} & \frac{1}{x^2} \\ \frac{1}{y^2} & \frac{-2x}{y^3} \end{vmatrix}} \\ &= \frac{1}{\left(\frac{-2y}{x^3}\right)\left(\frac{-2x}{y^3}\right) - \left(\frac{1}{x^2}\right)\left(\frac{1}{y^2}\right)} = \frac{1}{\frac{4}{x^2y^2} - \frac{1}{x^2y^2}} = \frac{x^2y^2}{3} = \frac{1}{3u^2v^2} \end{aligned}$$

$$\Rightarrow dx dy = |J| du dv = \frac{1}{3u^2v^2} du dv$$

$$\begin{aligned} \therefore \iint_R \left(\frac{y^2}{x^4} + \frac{x^2}{y^4} \right) dx dy &= \int_1^2 \left(\int_1^4 (u^2 + v^2) \cdot \frac{1}{3u^2v^2} dv \right) du \\ &= \frac{1}{3} \int_1^2 \left(\int_1^4 \left(\frac{1}{v^2} + \frac{1}{u^2} \right) dv \right) du \\ &= \frac{1}{3} \int_1^2 \left[-\frac{1}{v} + \frac{v}{u^2} \right]_1^4 du \\ &= \frac{1}{3} \int_1^2 \left(-\frac{1}{4} + \frac{4}{u^2} \right) - \left(-1 + \frac{1}{u^2} \right) du \\ &= \frac{1}{3} \int_1^2 \left(\frac{3}{4} + \frac{3}{u^2} \right) du \\ &= \frac{1}{3} \left[\frac{3}{4}u - \frac{3}{u} \right]_1^2 \\ &= \frac{1}{3} \left[\left(\frac{3}{2} - \frac{3}{2} \right) - \left(\frac{3}{4} - 3 \right) \right] \\ &= \frac{3}{4} \end{aligned}$$

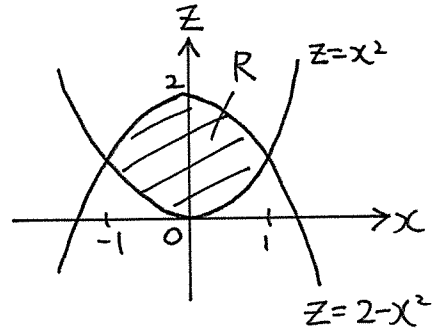
$$\begin{aligned}
 2(a) \quad \begin{cases} z = x^2 \\ z = 2 - x^2 \end{cases} &\Rightarrow x^2 = 2 - x^2 \\
 &\Rightarrow 2x^2 = 2 \\
 &\Rightarrow x = \pm 1
 \end{aligned}$$



Let R be the projection of D onto xz -plane.

$$R: \begin{cases} -1 \leq x \leq 1 \\ x^2 \leq z \leq 2 - x^2 \end{cases}$$

Then $0 \leq y \leq 4 - z$.



$$\begin{aligned}
 \iiint_D x^2 \, dx \, dy \, dz &= \int_{-1}^1 \left[\int_{x^2}^{2-x^2} \left(\int_0^{4-z} x^2 \, dy \right) dz \right] dx \\
 &= \int_{-1}^1 \left[\int_{x^2}^{2-x^2} x^2 (4-z) \, dz \right] dx \\
 &= \int_{-1}^1 x^2 \left[4z - \frac{z^2}{2} \right]_{x^2}^{2-x^2} dx \\
 &= \int_{-1}^1 (6x^2 - 6x^4) \, dx \\
 &= \left[2x^3 - \frac{6x^5}{5} \right]_{-1}^1 \\
 &= \left(2 - \frac{6}{5} \right) - \left(-2 - \left(-\frac{6}{5} \right) \right) \\
 &= \frac{8}{5}
 \end{aligned}$$

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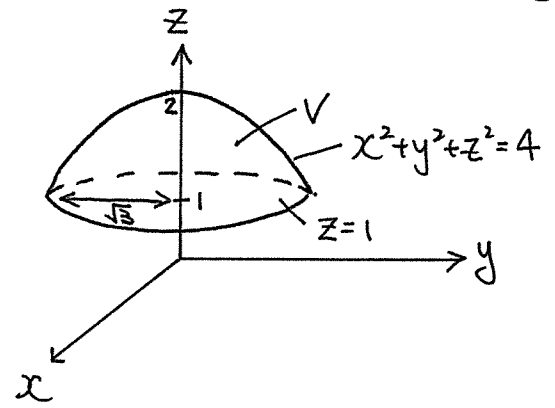
2(b) In cylindrical coordinates,

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases}$$

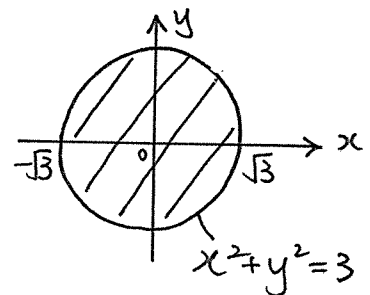
where

$$\begin{cases} 0 \leq r \leq \sqrt{3} \\ 0 \leq \theta \leq 2\pi \\ 1 \leq z \leq \sqrt{4-x^2-y^2} = \sqrt{4-r^2} \end{cases}$$

$$\begin{aligned} & \iiint_V \frac{1}{\sqrt{4-x^2-y^2}} dx dy dz \\ &= \int_0^{\sqrt{3}} \left[\int_0^{2\pi} \left(\int_1^{\sqrt{4-r^2}} \frac{1}{\sqrt{4-r^2}} \cdot r dz \right) d\theta \right] dr \\ &= \int_0^{\sqrt{3}} \left[\int_0^{2\pi} \frac{r}{\sqrt{4-r^2}} (\sqrt{4-r^2} - 1) d\theta \right] dr \\ &= \int_0^{\sqrt{3}} 2\pi \left(r - \frac{r}{\sqrt{4-r^2}} \right) dr \\ &= 2\pi \left\{ \left[\frac{r^2}{2} \right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} (4-r^2)^{-\frac{1}{2}} \cdot (-\frac{1}{2}) d(4-r^2) \right\} \\ &= 2\pi \left\{ \frac{3}{2} + \frac{1}{2} \left[\frac{(4-r^2)^{\frac{1}{2}}}{\frac{1}{2}} \right]_0^{\sqrt{3}} \right\} \\ &= 2\pi \left[\frac{3}{2} + (1-2) \right] \\ &= \pi \end{aligned}$$



Projection of V onto xy -plane:



$$\begin{cases} x^2 + y^2 + z^2 = 4 \\ z = 1 \end{cases} \Rightarrow x^2 + y^2 = 3$$

$$3. \quad \vec{F} = (ax \cos z - x^2) \vec{i} + (bz - 2y) \vec{j} + (4y + z - x^2 \sin z) \vec{k}$$

$$\begin{aligned} (a) \quad \text{curl } \vec{F} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ax \cos z - x^2 & bz - 2y & 4y + z - x^2 \sin z \end{vmatrix} \\ &= \vec{i} (4 - b) - \vec{j} [-2x \sin z - (-ax \sin z)] + \vec{k} (0 - 0) \\ &= (4 - b) \vec{i} + (2 - a)x \sin z \vec{j} \end{aligned}$$

$\vec{F}$ is irrotational if $\text{curl } \vec{F} = \vec{0}$

$$\Rightarrow 4 - b = 0 \quad \& \quad 2 - a = 0$$

$$\Rightarrow b = 4 \quad \& \quad a = 2$$

$$(b) \quad \nabla \varphi = \vec{F}$$

$$\therefore \begin{cases} \frac{\partial \varphi}{\partial x} = 2x \cos z - x^2 & \text{--- ①} \\ \frac{\partial \varphi}{\partial y} = 4z - 2y & \text{--- ②} \\ \frac{\partial \varphi}{\partial z} = 4y + z - x^2 \sin z & \text{--- ③} \end{cases}$$

$$\text{Int. ① w.r.t } x: \quad \varphi(x, y, z) = x^2 \cos z - \frac{x^3}{3} + g(y, z)$$

$$\text{Partially differentiate w.r.t. } y: \quad \frac{\partial \varphi}{\partial y} = \frac{\partial g}{\partial y}$$

$$\text{Compare with ②:} \quad \frac{\partial g}{\partial y} = 4z - 2y$$

$$\Rightarrow g(y, z) = 4yz - y^2 + h(z)$$

$$\therefore \varphi(x, y, z) = x^2 \cos z - \frac{x^3}{3} + 4yz - y^2 + h(z)$$

$$\text{Partially diff. w.r.t } z: \quad \frac{\partial \varphi}{\partial z} = -x^2 \sin z + 4y + h'(z)$$

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Compare with ③: $4y + z - x^2 \sin z = -x^2 \sin z + 4y + h'(z)$

$$\Rightarrow h'(z) = z$$

$$\Rightarrow h(z) = \frac{z^2}{2} + C$$

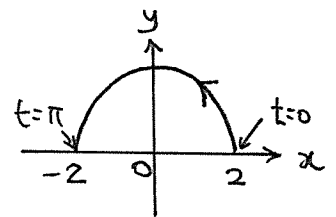
$$\therefore \varphi(x, y, z) = x^2 \cos z - \frac{x^3}{3} + 4yz - y^2 + \frac{z^2}{2} + C$$

where C is an arbitrary constant.

4. (a) $C: y = \sqrt{4-x^2}$ from $(2, 0)$ to $(-2, 0)$.

Parameterize C as

$$C: \begin{cases} x = 2 \cos t \\ y = 2 \sin t \end{cases}, \quad 0 \leq t \leq \pi$$



Then $x'(t) = -2 \sin t$, $y'(t) = 2 \cos t$.

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2} dt = \sqrt{(-2 \sin t)^2 + (2 \cos t)^2} dt = 2 dt$$

$$\begin{aligned} \text{mass} &= \int_C \rho(x, y) ds \\ &= \int_0^\pi (2 \sin t)^2 \cdot 2 dt \\ &= 4 \int_0^\pi (1 - \cos 2t) dt \\ &= 4 \left[t - \frac{\sin 2t}{2} \right]_0^\pi \\ &= 4 (\pi - 0) \\ &= 4\pi \end{aligned}$$

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4(b) Parameterize C as

$$C: \begin{cases} x = t \\ y = t^3 \\ z = t^2 \end{cases}, \quad 1 \leq t \leq 2$$

$$\frac{d\vec{r}}{dt} = x'(t) \vec{i} + y'(t) \vec{j} + z'(t) \vec{k} = \vec{i} + 3t^2 \vec{j} + 2t \vec{k}$$

$$\therefore \vec{F} = z \vec{i} - x \vec{j} + y \vec{k} = t^2 \vec{i} - t \vec{j} + t^3 \vec{k}$$

$$\text{Work done} = \int_C \vec{F} \cdot d\vec{r}$$

$$= \int_1^2 (t^2 \vec{i} - t \vec{j} + t^3 \vec{k}) \cdot (\vec{i} + 3t^2 \vec{j} + 2t \vec{k}) dt$$

$$= \int_1^2 (t^2 - 3t^3 + 2t^4) dt$$

$$= \left[\frac{t^3}{3} - \frac{3t^4}{4} + \frac{2t^5}{5} \right]_1^2$$

$$= \left(\frac{8}{3} - 12 + \frac{64}{5} \right) - \left(\frac{1}{3} - \frac{3}{4} + \frac{2}{5} \right)$$

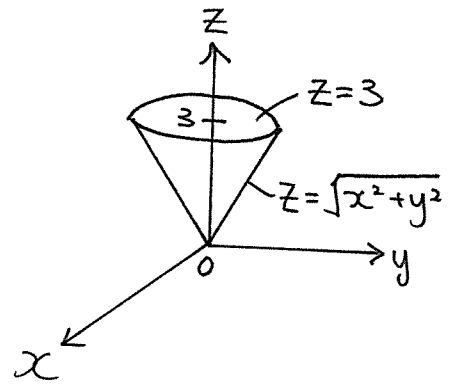
$$= \frac{209}{60}$$

5(a) $S: z = \sqrt{x^2 + y^2} \Rightarrow z - \sqrt{x^2 + y^2} = 0$

Let $\phi(x, y, z) = z - \sqrt{x^2 + y^2}$.

Then $\phi_x = \frac{-x}{\sqrt{x^2 + y^2}}$,

$\phi_y = \frac{-y}{\sqrt{x^2 + y^2}}$, $\phi_z = 1$.



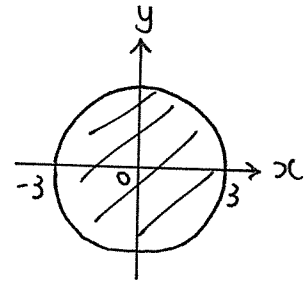
Project S onto xy -plane.

$$dS = \frac{\sqrt{\phi_x^2 + \phi_y^2 + \phi_z^2}}{|\phi_z|} dx dy = \sqrt{\frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2} + 1} dx dy$$

$$= \sqrt{2} dx dy$$

The projection of S onto xy -plane is

$\sigma_{xy}: x^2 + y^2 = 3^2$



In polar coordinates:

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases}, \text{ where } \begin{cases} 0 \leq r \leq 3 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\iint_S z dS = \iint_{\sigma_{xy}} \sqrt{x^2 + y^2} \cdot \sqrt{2} dx dy$$

$$= \sqrt{2} \int_0^{2\pi} \left(\int_0^3 r \cdot r dr \right) d\theta, \text{ using polar coord.}$$

$$= \sqrt{2} \int_0^{2\pi} \left[\frac{r^3}{3} \right]_0^3 d\theta$$

$$= 9\sqrt{2} \cdot 2\pi$$

$$= 18\sqrt{2} \pi$$

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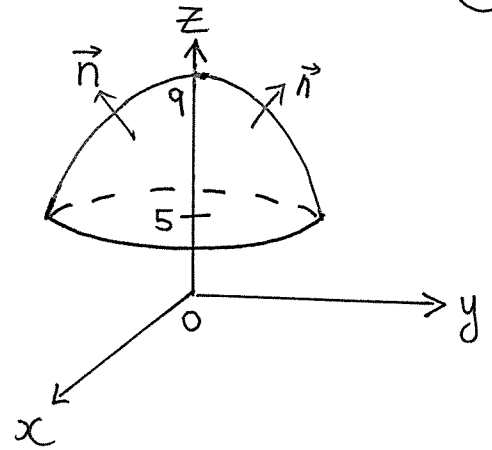
5(b) Let $\phi(x, y, z) = x^2 + y^2 + z - 9$.

Then $\phi_x = 2x$, $\phi_y = 2y$, $\phi_z = 1$.

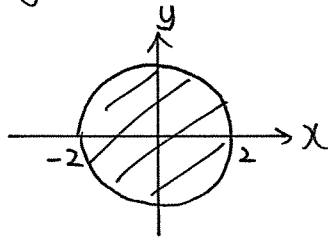
$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + \vec{k}.$$

The required normal to S is pointing in the direction of

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + \vec{k}. \quad (\text{The } \vec{k}\text{-component is positive})$$



Let σ_{xy} be the projection of S onto the xy -plane.



$$\begin{cases} z = 9 - x^2 - y^2 \\ z = 5 \end{cases} \Rightarrow x^2 + y^2 = 4$$

Using polar coordinates,

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \quad \text{where} \quad \begin{cases} 0 \leq r \leq 2 \\ 0 \leq \theta \leq 2\pi \end{cases}$$

$$\begin{aligned} \iint_S \vec{F} \cdot d\vec{S} &= \iint_{\sigma_{xy}} \frac{\vec{F} \cdot \nabla\phi}{|\phi_z|} dx dy \\ &= \iint_{\sigma_{xy}} \frac{(x\vec{i} + y\vec{j} + z\vec{k}) \cdot (2x\vec{i} + 2y\vec{j} + \vec{k})}{|1|} dx dy \\ &= \iint_{\sigma_{xy}} (2x^2 + 2y^2 + z) dx dy \\ &= \iint_{\sigma_{xy}} (2x^2 + 2y^2 + (9 - x^2 - y^2)) dx dy \\ &= \iint_{\sigma_{xy}} (9 + x^2 + y^2) dx dy \end{aligned}$$

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$$= \int_0^2 \left[\int_0^{2\pi} (9 + r^2) \cdot r \, d\theta \right] dr, \text{ using polar coordinates.}$$

$$= 2\pi \int_0^2 (9r + r^3) \, dr$$

$$= 2\pi \left[\frac{9r^2}{2} + \frac{r^4}{4} \right]_0^2$$

$$= 2\pi (18 + 4 - 0)$$

$$= 44\pi$$