

EF2452: Mathematics for Economics and Finance

Lecture 6

Part I of II

Department of Economics and Finance
City University of Hong Kong
Sem A, 2024-2025



Main Topics

- Function of two variables
- Partial derivatives
- Chain rule
- Implicit function; derivative of an implicit function
- Contours and isoquants
- Homogeneous functions and Euler's theorem

Functions of Several Variables

- Previously, we discussed functions of one variable: $f = f(x)$. Now, we consider functions of several variables.
- For example, a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ of two variables assigns to each $(x, y) \in \mathbb{R}^2$ a value $f(x, y) \in \mathbb{R}$.
- Examples:

$$f(x, y) = x^3 + y^3$$

$$f(x, y) = x^2y + 7xy + y^2$$

- A typical example from economics is the production function of a firm, $q = q(k, l)$. Here, l denotes the amount of labor, k denotes the amount of capital, and q is the quantity of the product that the firm can produce using labor and capital.

Partial Derivatives

- Consider $f = f(x, y)$. The partial derivative of f , with respect to the first variable, is obtained by treating the second variable as a constant and taking the derivative with respect to the first variable.
- For example, suppose $f(x, y) = x^2y + 7xy + y^2$.
- Treating y as a constant, differentiate f w.r.t. x to get

$$\frac{\partial f}{\partial x} = f_1(x, y) = 2xy + 7y$$

Partial Derivatives

- Similarly, the partial derivative of f , with respect to the second variable, is obtained by treating the first variable as a constant and taking the derivative with respect to the second variable.
- For example, suppose $f(x, y) = x^2y + 7xy + y^2$.
- Treating x as a constant, differentiate f w.r.t. y to get

$$\frac{\partial f}{\partial y} = f_2(x, y) = x^2 + 7x + 2y$$

Second Partial Derivatives

- Since f_1 and f_2 are themselves functions of (x, y) , we can further define their partial derivatives w.r.t x and y .

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = f_{11}(x, y) = \frac{\partial}{\partial x} (2xy + 7y) = 2y$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = f_{12}(x, y) = \frac{\partial}{\partial y} (2xy + 7y) = 2x + 7$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = f_{21}(x, y) = \frac{\partial}{\partial x} (x^2 + 7x + 2y) = 2x + 7$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = f_{22}(x, y) = \frac{\partial}{\partial y} (x^2 + 7x + 2y) = 2$$

Cobb-Douglas Production Function

- A function often used in economics is the Cobb-Douglas production function

$$q(k, l) = Ak^{\alpha}l^{\beta}$$

- A, α, β are constants that can be chosen to reflect the characteristics of a particular situation. For example, we might consider a specific firm with production function

$$q(k, l) = 6k^{1/4}l^{3/4}$$

with $A = 6, \alpha = 1/4, \beta = 3/4$.

Marginal Products

- For a given production function, the partial derivative $\frac{\partial q}{\partial k}$ measures the rate of change of output with respect to capital when labor is kept fixed: this is called the **marginal product of capital**. It gives the change in production when one additional unit of capital is used, keeping labor fixed.
- For a given production function, the partial derivative $\frac{\partial q}{\partial l}$ measures the rate of change of output with respect to labor when capital is kept fixed: this is called the **marginal product of labor**. It gives the change in production when one additional unit of labor is used, keeping capital fixed.

Cobb-Douglas Production Function

- For the Cobb-Douglas production function $q(k, l) = Ak^\alpha l^\beta$, we have the following partial derivatives:

$$q_1(k, l) = A\alpha k^{\alpha-1} l^\beta \qquad q_2(k, l) = A\beta k^\alpha l^{\beta-1}$$

- The second partial derivatives are

$$q_{11}(k, l) = A\alpha(\alpha - 1)k^{\alpha-2} l^\beta \qquad q_{12}(k, l) = A\alpha\beta k^{\alpha-1} l^{\beta-1}$$

$$q_{21}(k, l) = A\alpha\beta k^{\alpha-1} l^{\beta-1} \qquad q_{22}(k, l) = A\beta(\beta - 1)k^\alpha l^{\beta-2}$$

Chain Rule

- Consider the following function f of two variables (x, y) , both of which are themselves functions of t :

$$F(t) = f(x(t), y(t))$$

- Chain Rule states that

$$\frac{dF}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

Chain Rule: Example

- Suppose

$$f(x, y) = xy, \quad x(t) = 3 + 4t, \quad y(t) = 6 - 2t$$

- Then

$$\begin{aligned} \frac{dF}{dt} &= \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \\ &= (y)(4) + (x)(-2) = 4y - 2x = 4(6 - 2t) - 2(3 + 4t) \\ &= 18 - 16t \end{aligned}$$

- In this example, you can also directly differentiate F w.r.t. t :

$$F(t) = (3 + 4t)(6 - 2t) = 18 + 18t - 8t^2$$

$$\frac{dF}{dt} = 18 - 16t$$

Diminishing Marginal Product

- Suppose that a firm has a production function

$$q(k, l) = Ak^{\alpha}l^{1-\alpha}$$

where $A > 0$ and $0 < \alpha < 1$.

- The marginal product of labor is

$$\frac{\partial q}{\partial l} = A(1 - \alpha)k^{\alpha}l^{1-\alpha-1} = A(1 - \alpha)\left(\frac{k}{l}\right)^{\alpha} > 0$$

$$\begin{aligned}\frac{\partial^2 q}{\partial l^2} &= \frac{\partial}{\partial l}\left(\frac{\partial q}{\partial l}\right) = \frac{\partial}{\partial l}\left(A(1-\alpha)k^{\alpha}l^{-\alpha}\right) = A(1-\alpha)(-\alpha)k^{\alpha}l^{-\alpha-1} \\ &\implies \frac{\partial^2 q}{\partial l^2} = -A\alpha(1 - \alpha)k^{\alpha}l^{-\alpha-1}\end{aligned}$$

Since $A > 0, \alpha > 0, 1 - \alpha > 0, \frac{\partial^2 q}{\partial l^2} < 0$. Therefore, there is diminishing marginal product of labor: as labor (or size of workforce) increases in size, production increases too, but at a slower rate.

Implicit Function

- Suppose we are given $g(x, y) = c$, for example, like the following equation

$$g(x, y) = x^2 - 4y = 0$$

- Then, you can write the above as

$$y = \frac{x^2}{4}$$

- We say that $g(x, y) = c$ defines y implicitly as a function of x .

Derivative of Implicit Function

- Consider the equation $g(x, y) = c$ where c is a constant. It defines y is a function of x , i.e., $y(x)$ (often termed as an implicit function of x). Then, we can write

$$g(x, y(x)) = c$$

- Taking differentiation of both sides w.r.t. x , we get (use chain rule)

$$\frac{dg}{dx} = \frac{\partial g}{\partial x} \frac{dx}{dx} + \frac{\partial g}{\partial y} \frac{dy}{dx} = 0$$

$$\implies \frac{\partial g}{\partial x} + \frac{\partial g}{\partial y} \frac{dy}{dx} = 0$$

$$\implies \frac{dy}{dx} = -\frac{\partial g / \partial x}{\partial g / \partial y} = -\frac{g_1}{g_2}$$

Derivative of Implicit Function

- Consider the following equation

$$g(x, y) = x^2 + 3xy + 2y^2 = 48$$

- Here we have

$$g_1 = \frac{\partial g}{\partial x} = 2x + 3y$$

$$g_2 = \frac{\partial g}{\partial y} = 3x + 4y$$

$$\frac{dy}{dx} = -\frac{g_1}{g_2} = -\frac{2x + 3y}{3x + 4y}$$

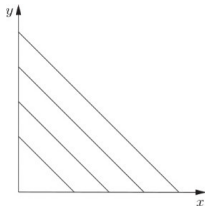
Contour

- For any function f of two variables we define a **contour** to be a set of the form

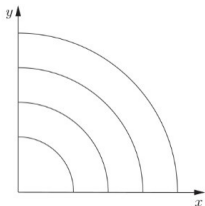
$$\{(x, y) | f(x, y) = c\}$$

for some constant c .

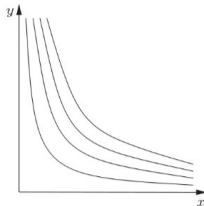
- The following graphs below plot typical contours for the functions $x + y$, $x^2 + y^2$ and xy in $\mathbb{R}_+^2$.



(a)



(b)



(c)

Slope of the Contour

- Suppose we want to find the slope of the contour $f(x, y) = c$ at the point (a, b) .
- Noting that the equation $f(x, y) = c$ implicitly defines y as a function of x , and the graph of this function is the contour, the slope is given by

$$\frac{dy}{dx} = -\frac{f_1}{f_2}$$

where the partial derivatives are evaluated at the point (a, b) .

Production Function and Isoquants

- When the function under consideration is a production function $q(k, l)$, the contours are called **isoquants** ('iso-quant' means 'same quantity').
- The points (k, l) lying on a particular isoquant correspond to a set of inputs of capital and labor resulting in a given fixed output.
- An example: consider $q(k, l) = 6k^{1/2}l^{1/4}$. The slope of an isoquant of this production function is given by

$$\frac{dl}{dk} = -\frac{q_1}{q_2} = -\frac{\partial q / \partial k}{\partial q / \partial l} = -\frac{6(1/2)k^{-1/2}l^{1/4}}{6(1/4)k^{1/2}l^{-3/4}} = -\frac{2l}{k}$$

Homogeneous of degree D

- If a function h is such that

$$h(cx, cy) = c^D h(x, y),$$

then we say that h is **homogeneous of degree D** .

- Example: $h(x, y) = 3x^2y + 7xy^2$. Then

$$h(cx, cy) = 3(cx)^2(cy) + 7(cx)(cy)^2 = c^3(3x^2y + 7xy^2) = c^3 h(x, y)$$

Therefore, this function is homogeneous of degree 3.

Returns to Scale

- The notion of homogeneous function is related to the idea of ‘returns to scale’ in economics.
- In the case of a production function, we say that there are **constant returns to scale** (CRS) if a proportional increase in k and l results in the same proportional increase in $q(k, l)$; that is if

$$q(ck, cl) = cq(k, l)$$

- We have increasing returns to scale (IRS) if

$$q(ck, cl) > cq(k, l)$$

- We have decreasing returns to scale (DRS) if

$$q(ck, cl) < cq(k, l)$$

Returns to Scale: Cobb-Douglas production function

- For the Cobb-Douglas production function $q(k, l) = Ak^\alpha l^\beta$, we have

$$q(ck, cl) = A(ck)^\alpha (cl)^\beta = c^{\alpha+\beta} (Ak^\alpha l^\beta) = c^{\alpha+\beta} q(k, l)$$

- Depending on the value of $\alpha + \beta$ (which is the degree of homogeneity for this Cobb-Douglas function), we will have either constant, increasing, or decreasing returns to scale.
- $\alpha + \beta = 1$: CRS
- $\alpha + \beta > 1$: IRS
- $\alpha + \beta < 1$: DRS

Euler's Theorem

- For any homogeneous function $h = h(x, y)$ of degree D , we have

$$x \frac{\partial h}{\partial x} + y \frac{\partial h}{\partial y} = Dh$$

References

Following chapters of the textbook:

11 Partial Derivatives

12 Applications of Partial Derivatives