

EE2104 Introduction to Electromagnetics
Assignment 3 Solution

Deadline (for uploading): 5:00 pm, 2 Nov 2021
Please put down your name and university ID.

1. Consider a copper wire with a cross-sectional area $S = 2 \text{ mm}^2$. Copper has $n = 10^{29}$ electrons in every 1 m^3 . It carries a current of $I = 4 \text{ A}$ for driving a light bulb $L = 50 \text{ m}$ away from a voltage source. The conductivity is $\sigma = 10^8 \Omega^{-1}\text{m}^{-1}$.
- (a) Find the time needed for an electron to drift from the source to the light bulb.
(b) Find the power dissipated in every 1 m^3 of the copper.

Solution:

- (a) The velocity of electron drifting is

$$V_{\text{drift}} = \frac{I}{Snq} = 125 \text{ (}\mu\text{m/s)}$$

The time needed is

$$\text{Time} = \frac{L}{V_{\text{drift}}} = \frac{50 \text{ (m)}}{125 \text{ (}\mu\text{m/s)}} = 4 \times 10^5 \text{ (s)}$$

- (b) We know that power

$$P = IV = IEL = \frac{I^2 L}{\sigma S}$$

Thus, the power dissipated in every 1 m^3 of the copper is

$$\frac{P}{LS} = \frac{I^2}{\sigma S^2} = 40000 \text{ (W/m}^3\text{)}$$

2. In a region of permittivity ϵ_0 , the potential field is given in cylindrical coordinates by

$$V(\rho, \phi, z) = V_0 \frac{\rho}{a} \cos \phi$$

where V_0 is a constant potential and a is a constant length.
Find the electric field.

Solution:

Calculating the gradient of V in spherical coordinates, we have

$$\nabla V = \frac{\partial V}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial V}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial V}{\partial z} \hat{\mathbf{a}}_z = \frac{V_0}{a} \cos \phi \hat{\mathbf{a}}_\rho - \frac{V_0}{a} \sin \phi \hat{\mathbf{a}}_\phi$$

Thus, the electric field is

$$\mathbf{E} = -\nabla V = -\frac{V_0}{a} \cos \phi \hat{\mathbf{a}}_\rho + \frac{V_0}{a} \sin \phi \hat{\mathbf{a}}_\phi$$

3. Given a field $\mathbf{G} = y\hat{\mathbf{x}} + x\hat{\mathbf{y}}$. There are points $A(1, 1)$ and $B(-1, 1)$.
 (a) Find

$$\int_{\text{Path } S} \mathbf{G} \cdot d\mathbf{l}$$

where the path S is a straight line from point A to point B .

- (b) Find

$$\int_{\text{Path } C} \mathbf{G} \cdot d\mathbf{l}$$

where the path C is the circular arc of radius $\sqrt{2}$ connecting point A and point B .

- (c) Is $\mathbf{G}$ conservative? Why?

Solution:

- (a) When the path is a straight line from point A to point B , we have $y = 1$ for the path.
 Since $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy$

$$\int_{\text{Path } S} \mathbf{G} \cdot d\mathbf{l} = \int_{\text{Path } S} (y\hat{\mathbf{x}} + x\hat{\mathbf{y}}) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy) = \int_{\text{Path } S} ydx + \int_{\text{Path } S} xdy$$

Thus, we have

$$\int_{\text{Path } S} \mathbf{G} \cdot d\mathbf{l} = \int_{x=1}^{-1} dx = -2$$

- (b) When the path is the circular arc connecting point A and point B , we have

$$x = \sqrt{2}\cos\varphi, \quad y = \sqrt{2}\sin\varphi \quad \text{for } \varphi: \frac{\pi}{4} \rightarrow \frac{3\pi}{4}$$

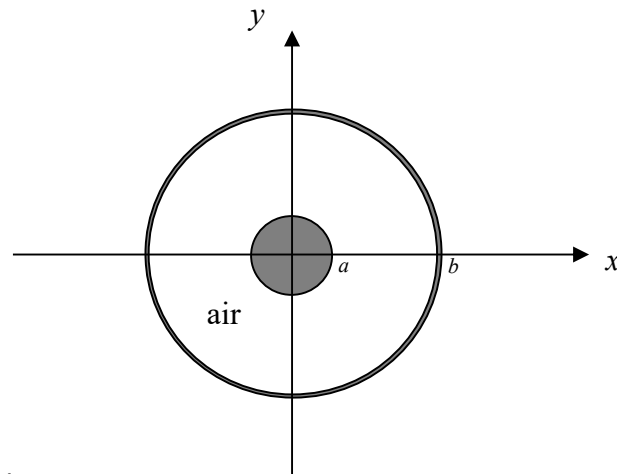
Therefore, $d\mathbf{l} = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy = -\sqrt{2}\sin\varphi d\varphi\hat{\mathbf{x}} + \sqrt{2}\cos\varphi d\varphi\hat{\mathbf{y}}$

Thus, we have

$$\int_{\text{Path } C} \mathbf{G} \cdot d\mathbf{l} = \int_{\varphi=\pi/4}^{3\pi/4} (2\cos^2\varphi - 2\sin^2\varphi)d\varphi = 2 \int_{\varphi=\pi/4}^{3\pi/4} \cos 2\varphi d\varphi = -2$$

- (c) $\mathbf{G}$ is conservative.
 Because the integration of $\mathbf{G}$ is independent on the path.

4. Consider a coaxial cable below consisting of cylinders of perfect conductors of length $l = 7$ m. In the cylindrical coordinate system, the inner conductor is at $0 < \rho < a$ and the outer conductor is at $\rho = b$, where $a = 3$ mm and $b = 8$ mm. The outer conductor is grounded. The region between the conductors is filled with air.



- Find the maximal $|\mathbf{E}|$ in the structure.
- Find the capacitance of the whole cable.
- Suggest one way to double the capacitance.

Solution:

- Inside the inner and outer conductor, the electric field is zero. For the electric field inside the dielectric, we can consider there is no leaking of the electric field and the charges in the inner/outer conductors are uniformly distributed. Because of the symmetric structure, the electric field can be expressed as

$$\mathbf{E}(\rho, \phi, z) = \mathbf{E}(\rho) = E(\rho)\hat{\rho}$$

Consider a cylindrical Gaussian surface S with a radius of ρ and a height of h , centered along z -axis and placed between the inner and outer conductor.

Using Gauss's law, we have

$$Q_{\text{enc}} = \oint_S \mathbf{D}(\rho, \phi, z) \cdot d\mathbf{S}$$

Thus,

$$Q_{\text{enc}} = \oint_S \epsilon_0 \mathbf{E}(\rho, \phi, z) \cdot d\mathbf{S} = \oint_S \epsilon_0 \mathbf{E}(\rho) \cdot d\mathbf{S}$$

Then we have

$$Q_{\text{enc}} = \int_{\text{Top Surf.}} \epsilon_0 \mathbf{E}(\rho) \cdot d\mathbf{S} + \int_{\text{Curved Surf.}} \epsilon_0 \mathbf{E}(\rho) \cdot d\mathbf{S} + \int_{\text{Bottom Surf.}} \epsilon_0 \mathbf{E}(\rho) \cdot d\mathbf{S}$$

Because $\mathbf{E}(\rho) = E(\rho)\hat{\rho}$, we have

$$\int_{\text{Top Surf.}} \epsilon_0 \mathbf{E}(\rho) \cdot d\mathbf{S} = \int_{\text{Bottom Surf.}} \epsilon_0 \mathbf{E}(\rho) \cdot d\mathbf{S} = 0$$

Thus,

$$Q_{\text{enc}} = \int_{\text{Curved Surf.}} \epsilon_0 E(\rho) (\hat{\rho} \cdot \hat{\rho}) dS$$

Therefore,

$$\frac{Q_{\text{enc}} h}{l} = 2\pi \rho h \epsilon_0 E(\rho) \Rightarrow E(\rho) = \frac{Q_{\text{enc}}}{2l\pi\epsilon_0} \frac{1}{\rho}$$

When $\rho = a$, we have

$$|E|_{\text{max}} = E_{\text{max}} = \frac{Q_{\text{enc}}}{2al\pi\epsilon_0}$$

(b) The potential difference between the conductors is

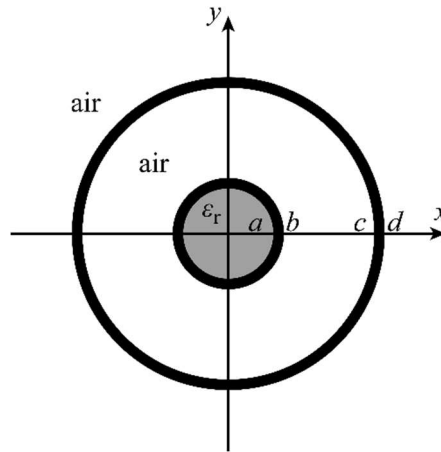
$$V = - \int_{\rho=b}^a \mathbf{E} \cdot d\mathbf{L} = - \int_{\rho=b}^a E(\rho) (\hat{\rho} \cdot \hat{\rho}) d\rho = - \int_{\rho=b}^a \frac{Q}{2l\pi\epsilon_0} \frac{1}{\rho} d\rho = \frac{Q}{2l\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

So the capacitance of the cable is

$$C = \frac{Q}{V} = \frac{2l\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} = 0.398 \text{ (nF)}$$

(c) To double the capacitance, we can double the length of the cable or fill the region between the conductors with a dielectric with relative permittivity $\epsilon_r = 2$.

5. A concentric spherical structure is placed at the origin of a spherical coordinate system (r, θ, φ) as shown by the figure below. The different layers contain the following materials: (i) inner conductor for $a < r < b$, (ii) outer conductor for $c < r < d$, (iii) dielectric with relative permittivity ϵ_r for $r < a$, and (iv) air for any other r . The inner conductor has potential $+V_0$ while the outer conductor is grounded.
- (a) Using Laplace equation, find the expressions for the potential V for $0 < r < a$ and $b < r < c$. Retain the constants of integration as unknowns.
- (b) Solve the unknown constants for V .
- (c) Find the electric field $\mathbf{E}$ for $b < r < c$.
- (d) Find the capacitance of the concentric conducting spheres.



Solution:

- (a) By symmetry, the electric potential only depends on r in the spherical coordinate system. Thus, we have $V(r, \theta, \varphi) = V(r)$.

Apply Laplace's equation for $0 < r < a$ first. Then we have

$$\nabla^2 V = \nabla \cdot \nabla V = 0 \Rightarrow \nabla \cdot \left[\frac{\partial V(r)}{\partial r} \hat{\mathbf{a}}_r \right] = 0$$

$$\frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} \left(r^2 \sin \theta \frac{\partial V}{\partial r} \right) \right] = 0$$

$$2r \sin \theta \frac{\partial V}{\partial r} + r^2 \sin \theta \frac{\partial^2 V}{\partial r^2} = 0$$

By letting $X = \frac{\partial V}{\partial r}$, we have

$$\frac{\partial X}{\partial r} = -\frac{2X}{r} \Rightarrow \ln(X) = -2 \ln(r) + A_0 \Rightarrow X = \frac{e^{A_0}}{r^2}$$

So we have

$$\frac{\partial V}{\partial r} = \frac{A}{r^2}, A = e^{A_0}$$

Thus,

$$V(r) = -\frac{A}{r} + B$$

where A_0 , A , and B are unknown constants.

For $b < r < c$, a similar derivation can be applied. Therefore, we have

$$V(r) = -\frac{C}{r} + D$$

where C and D are unknown constants.

So we have

$$V(r) = -\frac{A}{r} + B, \quad \text{for } 0 < r < a$$

$$V(r) = -\frac{C}{r} + D, \quad \text{for } b < r < c$$

(b) For $0 < r < a$, $V(r) = -\frac{A}{r} + B$.

By symmetry, $\mathbf{E}(r = 0) = 0$. Since we know $\mathbf{E} = -\nabla V$, we have

$$\frac{\partial V}{\partial r}(r = 0) = \frac{A}{r^2}(r = 0) = 0$$

Thus, $A = 0, B = V_0$.

By boundary condition, $V(r = a) = V_0$. So $V(r) = V_0$, for $0 < r < a$.

For $b < r < c$, $V(r) = -\frac{C}{r} + D$.

By boundary condition, we have

$$V_0 = -\frac{C}{b} + D, 0 = -\frac{C}{c} + D \Rightarrow C = -\frac{bc}{c-b}V_0, D = -\frac{b}{c-b}V_0$$

Therefore, we have

$$V(r) = V_0, \quad \text{for } 0 < r < a$$

$$V(r) = \frac{bV_0}{c-b} \left(\frac{c}{r} - 1 \right), \quad \text{for } b < r < c$$

(c) For $b < r < c$, $V(r) = \frac{bV_0}{c-b} \left(\frac{c}{r} - 1 \right)$.

$$\mathbf{E}(r, \theta, \varphi) = -\nabla V(r) = \frac{bcV_0}{(c-b)r^2} \hat{\mathbf{a}}_r$$

(d) Consider a spherical surface, centered at the origin as the Gaussian surface S , which encloses the inner conductor ($b < r < c$).

By Gauss's law, we have

$$Q_{\text{enc}} = \int_{\theta=0}^{\pi} \int_{\varphi}^{\varphi+2\pi} \epsilon_0 \mathbf{E} \cdot \hat{\mathbf{a}}_r \cdot r^2 \sin\theta d\theta d\varphi = 4\pi\epsilon_0 \frac{bcV_0}{c-b}$$

Thus, the capacitance is

$$C = \frac{Q_{\text{enc}}}{V_0} = 4\pi\epsilon_0 \frac{bc}{c-b}$$