

EE 1004 Foundation of Information Systems and Data Analysis

Part II

Chapter 8: Introduction of Matrix

Contents

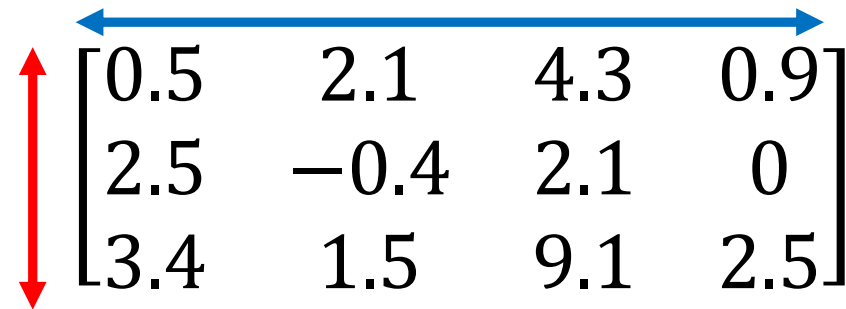
- Introduction
- Examples of matrices
- Basic definitions
- Basic operations
- Advanced topics

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- Introduction
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What is matrix?

- Matrix is a rectangular array of numbers, e.g.,



A 3x4 matrix is shown with a red vertical arrow on the left and a blue horizontal arrow on top. The matrix is:

$$\begin{bmatrix} 0.5 & 2.1 & 4.3 & 0.9 \\ 2.5 & -0.4 & 2.1 & 0 \\ 3.4 & 1.5 & 9.1 & 2.5 \end{bmatrix}$$

- Its size is given by the number of **row** dimensions $\times$ **column** dimensions;
- Each element ($a_{i,j}$) in the matrix (A) is called entry, i and j are the row and column index, respectively.

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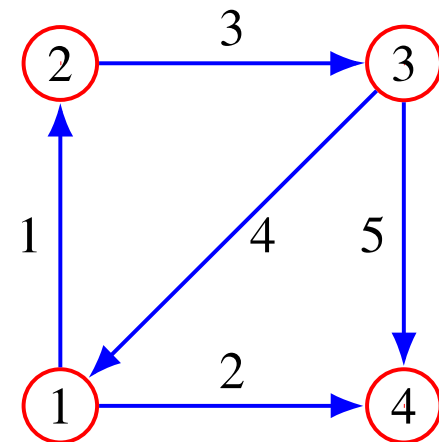
- Introduction
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Describing graph

- Graph with n vertices or nodes
- Then, the adjacent matrix is $n \times n$ matrix

- $A_{ij} = \begin{cases} 1 & \text{node } i \text{ points to node } j \\ -1 & \text{node } j \text{ points from node } i \\ 0 & \text{otherwise} \end{cases}$
- Example with $n = 4$:

$$A = \begin{matrix} & \begin{matrix} \text{Node } j \\ \textcircled{1} & \textcircled{2} & \textcircled{3} & \textcircled{4} \end{matrix} \\ \begin{matrix} \text{Node } i \\ \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \\ \textcircled{4} \end{matrix} & \begin{bmatrix} 0 & 1 & -1 & 1 \\ -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ -1 & 0 & -1 & 0 \end{bmatrix} \end{matrix}$$



Representing linear equations

- Think of the equation systems:

$$3x + 2y + 1z = 39$$

$$2x + 5y + 4z = 34$$

$$1x + 4y + 2z = 26$$

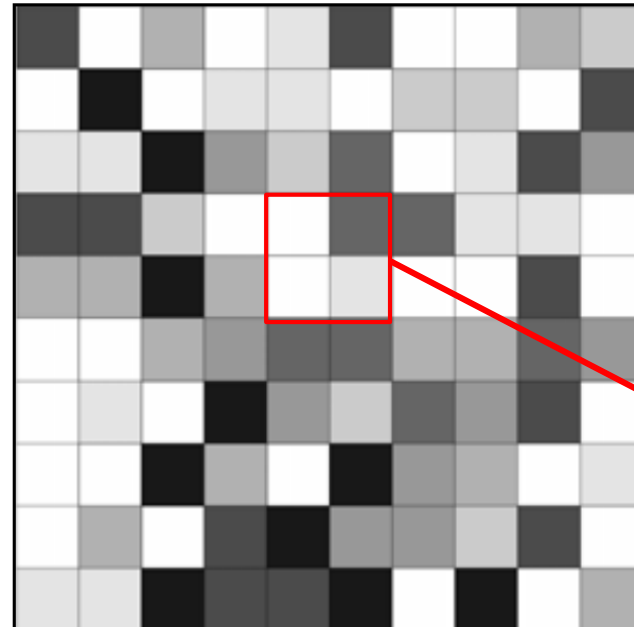
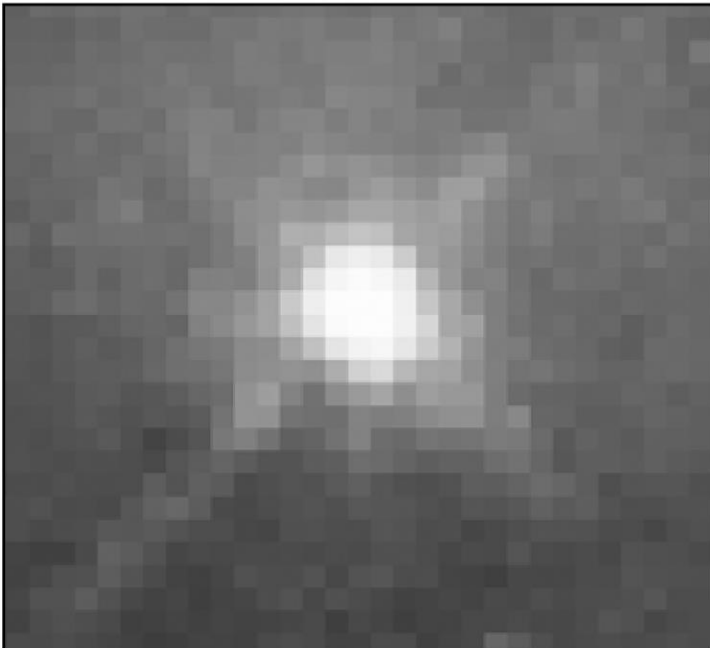
- We can represent it using the matrix form:

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 5 & 4 \\ 1 & 4 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 39 \\ 34 \\ 26 \end{bmatrix}$$

Matrix-vector product: $Ax = b$

Storing and representing images

- Images are composed of matrices of square pixels;
- In the shown examples:
 - Grids arranged in columns and rows -> matrix;
 - Each pixel has a value from **0 (black)** to **255 (white)**.



Value

254	107
255	165

[Reference link](#)

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- Introduction
- Examples of matrices
- **Basic definitions**
 - Column and row vectors;
 - Block matrix;
 - Special matrices (square, diagonal, identity, zero)
- Basic operations
- Advanced topics

Column and row vectors

- Column vector : $n \times 1$ matrix;

e.g., $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

- Row vector : $1 \times n$ matrix;

e.g., $[1 \ 2 \ 3]$

Block matrices

- In block matrices, the entries are some sub-matrices, e.g.,

$$A = \begin{bmatrix} B & C \\ D & E \end{bmatrix}$$

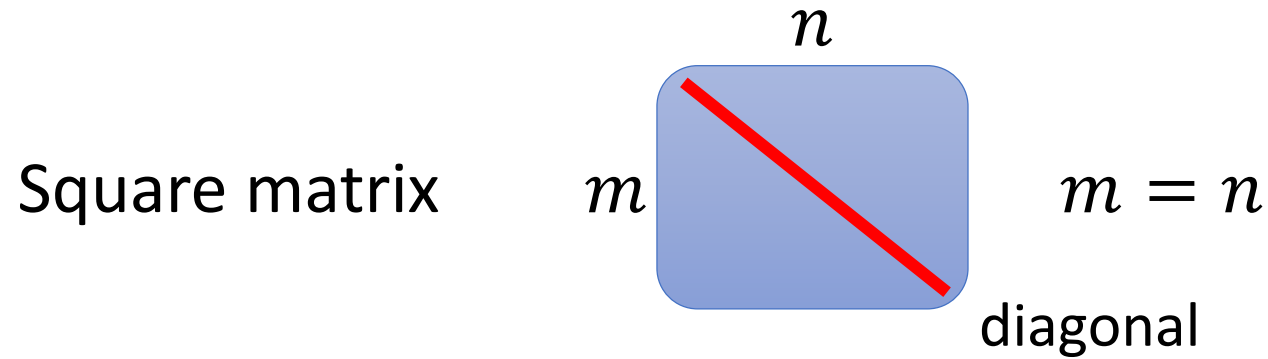
where B, C, D, E are all matrices.

Examples:

$$B = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}, C = \begin{bmatrix} -1 \end{bmatrix}, D = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 5 \end{bmatrix}, E = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

$$\text{Hence, } A = \begin{bmatrix} B & C \\ D & E \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 & -1 \\ 2 & 2 & 1 & 4 \\ 1 & 3 & 5 & 4 \end{bmatrix}$$

Special matrices



$$\begin{bmatrix} 2 & 3 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

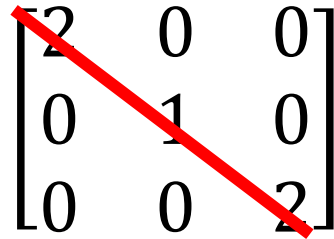
Upper Triangular
Matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

Lower Triangular
Matrix

Special matrices (Con't)

- Diagonal Matrix

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$


All non-diagonal elements are “0”.

- Identity Matrix

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Denoted by I (any size)
or I_n

- Zero Matrix

$$O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Denoted by O (any size)
or $O_{m \times n}$

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- **Basic operations**
 - Matrix addition
 - Matrix multiplication
 - Matrix transpose
- Advanced topics

Matrix addition

- For two matrices of **the same size**, $A, B \in \mathbb{R}^{m \times n}$, the addition is the sum of corresponding elements ($\mathbb{R}^{m \times n}$ is the real coordinate space of dimension $m \times n$)

$$A + B = C \in \mathbb{R}^{m \times n} \Leftrightarrow C_{ij} = A_{ij} + B_{ij}$$

- A 2×2 example:

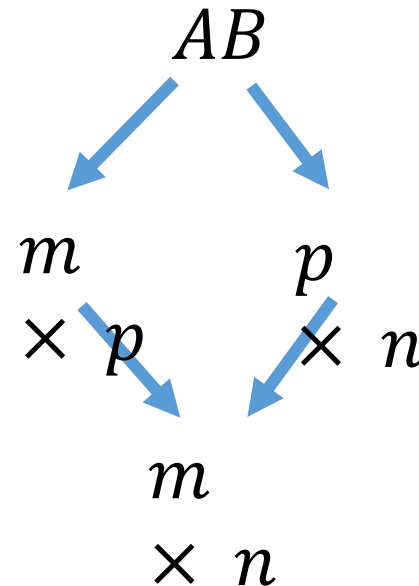
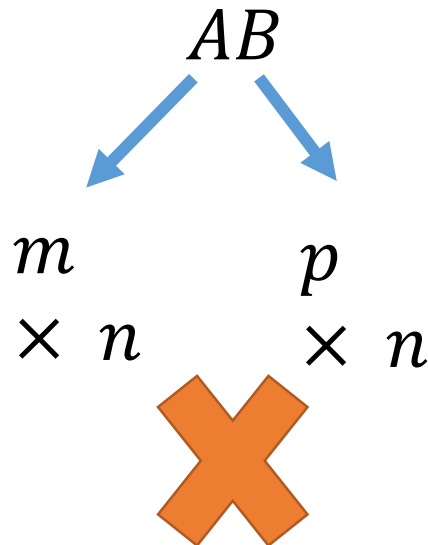
$$\text{e.g., } \begin{bmatrix} 1 & 1 \\ 3 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 2 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 9 \end{bmatrix}$$

Matrix multiplication - Definition

- For two matrices $A \in \mathbb{R}^{m \times p}$, $B \in \mathbb{R}^{p \times n}$, their product is

$$AB = C \in \mathbb{R}^{m \times n} \Leftrightarrow C_{ij} = \sum_{k=1}^p a_{ik} b_{kj}$$

- The number of columns of A should be equal to the number of rows of B :



Matrix multiplication – How to compute?

The $(i, j)^{th}$ entry of AB is formed by adding the product of some entries in i^{th} row of A and j^{th} column of B .

$$AB = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ \vdots & \ddots & \ddots & \vdots \\ \textcolor{red}{a_{i1}} & \textcolor{red}{a_{i2}} & \cdots & \textcolor{red}{a_{ip}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & \textcolor{teal}{b_{1j}} & \cdots & b_{1n} \\ b_{21} & \ddots & \textcolor{teal}{b_{2j}} & \ddots & b_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{p1} & \cdots & \textcolor{teal}{b_{pj}} & \cdots & b_{pn} \end{bmatrix}$$

$$[AB]_{ij} = \sum_{k=1}^p a_{ik} b_{kj} = \textcolor{red}{a_{i1}} \textcolor{teal}{b_{1j}} + \textcolor{red}{a_{i2}} \textcolor{teal}{b_{2j}} + \textcolor{red}{a_{i3}} \textcolor{teal}{b_{3j}} + \cdots + \textcolor{red}{a_{ip}} \textcolor{teal}{b_{pj}}$$

Matrix multiplication - Example

Compute the product $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix}$

- Step 1: Determine the size, the product $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix}$ is a 2×2 matrix $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$
 $\begin{matrix} 2 \\ \times 3 \text{ matrix} \end{matrix}$ $\begin{matrix} 3 \\ \times 2 \text{ matrix} \end{matrix}$

Matrix multiplication - Example

Compute the product $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix}$

- Step 2: Determine the entries.

$$\begin{bmatrix} \color{red}{1} & \color{red}{0} & \color{red}{-1} \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} \color{blue}{0} & 1 \\ \color{blue}{2} & 3 \\ \color{blue}{1} & -1 \end{bmatrix}$$

Compute (1,1)th entry (Entry-A): $\left[\color{red}{1} \times \color{blue}{0} + \color{red}{0} \times \color{blue}{2} + (\color{red}{-1}) \times \color{blue}{1} \right] \begin{matrix} B \\ D \end{matrix} \begin{matrix} \color{blue}{1} \\ \color{blue}{-1} \end{matrix} \begin{matrix} B \\ D \end{matrix}$

Matrix multiplication - Example

Compute the product $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix}$

- Step 2: Determine the entries.

$$\begin{bmatrix} \color{red}{1} & \color{red}{0} & \color{red}{-1} \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & \color{blue}{1} \\ 2 & \color{blue}{3} \\ 1 & \color{blue}{-1} \end{bmatrix}$$

Compute (1,2)th entry (Entry-B): $\begin{bmatrix} A & \color{red}{1} \times \color{blue}{1} + \color{red}{0} \times \color{blue}{3} + (\color{red}{-1}) \times (\color{blue}{-1}) \\ C & \color{blue}{-1} \end{bmatrix} \begin{bmatrix} -1 & 2 \\ \color{blue}{-1} & \color{blue}{-1} \end{bmatrix}$

Matrix multiplication - Example

Compute the product $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix}$

- Step 2: Determine the entries.

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix}$$

Compute (2,1)th entry (Entry-C): $\begin{bmatrix} & A \\ (-2) \times 0 + 1 \times 2 + 2 \times 1 & B \\ & D \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & D \end{bmatrix}$

Matrix multiplication - Example

Compute the product $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix}$

- Step 2: Determine the entries.

$$\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 1 & -1 \end{bmatrix}$$

Compute (2,2)th entry (Entry-D): $\begin{bmatrix} A & B \\ C & (-2) \times 1 + 1 \times 3 + 2 \times (-1) \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & -1 \end{bmatrix}$

Matrix multiplication - Properties

- Associative: $(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{p \times q})$

$$A(BC) = (AB)C$$

- Distributive: $(A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{p \times q})$

$$A(BC) = (AB)C$$

- **NOT** commutative: $AB \neq BA$

Matrix multiplication - Scalar-matrix product

Suppose c is a scalar and A is a $m \times n$ matrix,

$$cA = c \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{bmatrix}$$

Example, $2 \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -6 & 2 \\ 2 & -2 \end{bmatrix}$

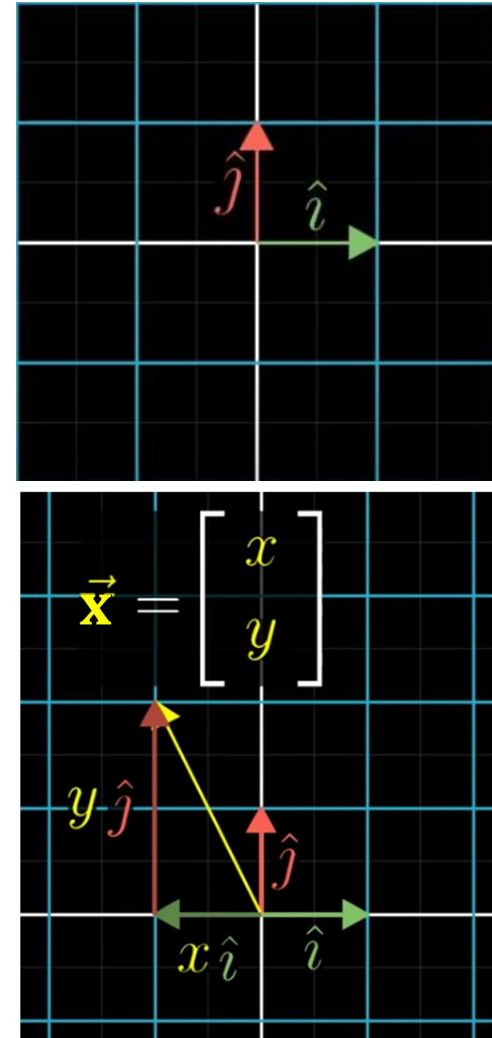
Matrix multiplication - Geometric intuition

- Step 1: Consider the basis $\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Every vector in the 2-D space can be represented as the linear combinations of $\hat{i}$ and $\hat{j}$,

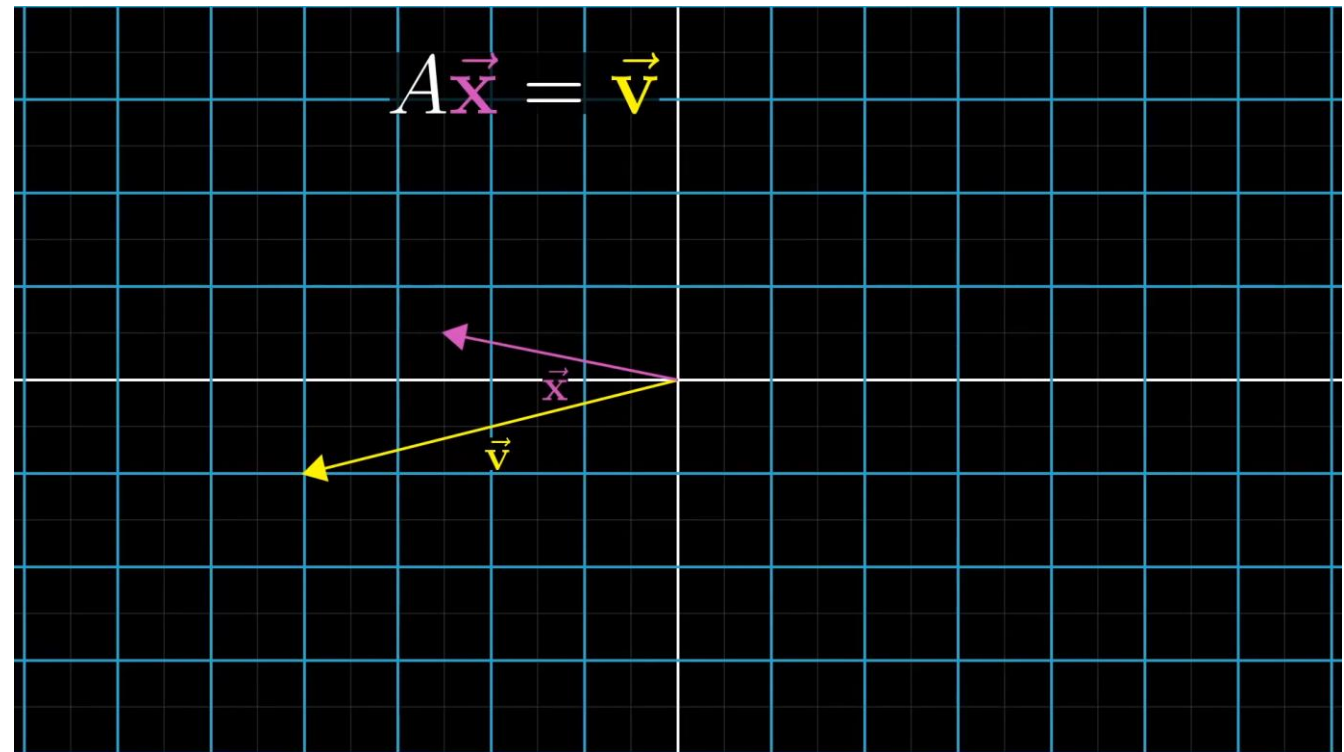
e.g., $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix} = x\hat{i} + y\hat{j}$.

x and y are the combination coefficients.



Matrix multiplication - Geometric intuition

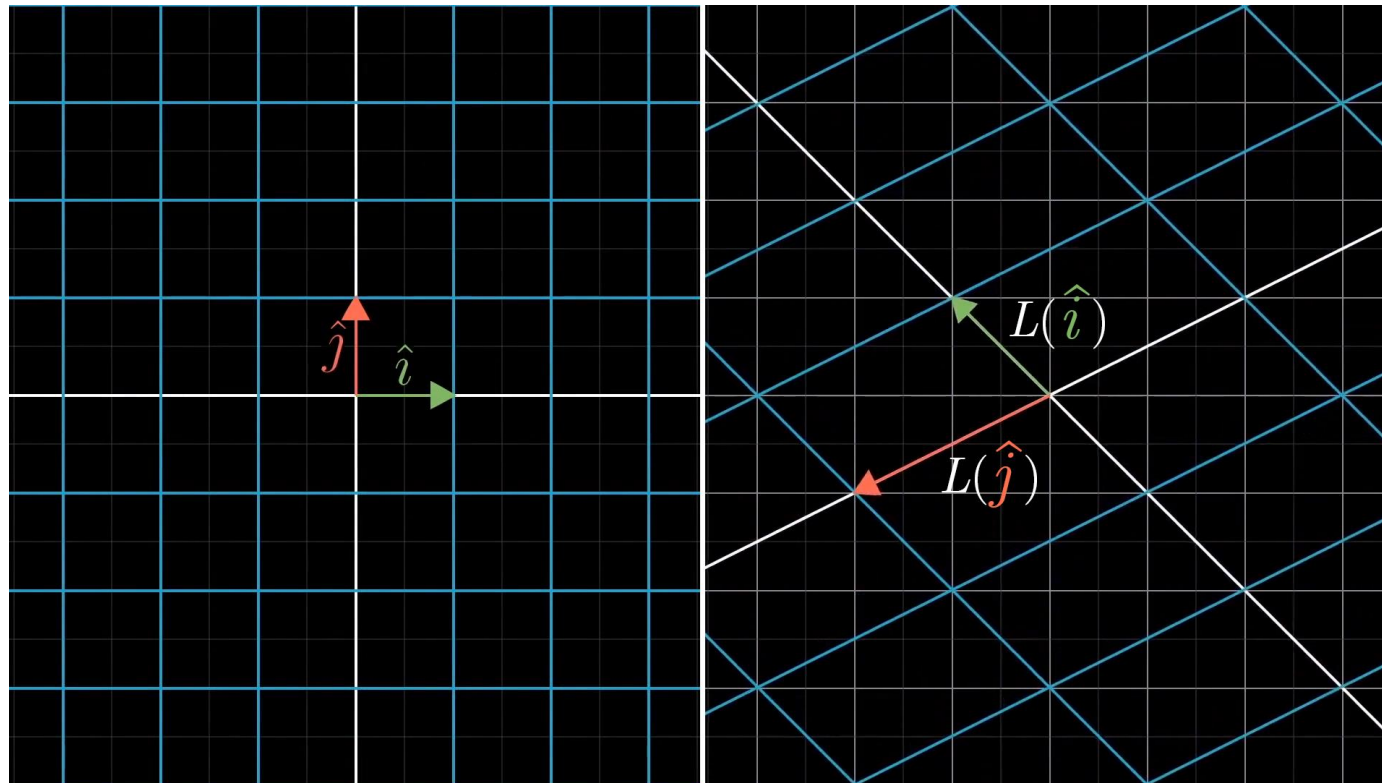
- Step 2: Consider a linear transformation A
 1. Linear transformations are functions, input and output are both vectors; Here the input vector is $\vec{x}$ and the output vector is $\vec{v}$.
 2. The transformed grids are parallel and evenly spaced;
 3. The origin is fixed.



How to find the coordinates of $\vec{v}$?

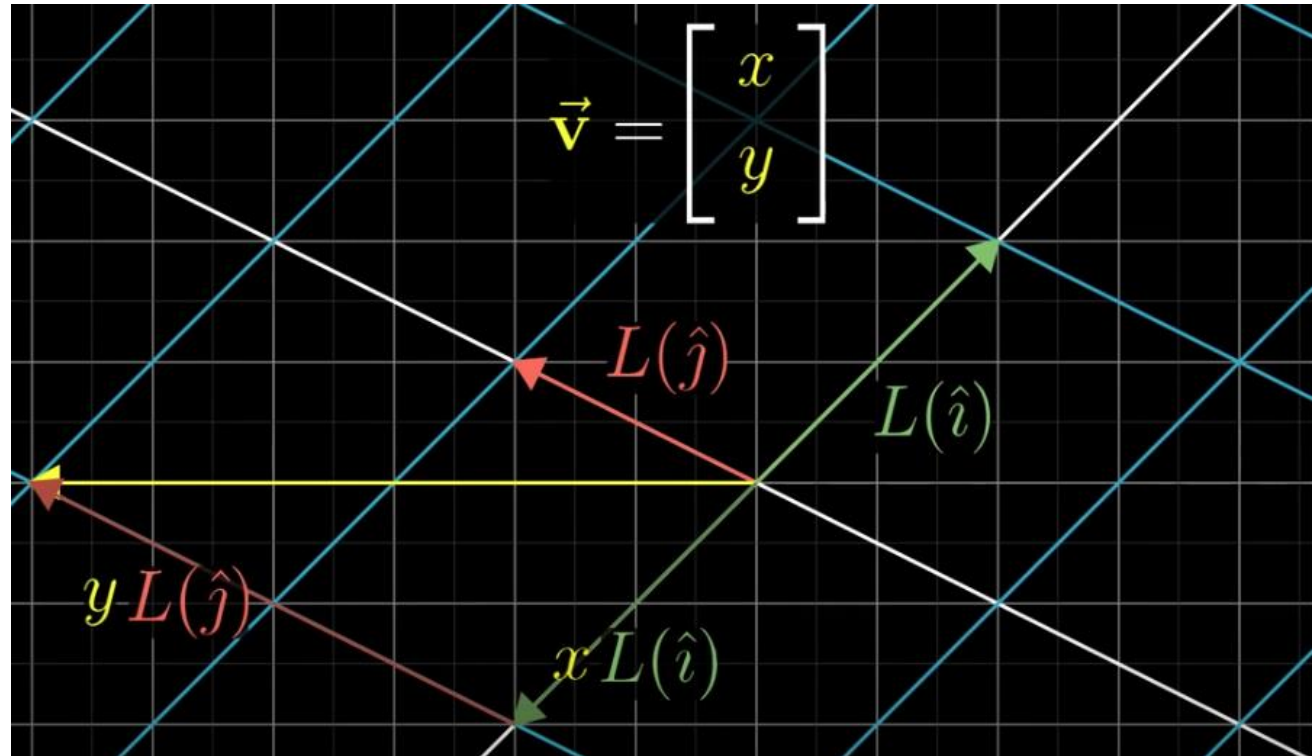
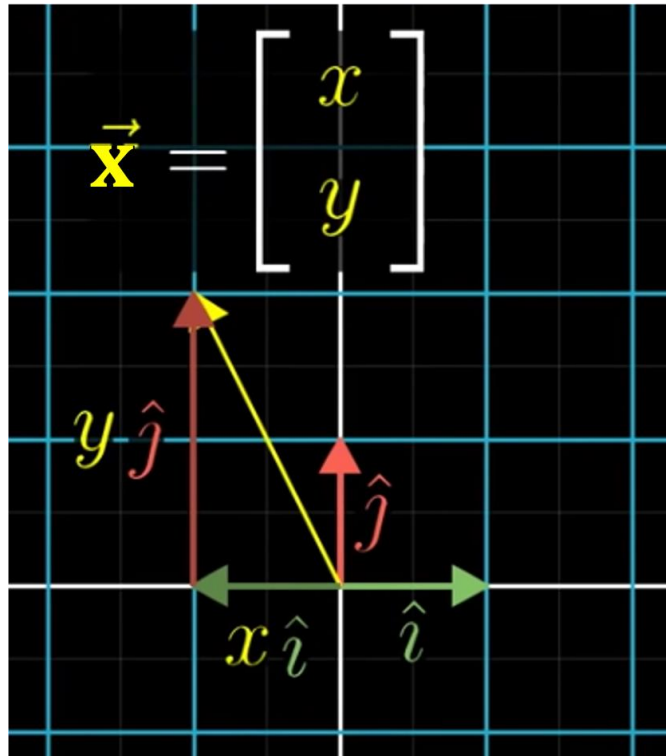
Matrix multiplication - Geometric view

- Step 3: Consider the new basis $L(\hat{i})$ and $L(\hat{j})$, where L is the linear transformation operator.



Matrix multiplication - Geometric intuition

- Step 4: The $\vec{v}$ will then become the linear combination of the new basis $L(\hat{i})$ and $L(\hat{j})$. The coefficient is still x and y .



Matrix multiplication - Geometric intuition

coefficients of the linear combination

The diagram illustrates the geometric intuition of matrix multiplication. It shows the equation $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$. The matrix is represented as a 2x2 grid of colored boxes: green for a and c , red for b and d , and white for the variables x and y . Two blue arrows point from the text 'coefficients of the linear combination' to the x and y terms in the middle part of the equation. Two orange arrows point from the $\begin{bmatrix} a \\ c \end{bmatrix}$ and $\begin{bmatrix} b \\ d \end{bmatrix}$ column vectors to the text 'where transformed basis $L(\hat{i})$ and $L(\hat{j})$ land in the coordinates'.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a \\ c \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

where transformed basis $L(\hat{i})$ and $L(\hat{j})$
land in the coordinates

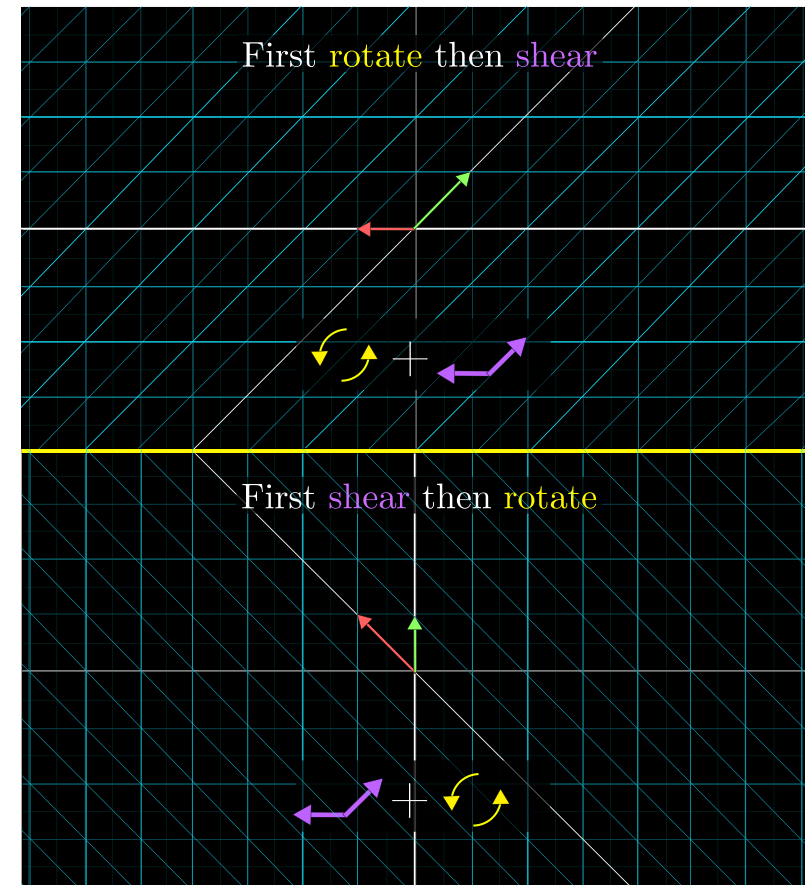
Matrix multiplication - Geometric intuition

Associativity:

Associativity: $(AB)C \stackrel{???}{=} A(BC)$



Not commutative:



Transpose - overview

- If A is an $m \times n$ matrix, A^T (transpose of A) is an $n \times m$ matrix whose (i, j) -entry is the (j, i) -entry of A

$$A = \begin{bmatrix} 6 & 9 \\ 8 & 0 \\ 9 & 2 \end{bmatrix} \xrightarrow{\text{Transpose}} A^T = \begin{bmatrix} 6 & 8 & 9 \\ 9 & 0 & 2 \end{bmatrix}$$

The diagram illustrates the transpose operation. Matrix A is a 3×2 matrix with elements 6, 9, 8, 0, 9, 2. The element 9 at row 1, column 2 is highlighted with a blue box and labeled $(1,2)$. The element 2 at row 3, column 2 is highlighted with a red box and labeled $(3,2)$. A large blue arrow labeled "Transpose" points to matrix A^T , which is a 2×3 matrix with elements 6, 8, 9, 9, 0, 2. The element 9 at row 2, column 1 is highlighted with a blue box and labeled $(2,1)$. The element 0 at row 2, column 2 is highlighted with a red box and labeled $(2,3)$.

Column $\rightarrow$ Row ; Row $\rightarrow$ Column

Transpose - properties

Suppose A and B are $m \times n$ matrices, and s is a scalar, we have:

- $(A^T)^T = A$
- $(sA)^T = sA^T$
- $(A + B)^T = A^T + B^T$

Symmetric Matrix

The **square** matrix C is a symmetric matrix if and only if $C^T = C$

Examples:

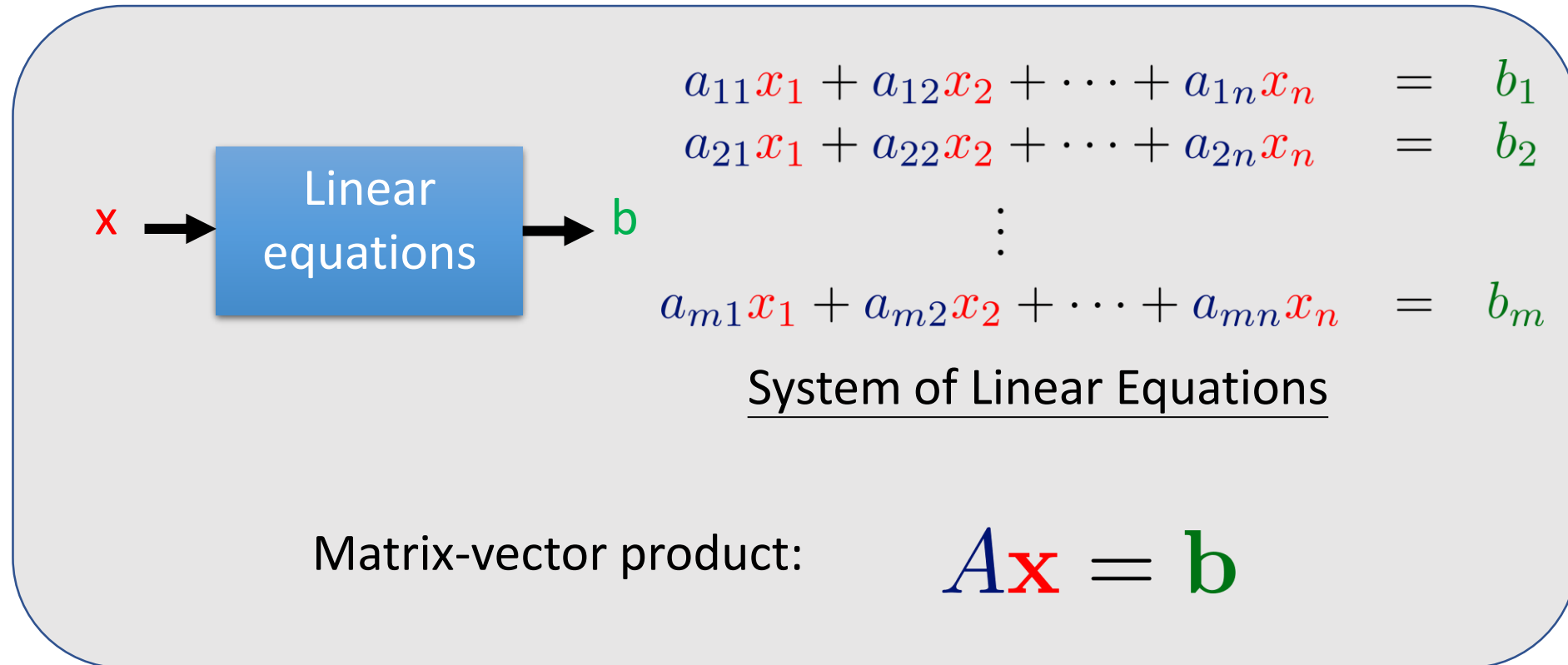
$$C = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 5 \end{bmatrix} = C^T$$

$$D = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq D^T$$

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 - Determinant of square matrix
 - Inverse of square matrix
 - Eigenvalue and eigenvectors

Solving linear equations - Definition



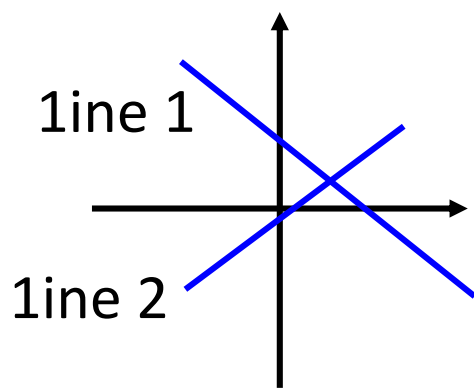
Given A and b , to find x

Solving linear equations - Solutions

- Considering any system of linear equations with 2 variables and 2 equations

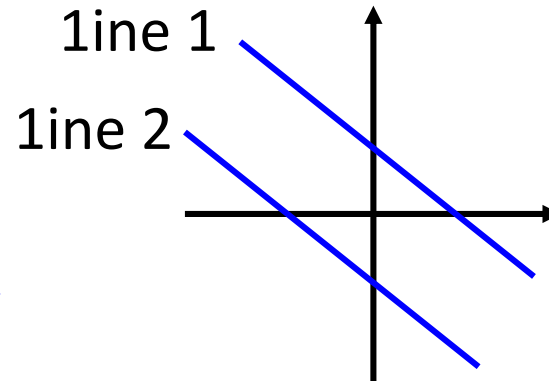
$$a_{11}x_1 + a_{12}x_2 = b_1 \quad \text{..... line 1}$$

$$a_{21}x_1 + a_{22}x_2 = b_2 \quad \text{..... line 2}$$



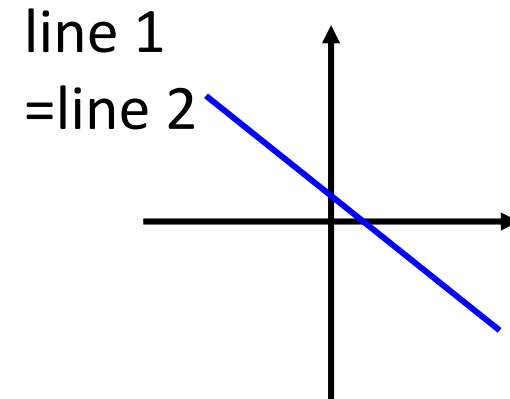
unique solution

consistent



no solution

inconsistent



infinite solutions

consistent

Determinant of the square matrix - Definition

- A determinant is a value associated to a square matrix. For example, here are determinants of a general 2×2 matrix.

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \quad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Hence, the determinant is a function set: $\{n \times n \text{ matrix}\} \rightarrow \mathbb{R}$.

The determinant of a square matrix A is usually denoted $|A|$ or $\det(A)$.

Determinant of the square matrix - Definition

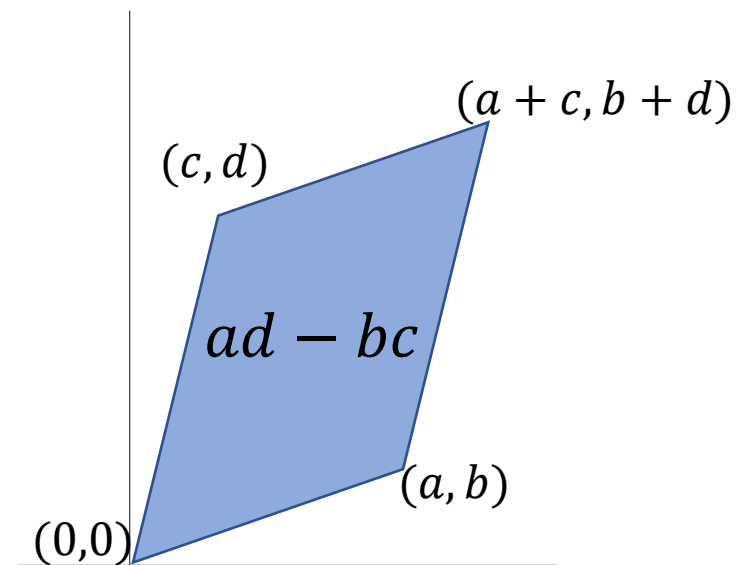
- Suppose we have $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
 (a, b) and (c, d) are the mapped basis by the linear transformation of A .
- The determinant measure the area of the parallelogram formed by transformed basis.
- Suppose we also have $\mathbf{u} = (a, b)$, $\mathbf{v} = (c, d)$

The signed area can be also calculated as:

$$|\mathbf{u}||\mathbf{v}| \sin \theta = ad - bc = \det(A)$$

Hence,

determinant can naturally link the matrix to the vector analysis



Determinant of the square matrix - Calculation, 2 x 2 case

- The determinant of a 2×2 square matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is defined as follows:

$$\det A = |A| = a_{11}a_{22} - a_{12}a_{21}.$$

- Example

$$\det \begin{pmatrix} 2 & 3 \\ 1 & -4 \end{pmatrix} = 2(-4) - (3)(1) = -11, \quad \det \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} = 1(3) - 2(0) = 3$$

Determinant of the square matrix – Calculation, $n \times n$ case

- Some essential definitions:
 - **Minor** of the element $a_{i,j}$, denoted by $M_{i,j}$, is the determinant of a $(n - 1) \times (n - 1)$ submatrix of A obtained by deleting row- i and column- j of A .

$$M_{11} = \det \begin{pmatrix} \textcolor{red}{1} & \textcolor{red}{2} & \textcolor{red}{3} \\ \textcolor{red}{4} & 5 & 6 \\ \textcolor{red}{7} & 8 & 9 \end{pmatrix} = \det \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} = 5(9) - 6(8) = -3$$

$$M_{23} = \det \begin{pmatrix} 1 & 2 & \textcolor{red}{3} \\ \textcolor{red}{4} & \textcolor{red}{5} & \textcolor{red}{6} \\ 7 & 8 & \textcolor{red}{9} \end{pmatrix} = \det \begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} = 1(8) - 2(7) = -6$$

- **Cofactor** of elements $a_{i,j}$, denoted by $A_{i,j}$: $A_{ij} = (-1)^{i+j} M_{ij}$

Determinant of the square matrix – Calculation, $n \times n$ case

- The determinant of A ($n \times n$ matrix, $n > 3$) is defined as:

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} + \cdots + a_{1n}A_{1n}$$

The above equation gives the case we expand it along the 1st row.

In general, we can calculate the determinant along any row or any column (see next page).

Determinant of the square matrix – Calculation, $n \times n$ case

- Pick row 1

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}$$

- Or pick row i

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

- Or pick column j

$$\det(A) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

Cofactor expansion again ...

Determinant of the square matrix – Calculation, $n \times n$ case

- Example $n=3$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Pick row 2

$$\det A = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$$

4

5

6

$$(-1)^{2+1}M_{21}$$

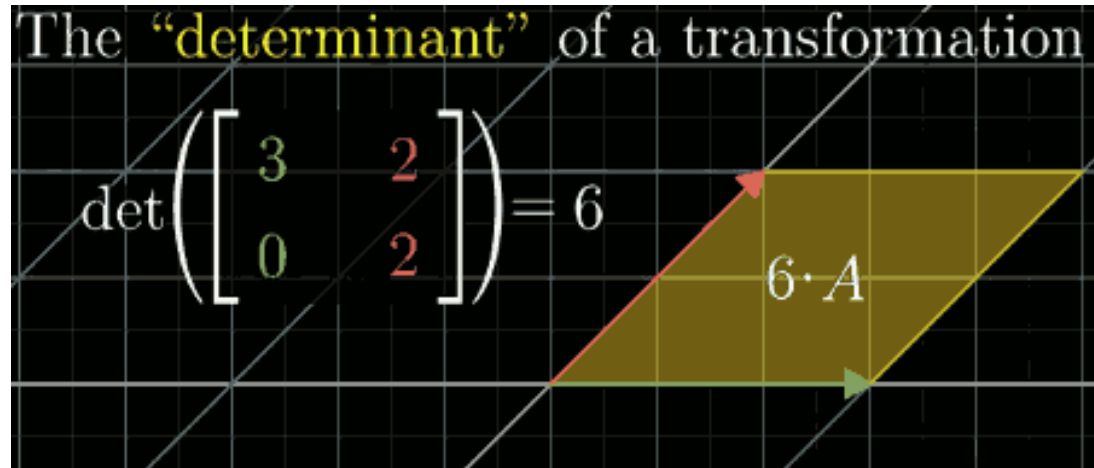
$$(-1)^{2+2}M_{22}$$

$$(-1)^{2+3}M_{23}$$

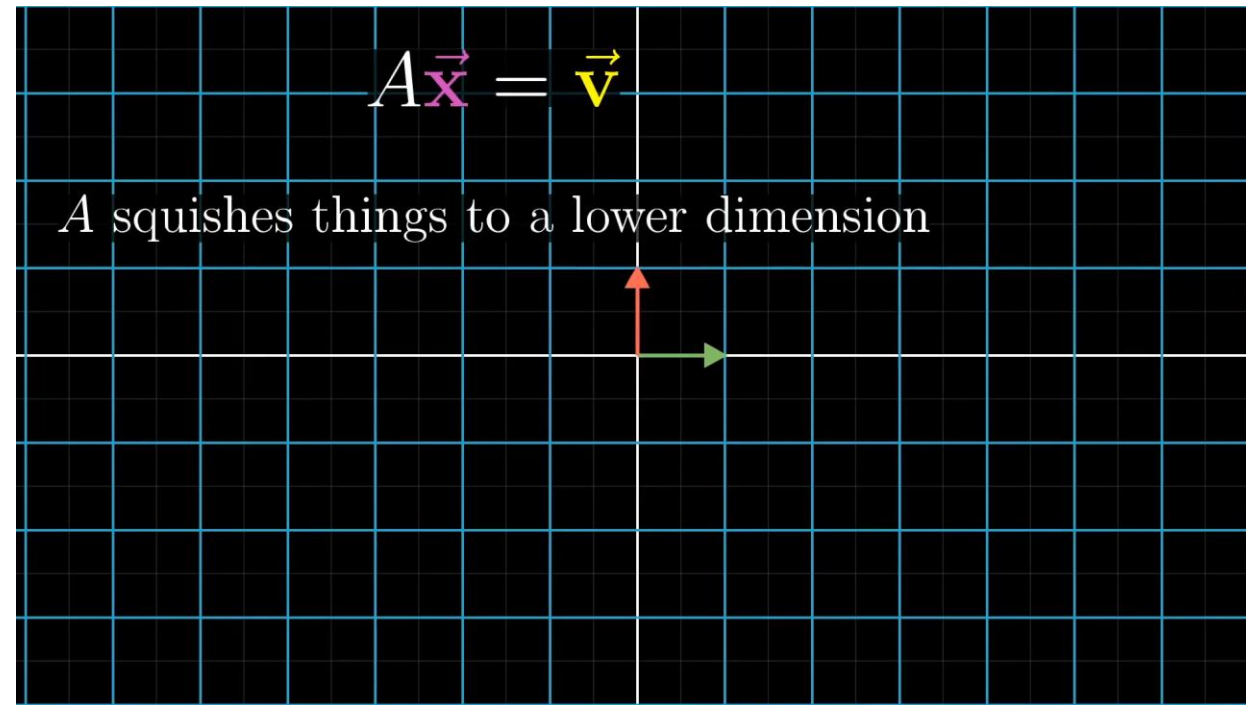
$$M_{21} = \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad M_{22} = \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \quad M_{23} = \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Determinant of the square matrix – Geometric visualization

- The factor by which a linear transformation changes the area



- A special case when $\det = 0$



Inverse - Definition

In the real number world with scalars, the reverse of a can be computed by $\frac{1}{a}$, hence $a^{-1} = \frac{1}{a}$.

While, in the matrix system, we **cannot** directly use $1/A$ to compute the inverse of A .

- **Definition:** A is called invertible if there is a matrix B such that $AB = I$ and $BA = I$.
- B is an inverse of A : $B = A^{-1}$ $A = B^{-1}$

e.g.,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \quad B = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \quad AB = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Inverse – Properties

- The inverse of matrix is defined on square matrix only. The inverse of a rectangular matrix (like 1×2 matrix) does not exist;
- The inverse of some matrix may not exist, we call them **singular** matrices; In the meanwhile, the **non-singular** matrices are those with inverse forms existing.

Example: there is no inverse for $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

For any square matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq I_2$$

-> Contradicts with the definition of inverse matrix.

Inverse – Properties and calculation

- Existence of inverse matrix

Given an $n \times n$ square matrix A , if

(1) $\det(A) = 0$, then the matrix is singular (not invertible);

(2) $\det(A) \neq 0$, then the matrix is non-singular (invertible)

- General formula of computing inverse

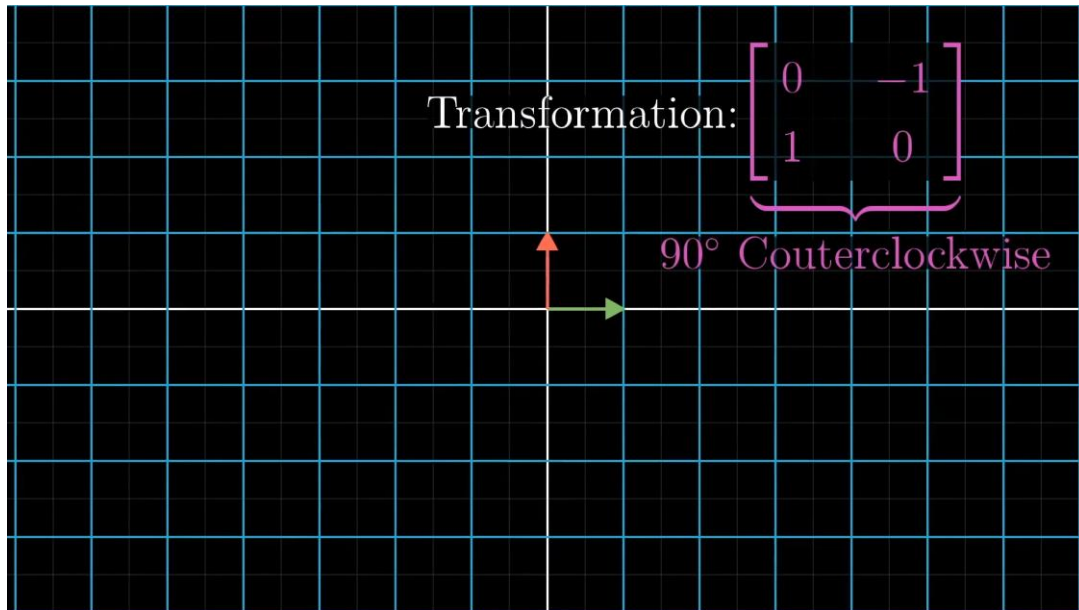
Let A is an $n \times n$ invertible square matrix, then A^{-1} is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}^T$$

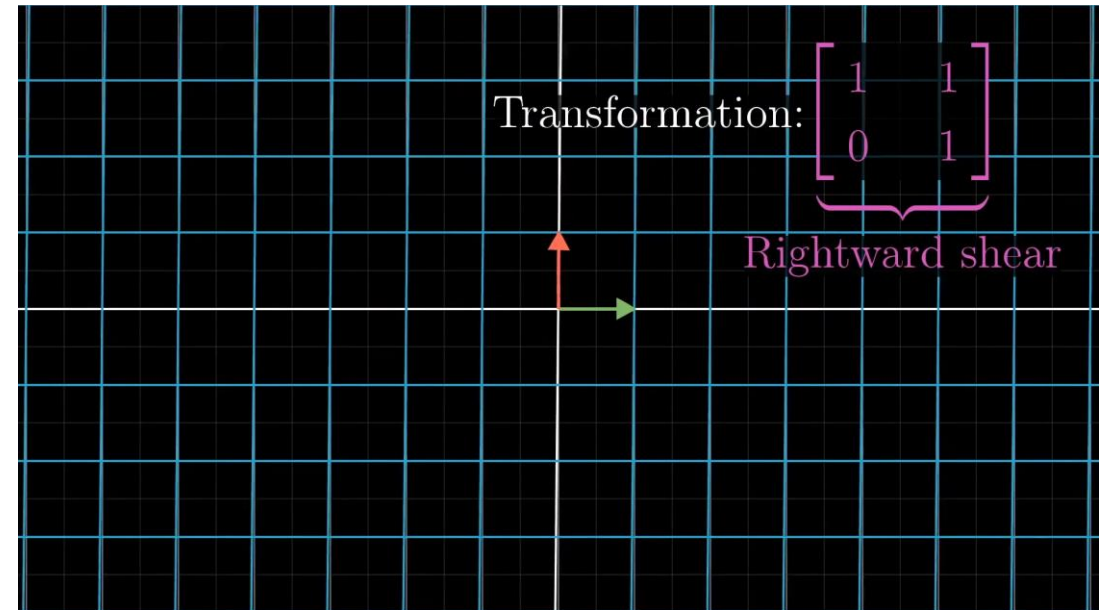
where A_{ij} is the cofactor of the entry a_{ij} .

Inverse – Geometric views

- Rotation inverse



- Shear inverse



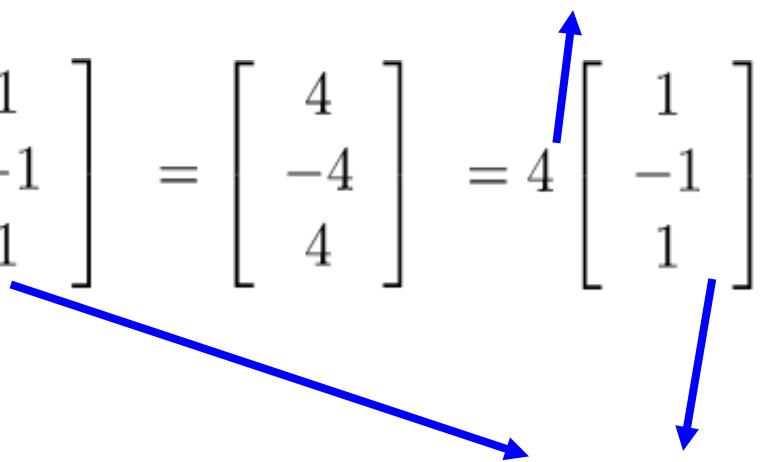
Eigenvalue and Eigenvectors - Definition

- If $Av = \lambda v$ (v is a vector, λ is a scalar, **A is a square matrix**)
 - v is an eigenvector of A (excluding the zero vectors)
 - λ is an eigenvalue of A that corresponds to v

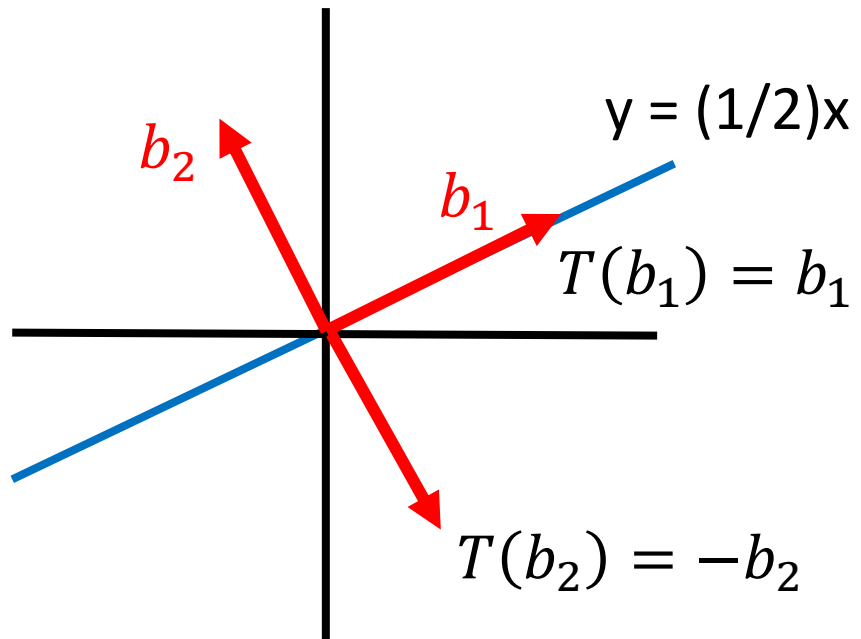
$$\begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Eigen value

Eigen vector



Eigenvalue and Eigenvectors - Geometric views



b_1 is an eigenvector of T
(T is a linear transformation)

b_2 is an eigenvector of T

To be discussed in the next lecture

Suggested references

- [Linear algebra done right](#) - Textbook
- [Notes from Stanford](#) - short
- [3blue1brown](#) -beautiful animated explanations
- [Linear Algebra lectures by Professor Gil Strang at MIT](#) - full course
- [A Tutorial on Linear Algebra by Professor C. T. Abdallah](#)
- [Linear Algebra Review by Professor Fernando Paganini, UCLA](#)
- [The Matrix Cookbook](#) - good toolbox for information engineering and data analysis projects and research.