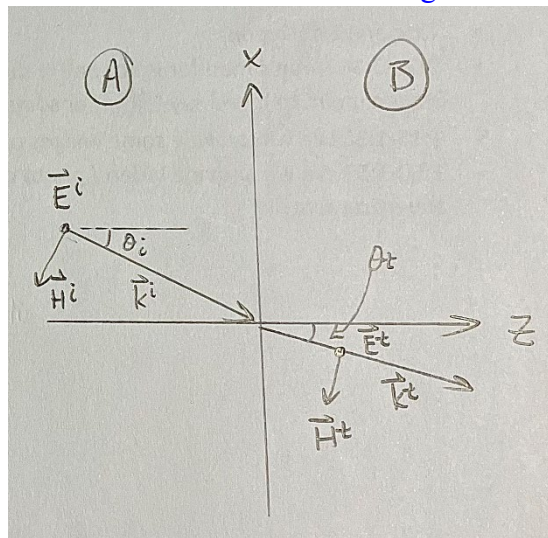


EE3109 2021/22 Sem. B Test 2 Solution

1. (23 Marks) A 10 GHz, TE-polarized plane wave, with amplitude 10 [V/m], is incident at 30° from an air-material interface, from the side of air. The incident plane wave travels in the xz-plane, in the -x and +z direction. The air-material interface is on the xy-plane (i.e. the plane $z = 0$). The material has $(\mu_r, \epsilon_r, \sigma) = (4, 1, 0)$.
- Defining the air as A and the material as B, find $n_A, n_B, \eta_A, \eta_B, k_A$ and k_B . (6)
 - Use Snell's law to find the angle of the transmitted wave θ_t . (2)
 - What is the direction of travel of the transmitted wave? Answer by indicating whether each of k_x, k_y , and k_z is positive, negative or zero (2)
 - Write the incident electric field phasor (4)
 - Write the transmitted electric field phasor (4)
 - If the incident wave carries 1W energy, what is the power of the transmitted wave? (3)
 - If the incident angle is swept 0° to just less than 90° , will we have an angle where there is total reflection and/or total transmission? Explain your answer. (2)

Solution:

The picture of plane wave incidence is drawn in the figure below.



(a)

$$n_A = 1; \eta_A = \eta_0 = 120\pi \Omega; k_A = \frac{\omega}{c} = \frac{2\pi \times 10^{10}}{3 \times 10^8} = \frac{200\pi}{3}$$

$$n_B = \sqrt{\mu_r \epsilon_r} = 2; \eta_B = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} = 2\eta_0 = 240\pi \Omega; k_B = \frac{\omega}{c} n_B = \frac{400\pi}{3}$$

(b) From the figure, $\theta_i = -30^\circ$.

$$n_A \sin \theta_i = n_B \sin \theta_t \Rightarrow \theta_t = \sin^{-1} \left(\frac{n_A}{n_B} \sin \theta_i \right) = \sin^{-1} \left(\frac{1}{2} \left(-\frac{1}{2} \right) \right) = -14.48^\circ$$

(c) The transmitted wave is in the +z and -x direction, i.e. $k_x < 0, k_y = 0, k_z > 0$.

(d)

$$k_{xA} = k_A \sin(-30^\circ) = \frac{200\pi}{3} \left(-\frac{1}{2}\right) = -\frac{100\pi}{3} \text{ [rad/m]}$$

$$k_{zA} = k_A \cos(-30^\circ) = \frac{200\pi}{3} \left(\frac{\sqrt{3}}{2}\right) = \frac{100\pi}{\sqrt{3}} \text{ [rad/m]}$$

$$\mathbf{E}^i = 10 e^{-j(k_{xA}x + k_{zA}z)} \hat{\mathbf{y}} = 10 e^{-j\left(-\frac{100\pi}{3}x + \frac{100\pi}{\sqrt{3}}z\right)} \hat{\mathbf{y}} \text{ [V/m]}$$

(e) We first find the relevant cosine terms: $\cos \theta_i = 0.8660$; $\cos \theta_t = 0.9682$.

The TE transmission coefficient is

$$\tau_{TE} = \frac{2\eta_B \cos \theta_i}{\eta_B \cos \theta_i - \eta_A \cos \theta_t} = \frac{2(2)(0.8660)}{2(0.8660) + (1)(0.9682)} = 1.283$$

The wave vector components for the transmitted wave are:

$$k_{xB} = k_B \sin \theta_t = -104.7 \text{ [rad/m]}; k_{zB} = k_B \cos \theta_t = 405.6 \text{ [rad/m]}$$

The transmitted electric field is

$$\mathbf{E}^t = \tau_{TE} 10 e^{-j(k_{xB}x + k_{zB}z)} = 12.83 e^{-j(-104.7x + 405.6z)} \hat{\mathbf{y}} \text{ [V/m]}$$

(f) The transmitted wave's power is

$$P_t = T P_i = |\tau_{TE}|^2 \left(\frac{\eta_A \cos \theta_t}{\eta_B \cos \theta_i}\right) P_i = (1.283)^2 \left(\frac{1}{2}\right) * \frac{0.9682}{0.8660} (1) = 0.92 \text{ [W]}$$

(g) There will be a Brewster angle (for TE waves, within a magnetic material) where $\eta_A < \eta_B$ and $\cos \theta_t > \cos \theta_i$, but $\eta_A \cos \theta_t = \eta_B \cos \theta_i$, so that the reflection coefficient Γ_{TE} goes to zero. At that incident angle, the wave will be totally transmitted into the material.

2. **(15 Marks)** A lossless air transmission line has a characteristic impedance of 50Ω and a length of 30mm. The transmission line is terminated with a 100Ω resistor in parallel with a 10 nH inductor. An RF wave of 15 GHz travels along the transmission line.

- (a) Find Z_L and Γ_L along the transmission line (4)
 (b) Find α and β along the transmission line (3)
 (c) Write the voltage phasor as a function of distance along the line. Substitute known numerical values into the equation and simplify as much as possible. (4)

Now the input of the transmission line is connected to a microwave source, modelled with an ideal 5V voltage source in series with a generator resistance of $R_g = 75\Omega$.

- (d) Write the voltage phasor as a function of distance along the line. Substitute known numerical values into the equation and simplify as much as possible. (4)

Solution:

- (a) The load is a resistor in parallel with an inductor:

$$Z_L = \frac{R_L(j\omega L_L)}{R_L + (j\omega L_L)} = \frac{100(j300\pi)}{100 + j300\pi} = 98.87 + j10.49 [\Omega]$$

The corresponding reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{98.87 + j10.49 - 50}{98.87 + j10.49 + 50} = 0.3317 + j0.0471$$

- (b)

$$\alpha = 0 \text{ (lossless TL); } \beta = \frac{\omega}{v_p} = \frac{2\pi(1.5 \times 10^{10})}{3 \times 10^8} = 100\pi = 314.2 [\text{rad/m}]$$

- (c) The voltage along the line is:

$$\begin{aligned} V(z) &= V_0^+ e^{-j\beta z} (1 + \Gamma_L e^{-j2\beta z}) \\ &= V_0^+ e^{-j100\pi z} (1 + (0.3317 + j0.0741)e^{-j200\pi z}) [\text{V}] \end{aligned}$$

- (d) The voltage along the line is:

$$V(z) = \frac{V_g Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l} e^{-j\beta z}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) (1 + \Gamma_L e^{j2\beta z})$$

We now find the known parameters:

$$\beta l = (100\pi)(30 \times 10^{-3}) = 3\pi \Rightarrow e^{-j\beta l} = e^{-j\pi} = -1$$

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} = \frac{75 - 50}{75 + 50} = 0.2$$

$$\begin{aligned} \frac{V_g Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) &= \left(\frac{5(50)}{125} \right) \left(\frac{-1}{1 - (0.3317 + j0.0471)(0.2)(1)} \right) \\ &= -2.142 - j0.022 \end{aligned}$$

Substituting these into the equation for voltage gives

$$V(z) = (-2.142 - j0.022)e^{-j100\pi z} (1 + (0.3317 + j0.0471)e^{j200\pi z}) [\text{V}]$$

3. **(7 Marks)** Three loads generate reflection coefficients of $\Gamma_A = 0.5 - j0.7$, $\Gamma_B = 0.7 + j0.3$, $\Gamma_C = -0.3 - j0.8$ when connected to a $100\ \Omega$ transmission line.
- (a) Plot the impedances as A B and C on the same Smith Chart, in impedance mode. (2)
 - (b) What is the admittance of Y_A ? What is the impedance Z_B ? (2)
 - (c) Find the VSWR of a TL with a load of Z_C (i.e. $\Gamma_L = \Gamma_C$) (1)
 - (d) Find the input impedance $Z_{in,C}$ of a TL with $l = 0.3\lambda$, terminated with Z_C . (2)

Solution:

Please see the Smith Chart

(a) See the points below.

(b) $Y_A' = 0.1 + j0.51$

$\Rightarrow Y_A = 1 + j5.1 \text{ [mS]};$

$Z_B' = 2.3 + j3.2$

$\Rightarrow Z_B = 230 + j320 [\Omega]$

The Complete Smith Chart

Black Magic Design

(c) $VSWR = 12.8$

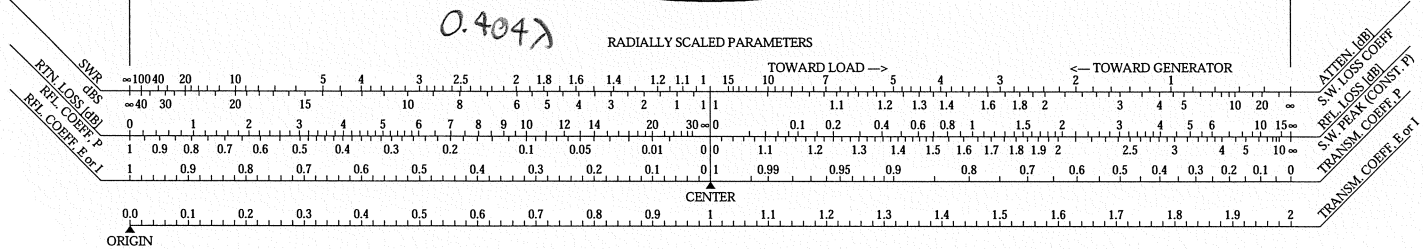
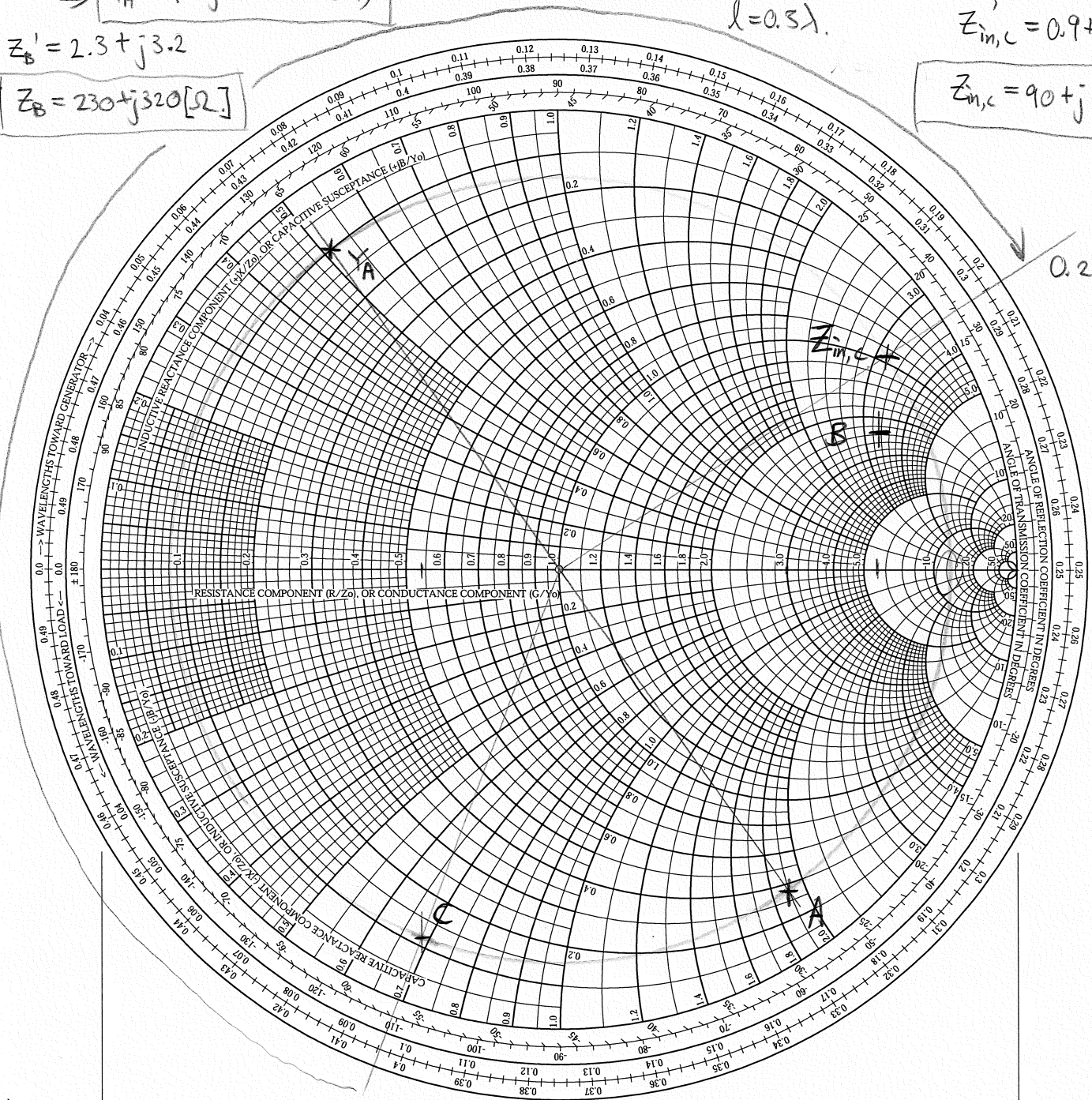
(d) $0.404\lambda + 0.3\lambda = 0.704\lambda$

$\rightarrow 0.204\lambda$

$l = 0.5\lambda$

$Z_{in,c} = 0.9 + j3.15$

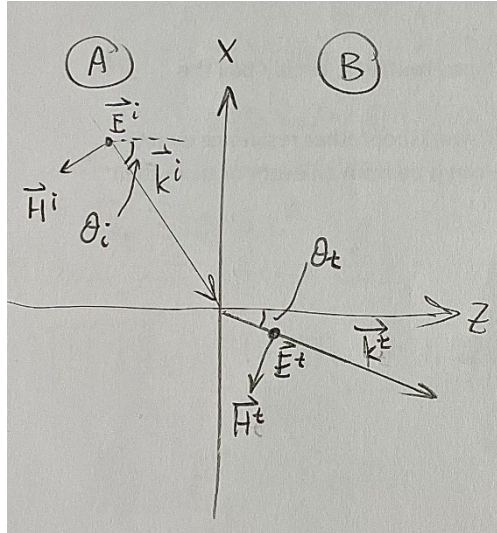
$Z_{in,c} = 90 + j315 [\Omega]$



1. **(23 Marks)** A 10 GHz, TE-polarized plane wave, with amplitude 15 [V/m], is incident at 60° from an air-material interface, from the side of air. The incident plane wave travels in the xz-plane, in the -x and +z direction. The air-material interface is on the xy-plane (i.e. the plane $z = 0$). The material has $(\mu_r, \epsilon_r, \sigma) = (9, 1, 0)$.
- Defining the material as A and air as B, find $n_A, n_B, \eta_A, \eta_B, k_A$ and k_B . (6)
 - Use Snell's law to find the angle of the transmitted wave θ_t . (2)
 - What is the direction of travel of the transmitted wave? Answer by indicating whether each of k_x, k_y , and k_z is positive, negative or zero (2)
 - Write the incident electric field phasor (4)
 - Write the transmitted electric field phasor (4)
 - If the incident wave carries 1W energy, what is the power of the transmitted wave? (3)
 - If the incident angle is swept 0° to just less than 90° , will we have an angle where there is total reflection and/or total transmission? Explain your answer. (2)

Solution:

The picture of plane wave incidence is drawn in the figure below.



(a)

$$n_A = 1; \eta_A = \eta_0 = 120\pi \Omega; k_A = \frac{\omega}{c} = \frac{2\pi \times 10^{10}}{3 \times 10^8} = \frac{200\pi}{3}$$

$$n_B = \sqrt{\mu_r \epsilon_r} = 3; \eta_B = \eta_0 \sqrt{\frac{\mu_r}{\epsilon_r}} = 3\eta_0 = 360\pi \Omega; k_B = \frac{\omega}{c} n_B = \frac{600\pi}{3}$$

(b) From the figure, $\theta_i = -60^\circ$.

$$n_A \sin \theta_i = n_B \sin \theta_t \Rightarrow \theta_t = \sin^{-1} \left(\frac{n_A}{n_B} \sin \theta_i \right) = \sin^{-1} \left(\frac{1}{3} \left(-\frac{\sqrt{3}}{2} \right) \right) = -16.78^\circ$$

(c) The transmitted wave is in the +z and -x direction, i.e. $k_x < 0, k_y = 0, k_z > 0$.

(d)

$$k_{xA} = k_A \sin(-60^\circ) = \frac{200\pi}{3} \left(-\frac{\sqrt{3}}{2} \right) = -\frac{100\pi}{\sqrt{3}} \text{ [rad/m]}$$

$$k_{zA} = k_A \cos(-60^\circ) = \frac{200\pi}{3} \left(\frac{1}{2} \right) = \frac{100\pi}{3} \text{ [rad/m]}$$

$$\mathbf{E}^i = 15 e^{-j(k_{xA}x + k_{zA}z)} \hat{\mathbf{y}} = 15 e^{-j\left(-\frac{100\pi}{\sqrt{3}}x + \frac{100\pi}{3}z\right)} \hat{\mathbf{y}} \text{ [V/m]}$$

(e) We first find the relevant cosine terms: $\cos \theta_i = 0.5$; $\cos \theta_t = 0.9574$.

The TE transmission coefficient is

$$\tau_{TE} = \frac{2\eta_B \cos \theta_i}{\eta_B \cos \theta_i + \eta_A \cos \theta_t} = \frac{2(3)(0.5)}{3(0.5) + (1)(0.9574)} = 1.22$$

The wave vector components for the transmitted wave are:

$$k_{xB} = k_B \sin \theta_t = -181.3 \text{ [rad/m]}; k_{zB} = k_B \cos \theta_t = 601.3 \text{ [rad/m]}$$

The transmitted electric field is

$$\mathbf{E}^t = \tau_{TE} 15 e^{-j(k_{xB}x + k_{zB}z)} = 18.3 e^{-j(-181.3x + 601.3z)} \hat{\mathbf{y}} \text{ [V/m]}$$

(f) The transmitted wave's power is

$$P_t = TP_i = |\tau_{TE}|^2 \left(\frac{\eta_A \cos \theta_t}{\eta_B \cos \theta_i} \right) P_i = (1.22)^2 \left(\frac{1}{3} \right) \left(\frac{0.9574}{0.5} \right) (1) = 0.9500 \text{ [W]}$$

(g) There will be a Brewster angle (for TE waves, within a magnetic material) where $\eta_A < \eta_B$ and $\cos \theta_t > \cos \theta_i$, but $\eta_A \cos \theta_t = \eta_B \cos \theta_i$, so that the reflection coefficient Γ_{TE} goes to zero. At that incident angle, the wave will be totally transmitted into the material.

2. **(15 Marks)** A lossless air transmission line has a characteristic impedance of 50Ω and a length of 150 mm. The transmission line is terminated with a 100Ω resistor in series with a 10 pF capacitor. An RF wave of 3.0 GHz travels along the transmission line.

- (a) Find Z_L and Γ_L along the transmission line (4)
 (b) Find α and β along the transmission line (3)
 (c) Write the current phasor as a function of distance along the line. Substitute known numerical values into the equation and simplify as much as possible. (4)

Now the input of the transmission line is connected to a microwave source, modelled with an ideal 3V voltage source in series with a generator resistance of $R_g = 75\Omega$.

- (d) Write the voltage phasor as a function of distance along the line. Substitute known numerical values into the equation and simplify as much as possible. (4)

Solution:

- (a) The load is a resistor in series with a capacitor:

$$Z_L = R_L + \frac{1}{j\omega C_L} = 100 - j \frac{1}{(0.06\pi)} = 100 - j5.305 [\Omega]$$

The corresponding reflection coefficient is:

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{100 - j5.305 - 50}{100 - j5.305 + 50} = 0.3342 - j0.0235$$

- (b)

$$\alpha = 0 \text{ (lossless TL); } \beta = \frac{\omega}{v_p} = \frac{2\pi(3 \times 10^9)}{3 \times 10^8} = 20\pi = 62.83 [\text{rad/m}]$$

- (c) The current along the line is:

$$\begin{aligned} I(z) &= \frac{V_0^+}{Z_0} e^{-j\beta z} (1 - \Gamma_L e^{-j2\beta z}) \\ &= \frac{V_0^+}{50} e^{-j20\pi z} (1 - (0.3342 - j0.0235)e^{-j40\pi z}) [\text{A}] \end{aligned}$$

- (d) The voltage along the line is:

$$V(z) = \frac{V_g Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l} e^{-j\beta z}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) (1 + \Gamma_L e^{j2\beta z})$$

We now find the known parameters:

$$\beta l = (20\pi)(150 \times 10^{-3}) = 3\pi \Rightarrow e^{-j\beta l} = e^{-j\pi} = -1$$

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} = \frac{75 - 50}{75 + 50} = 0.2$$

$$\begin{aligned} \frac{V_g Z_0}{R_g + Z_0} \left(\frac{e^{-j\beta l}}{1 - \Gamma_L \Gamma_G e^{-j2\beta l}} \right) &= \left(\frac{3(50)}{125} \right) \left(\frac{-1}{1 - (0.3342 - j0.0235)(0.2)(1)} \right) \\ &= -1.286 + j0.007 \end{aligned}$$

Substituting these into the equation for voltage gives

$$V(z) = (-1.286 + j0.007)e^{-j20\pi z} (1 + (0.3342 - j0.0235)e^{j40\pi z}) [\text{V}]$$

3. (7 Marks) Three loads generate reflection coefficients of $\Gamma_A = +0.3 - j0.8$, $\Gamma_B = -0.5 - j0.7$, $\Gamma_C = -0.7 + j0.3$ when connected to a $100\ \Omega$ transmission line.
- (a) Plot the impedances as A B and C on the same Smith Chart, in impedance mode. (2)
 - (b) What is the admittance of Y_A ? What is the impedance Z_B ? (2)
 - (c) Find the VSWR of a TL with a load of Z_C (i.e. $\Gamma_L = \Gamma_C$) (1)
 - (d) Find the input impedance $Z_{in,C}$ of a TL with $l = 0.3\lambda$, terminated with Z_C . (2)

Solution:

Please see the Smith Chart

(a) See the points A, B + C below.

(b). $Y_A' = 0.12 + j0.68$

$Y_A = 1.2 + j6.8 \text{ [mS]}.$

$Z_B' = 0.1 - j0.51$

$Z_B = 10 - j51 \text{ [}\Omega\text{]}.$

The Complete Smith Chart

Black Magic Design

(c). $V_{SWR} = 7.3.$

(d) $0.033\lambda + 0.3\lambda$
 $= 0.333\lambda.$

$Z_{in,c} = 0.51 - j1.62$

$Z_{in,c} = 51 - j162 \text{ }\Omega$

