

Test 1 Solution

7 Oct 2022: 6:00pm~6:40pm (YEUNG LT-7)

- As a closed-book test, you are not allowed to use any written materials online/offline.
- Write your answers on the question papers directly.
- Non-graphical calculators are allowed. Laptops/phones must be switched off.
- Unless otherwise specified, SI units are assumed in the questions and 3 significant figures should be used in numerical answers.
- The total score is 100. There are a total of **5 pages**.

The following formulas may or may not be useful.

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} & \mathbf{E}(x, y, z) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{a}}_R & d\mathbf{B} &= \frac{\mu_0}{4\pi} \frac{Id\mathbf{L} \times \hat{\mathbf{a}}_R}{R^2} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} & \oint_S \mathbf{D} \cdot d\mathbf{S} &= Q & \oint_L \mathbf{H} \cdot d\mathbf{L} &= I \\ \nabla \cdot \mathbf{D} &= \rho_v & \mathbf{D} &= \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E} & \mathbf{B} &= \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 \mu_r \mathbf{H} \\ \nabla \cdot \mathbf{B} &= 0 & \epsilon_0 &= 8.854 \times 10^{-12} \text{ (F/m)} & \mu_0 &= 4\pi \times 10^{-7} \text{ (H/m)}\end{aligned}$$

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{\mathbf{a}}_\rho \quad \mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{\mathbf{a}}_r + \sin\theta \hat{\mathbf{a}}_\theta) \quad \mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\mathbf{a}}_\phi \quad \mathbf{B} = \mu_0 n I \hat{\mathbf{a}}_z$$

$$\begin{aligned}V_{AB} &= -\int_B^A \mathbf{E} \cdot d\mathbf{L} & \nabla^2 V &= -\frac{\rho_v}{\epsilon} & V &= -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S} \\ \nabla^2 &= \nabla \cdot \nabla\end{aligned}$$

$$\begin{aligned}\nabla &= \frac{\partial}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{\rho} \begin{vmatrix} \hat{\mathbf{a}}_\rho & \rho \hat{\mathbf{a}}_\phi & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin\theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix} \\ \nabla f &= \frac{\partial f}{\partial r} \hat{\mathbf{a}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{a}}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi & \nabla \times \mathbf{F} &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin\theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin\theta F_\phi \end{vmatrix}\end{aligned}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{\partial}{\partial \phi} F_\phi + \frac{\partial}{\partial z} (\rho F_z) \right]$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r} (r^2 \sin\theta F_r) + \frac{\partial}{\partial \theta} (r \sin\theta F_\theta) + \frac{\partial}{\partial \phi} (r F_\phi) \right]$$

Question 1 (20 points)

Fill in the blanks.

- (a) Electromagnetics are governed by a set of four equations that are called “_____ Equations”.
- (b) An atom contains a positively charged _____ and negatively charged _____.
- (c) A scalar field is mathematically a _____ of position and time.
- (d) A unit vector must have a length of _____.

Solution: Maxwell’s//nuclei//electrons//function//1.

Question 2 (20 points)

In rectangular coordinates, consider a point $P: (x, y, z) = (0, 3, 4)$. The unit vectors are denoted as $\hat{x}$, $\hat{y}$, and $\hat{z}$.

- (a) Express P in the spherical coordinates (r, θ, ϕ) .
(Hint: Remember that θ and ϕ are the polar angle and the azimuthal angle, respectively.)

Solution:

$$r = \sqrt{3^2 + 4^2} = 5$$

$$\cos \theta = \frac{4}{5} \Rightarrow \theta = 36.9^\circ$$

$$\tan \phi = \frac{3}{0} \Rightarrow \phi = 90^\circ$$

Other versions would have $r = 50$ and/or $\theta = 53$ degrees. Answers in radians are also acceptable.

- (b) At P , express $\hat{r}$ of the spherical coordinate system in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$.

Solution:

$$\hat{r} = (3\hat{y} + 4\hat{z})/r = \frac{3}{5}\hat{y} + \frac{4}{5}\hat{z}$$

(Other versions would have the components swapped).

- (c) At P , express $\hat{\theta}$ of the spherical coordinate system in terms of $\hat{x}$, $\hat{y}$, and $\hat{z}$.

Solution:

Note that $\hat{\theta}$ can be drawn on the yz -plane. It is also perpendicular to $\hat{r}$. Thus,

$$\hat{\theta} = +\frac{4}{5}\hat{y} - \frac{3}{5}\hat{z}$$

Or formally one can write:

$$\hat{\theta} = \cos \theta \hat{y} - \sin \theta \hat{z} = +\frac{4}{5}\hat{y} - \frac{3}{5}\hat{z}$$

(Other versions would have the components swapped).

Question 3 (20 points)

In the cylindrical coordinate system, a field is given as:

$$\mathbf{F}(\rho, \phi, z) = 2 \exp\left(-\frac{\rho}{a}\right) \hat{\rho} + \frac{\rho}{a} \hat{\phi}$$

for some positive constant a (in meters). At point $(\rho, \phi, z) = (a, 0, 0)$:

(a) Find $\text{div } \mathbf{F}$.

Solution

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{1}{\rho} \left[2 \frac{d}{d\left(\frac{\rho}{a}\right)} \left(\frac{\rho}{a} \exp\left(-\frac{\rho}{a}\right) \right) \right] \\ &= \frac{2}{\rho} \left[\exp\left(-\frac{\rho}{a}\right) - \frac{\rho}{a} \exp\left(-\frac{\rho}{a}\right) \right] \end{aligned}$$

At $(\rho, \phi, z) = (a, 0, 0)$, we have

$$\nabla \cdot \mathbf{F} = \frac{2}{a} \left(\frac{1}{e} - \frac{1}{e} \right) = 0$$

(Other versions would have different coefficients, but the result may be different).

(b) Find $\text{curl } \mathbf{F}$.

Solution:

$$\begin{aligned} \nabla \times \mathbf{F} &= \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & 0 & 0 \\ 2 \exp\left(-\frac{\rho}{a}\right) & \rho \frac{\rho}{a} & 0 \end{vmatrix} \\ &= \frac{1}{\rho} \frac{\partial(\rho^2/a)}{\partial \rho} \hat{z} \\ &= \frac{2}{a} \hat{z} \end{aligned}$$

at any point.

(Other versions may have different coefficients, but it only changes the magnitude).

Question 4 (20 points)

Consider a ball of charge with a uniform charge density of $+4 \text{ (C/m}^3\text{)}$. The ball has a radius b of 3 mm. It is centered at the origin.

(a) Find the total charge.

Solution

$$Q = 4 \times \frac{4\pi}{3} b^3 = 452 \text{ nC}$$

(b) Find, or estimate, the electric field measured at $x = 22 \text{ m}$ and $y = z = 0$.

Solution

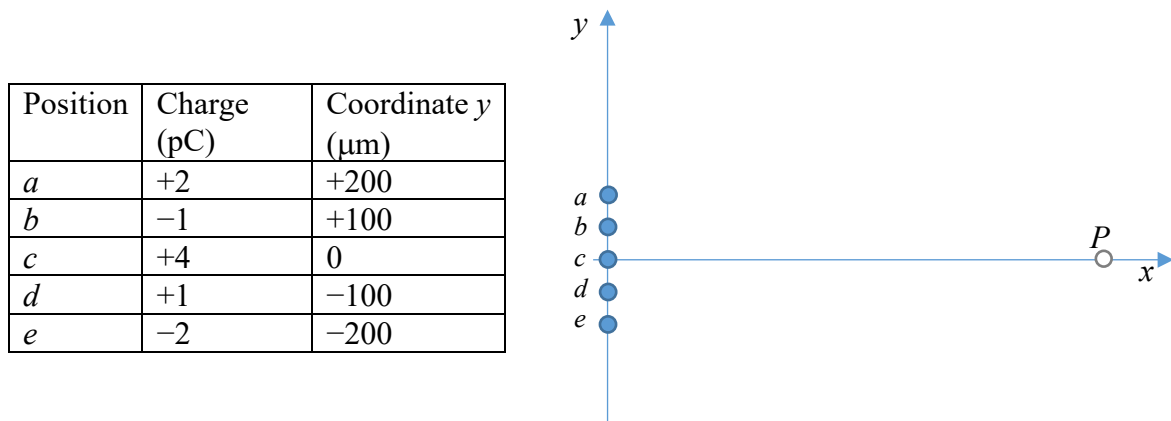
The magnitude of the electric field at $(x, y, z) = (22, 0, 0)$ is

$$E = \frac{Q}{4\pi\epsilon_0 R^2} = 8.39 \text{ (V/m)}$$

where the direction is towards $+x$.

Question 5 (20 points)

Consider 5 point charges positioned on the y -axis as follows:



The electric field $\mathbf{E}$ is measured at position P : $(x, y) = (3 \text{ m}, 0)$.

(a) Find E (i.e. magnitude of $\mathbf{E}$). Express the answer in 2 significant figures. State the reasoning.

Solution

For the charges spans only $\pm 0.2 \text{ mm}$, which is much smaller than 3 m . The resultant field is approximately the same as that by lumping all the 5 charges into one with a total charge $Q = 4 \text{ pC}$.

The magnitude of the field is simply:

$$\frac{4 \text{ pC}}{4\pi\epsilon_0(3 \text{ m})^2} = 3.99 \text{ mV/m}$$

(Other versions would have different total charge/distance).

(b) Find E_y (i.e. the y -component of $\mathbf{E}$). Express the answer in 2 significant figures. State the reasoning.

Solution

The pair at a and e form a dipole with moment $(2 \text{ pC})(400 \mu\text{m})$ that points towards $+z$. The pair at b and d form a dipole with moment $(1 \text{ pC})(200 \mu\text{m})$ that points towards $-z$. The charge at c does not contribute to any y -component of the electric field at P . Thus, E_y at P comes from an effective dipole of moment $(1 \text{ pC})(600 \mu\text{m})$ that points towards $+z$.

Consider the equivalent dipole with $\pm 1 \text{ pC}$ placed at $z = \pm 300 \mu\text{m}$. Such an effective dipole spans a total angle $2\alpha \sim 2(300 \mu\text{m})/3\text{m}$. As a result,

$$\begin{aligned} E_y &= -\frac{1 \text{ pC}}{4\pi\epsilon_0(3 \text{ m})^2} \times 2 \times \sin \alpha \\ &\approx -\frac{1 \text{ pC}}{4\pi\epsilon_0(3 \text{ m})^2} \times 2\alpha \\ &= \frac{-1 \text{ pC} \times 600 \mu\text{m}}{4\pi\epsilon_0(3 \text{ m})^3} \\ &= -2 \times 10^{-7} \text{ V/m} \end{aligned}$$

Alternatively, the dipole formula for a moment of $600 \text{ pC}\cdot\mu\text{m}$ can be used at $\theta = \pi/2$.

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