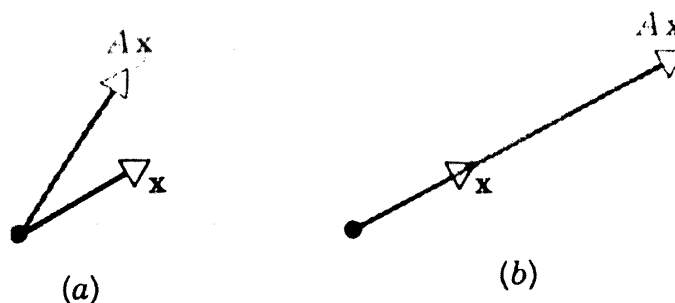


## MA2001 Eigenvalues and Eigenvectors

### 1. Introduction

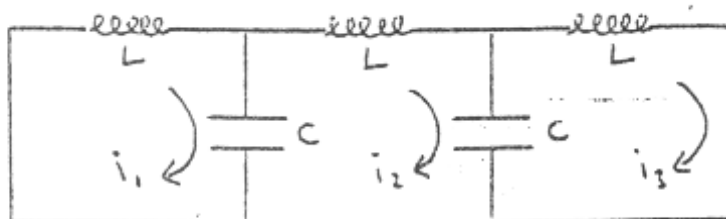
If  $A$  is an  $3 \times 3$  matrix and  $\vec{x}$  is a vector in  $R^3$ , then there is usually no general geometric relationship between the vector  $\vec{x}$  and the vector  $A\vec{x}$ , see figure (a). However, there are often certain nonzero vectors  $\vec{x}$  such that  $\vec{x}$  and  $A\vec{x}$  are scalar multiples of one another, that is,  $A\vec{x} = \lambda\vec{x}$  for some real number  $\lambda$ , see figure (b). Such vectors arise naturally in the study of vibrations, electrical systems, genetics, chemical reactions, quantum mechanics, mechanical stress, economics, and geometry.



Let  $A$  be an  $n \times n$  square matrix and consider the equation  $A\vec{x} = \lambda\vec{x}$ . A value of  $\lambda$  for which  $A\vec{x} = \lambda\vec{x}$  has a non-trivial solution  $\vec{x} \neq \vec{0}$  is called an eigenvalue (or characteristic value) of  $A$  and the corresponding solution  $\vec{x} \neq \vec{0}$  is an eigenvector (or characteristic vector). The set of all eigenvalues of  $A$  is known as the spectrum of  $A$ .

### Example – Electric Circuit

Consider the electrical circuit



The currents flowing in the loops satisfy

$$L \frac{di_1}{dt} + \frac{1}{C} \int (i_1 - i_2) dt = 0 \Rightarrow LCi_1'' + i_1 - i_2 = 0$$

$$L \frac{di_2}{dt} + \frac{1}{C} \int (i_2 - i_3) dt + \frac{1}{C} \int (i_2 - i_1) dt = 0 \Rightarrow LCi_2'' + 2i_2 - i_3 - i_1 = 0$$

$$L \frac{di_3}{dt} + \frac{1}{C} \int (i_3 - i_2) dt = 0 \Rightarrow LCi_3'' + i_3 - i_2 = 0$$

Or

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = -LC \begin{pmatrix} i_1'' \\ i_2'' \\ i_3'' \end{pmatrix}$$

Look for solutions which are sinusoidal and all have the same frequency, that is,

$$i_1 = I_1 \sin(\omega t + \theta_1), i_2 = I_2 \sin(\omega t + \theta_2), i_3 = I_3 \sin(\omega t + \theta_3).$$

Then  $i_1'' = -\omega^2 i_1$ ,  $i_2'' = -\omega^2 i_2$  and  $i_3'' = -\omega^2 i_3$  and the problem becomes

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = \omega^2 LC \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix}, \text{ which is an eigenvalue problem with } \lambda = \omega^2 LC$$

□

## 2. Determination of Eigenvalues and Eigenvectors

$$A\vec{x} = \lambda\vec{x} \Leftrightarrow A\vec{x} = \lambda I\vec{x} \Leftrightarrow A\vec{x} - \lambda\vec{x} = \vec{0} \Leftrightarrow (A - \lambda I)\vec{x} = \vec{0}.$$

For this equation  $(A - \lambda I)\vec{x} = \vec{0}$  to have non-trivial solutions  $\vec{x} \neq \vec{0}$  we must have  $(A - \lambda I)^{-1}$  not existing,

$$\text{for otherwise } (A - \lambda I)\vec{x} = \vec{0} \Rightarrow (A - \lambda I)^{-1}(A - \lambda I)\vec{x} = (A - \lambda I)^{-1}\vec{0} \Rightarrow I\vec{x} = \vec{0} \Rightarrow \vec{x} = \vec{0}$$

is the unique solution. Hence  $\lambda$  is an eigenvalue if and only if  $(A - \lambda I)^{-1}$  does not exist if and only if  $\det(A - \lambda I) = 0$ .

Consider  $\det(A - \lambda I)$ . After expansion  $\det(A - \lambda I)$  is a polynomial of degree  $n$  in  $\lambda$ , known as the characteristic polynomial of  $A$ . And  $\det(A - \lambda I) = 0$  is the characteristic equation of  $A$ .

Clearly, if multiplicity roots are counted, an  $n \times n$  square matrix  $A$  has  $n$  eigenvalues including real and complex eigenvalues.

### Example

Find eigenvalues of  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$  and the corresponding eigenvectors.

Solution:

$$A - \lambda I = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{pmatrix}.$$

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 4 \\ 2 & 3 - \lambda \end{vmatrix} = (1 - \lambda)(3 - \lambda) - 8 = \lambda^2 - 4\lambda + 3 - 8 = \lambda^2 - 4\lambda - 5 = (\lambda - 5)(\lambda + 1).$$

Hence the eigenvalues are  $\lambda = 5$  and  $\lambda = -1$

To find the corresponding eigenvectors we solve:

When  $\lambda = 5$

$$\begin{pmatrix} -4 & 4 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} -4 & 4 & | & 0 \\ 2 & -2 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \Rightarrow x_1 = x_2$$

Let  $x_2 = k \neq 0$ , then the corresponding eigenvectors of  $A$  for  $\lambda = 5$  are  $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}, k \neq 0$ .

And one of the eigenvectors for  $k = 1$  is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

When  $\lambda = -1$

$$\begin{pmatrix} 2 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \Leftrightarrow \begin{pmatrix} 2 & 4 & 0 \\ 2 & 4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_1 = -2x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} -2 \\ 1 \end{pmatrix}, k \neq 0.$$

Let  $k = 1$ , then  $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$  is one of the corresponding eigenvectors of  $A$  for  $\lambda = -1$ .

□

### Example

Find eigenvalues of  $A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$  and the corresponding eigenvectors.

Solution:

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(2-\lambda) - (1-\lambda) - (1-\lambda) \\ &= (1-\lambda)(\lambda^2 - 3\lambda + 2 - 2) = \lambda(1-\lambda)(\lambda-3) \end{aligned}$$

When  $\lambda = 0$

$$\begin{aligned} \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 &\Leftrightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\Rightarrow x_2 = x_3, x_1 = x_2 \Rightarrow x_1 = x_2 = x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, k \neq 0 \end{aligned}$$

Let  $k = 1$ , then one of the eigenvectors for  $\lambda = 0$  is  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

When  $\lambda = 1$

$$\begin{aligned} \begin{pmatrix} 0 & -1 & 0 \\ -1 & 1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 &\Rightarrow \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\ &\Rightarrow x_2 = 0, x_1 = x_2 - x_3 = -x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, k \neq 0 \end{aligned}$$

One of the eigenvectors for  $\lambda = 1$  is  $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ .

When  $\lambda = 3$

$$\begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} -2 & -1 & 0 & | & 0 \\ -1 & -1 & -1 & | & 0 \\ 0 & -1 & -2 & | & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & -1 & -1 & | & 0 \\ -2 & -1 & 0 & | & 0 \\ 0 & -1 & -2 & | & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} -1 & -1 & -1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & -1 & -2 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow x_2 = -2x_3, \quad x_1 = -x_2 - x_3 = x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, k \neq 0$$

One of the eigenvectors for  $\lambda = 3$  is  $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ .

□

### Example

Find eigenvalues of  $A = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}$  and the corresponding eigenvectors.

Solution:

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & -1 \\ 4 & 1 - \lambda \end{vmatrix} = \lambda^2 - 2\lambda + 5 \Rightarrow \lambda = 1 \pm 2i$$

For  $\lambda = 1 + 2i$ , we have

$$\begin{pmatrix} -2i & -1 \\ 4 & -2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -2i & -1 & | & 0 \\ 4 & -2i & | & 0 \end{pmatrix} \xrightarrow{\frac{1}{-2i}r_1} \begin{pmatrix} 1 & \frac{1}{2i} & | & 0 \\ 4 & -2i & | & 0 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{i}{2} & | & 0 \\ 4 & -2i & | & 0 \end{pmatrix} \xrightarrow{r_2 - 4r_1} \begin{pmatrix} 1 & -\frac{i}{2} & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \Rightarrow$$

$$x_1 = \frac{i}{2}x_2 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{i}{2}k \\ k \end{pmatrix} = k \begin{pmatrix} \frac{i}{2} \\ 1 \end{pmatrix}, k \neq 0$$

Notice  $\frac{1}{2i} = \frac{-i}{-2i^2} = -\frac{i}{2}$ .

An eigenvector for  $\lambda = 1 + 2i$  is  $\begin{pmatrix} \frac{i}{2} \\ 1 \end{pmatrix}$ .

For  $\lambda = 1 - 2i$ , we have

$$\begin{pmatrix} 2i & -1 \\ 4 & 2i \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 2i & -1 & | & 0 \\ 4 & 2i & | & 0 \end{pmatrix} \xrightarrow{\frac{1}{2i}r_1} \begin{pmatrix} 1 & -\frac{1}{2i} & | & 0 \\ 4 & 2i & | & 0 \end{pmatrix} = \begin{pmatrix} 1 & \frac{i}{2} & | & 0 \\ 4 & 2i & | & 0 \end{pmatrix} \xrightarrow{r_2 - 4r_1} \begin{pmatrix} 1 & \frac{i}{2} & | & 0 \\ 0 & 0 & | & 0 \end{pmatrix} \Rightarrow x_1 = -\frac{i}{2}x_2$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -\frac{i}{2}k \\ k \end{pmatrix} = k \begin{pmatrix} -\frac{i}{2} \\ 1 \end{pmatrix}, k \neq 0$$

Notice  $-\frac{1}{2i} = -\frac{i}{-2i^2} = -\frac{i}{-2} = \frac{i}{2}$ .

An eigenvector for  $\lambda = 1 - 2i$  is  $\begin{pmatrix} -\frac{i}{2} \\ 1 \end{pmatrix}$ .

□

Example

Find eigenvalues of  $A = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}$  and the corresponding eigenvectors.

Solution:

$$|A - \lambda I| = \begin{vmatrix} 5 - \lambda & 3 \\ -3 & -1 - \lambda \end{vmatrix} = (\lambda - 2)^2, \lambda = 2 \text{ is the only eigenvalue.}$$

$$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}, k \neq 0$$

□

3 Eigenvalue ResultsTheorem

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Proof:

The proof is by induction on the number of distinct eigenvalues.

Any single eigenvector by itself, being nonzero, is linearly independent.

Assume any  $k-1$  eigenvectors associated with  $k-1$  distinct eigenvalues are linearly independent.

Suppose  $\vec{v}_1, \dots, \vec{v}_k$  are eigenvectors associated, respectively, with  $k$  distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ .

Consider  $c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$ , claim  $c_1 = \dots = c_k = 0$ .

Suppose not, then without loss of generality suppose, say  $c_1 \neq 0$ .

Then

$$\begin{aligned} (A - \lambda_1 I)(c_1 \vec{v}_1 + \dots + c_k \vec{v}_k) &= (A - \lambda_1 I)c_1 \vec{v}_1 + \dots + (A - \lambda_1 I)c_k \vec{v}_k = (A - \lambda_1 I)\vec{0} \\ \Rightarrow (Ac_1 \vec{v}_1 - \lambda_1 c_1 \vec{v}_1) &+ (Ac_2 \vec{v}_2 - \lambda_1 c_2 \vec{v}_2) + \dots + (Ac_k \vec{v}_k - \lambda_1 c_k \vec{v}_k) = \vec{0} \\ \Rightarrow (c_1 A \vec{v}_1 - c_1 \lambda_1 I \vec{v}_1) &+ (c_2 A \vec{v}_2 - \lambda_1 c_2 \vec{v}_2) + \dots + (c_k A \vec{v}_k - c_k \lambda_1 I \vec{v}_k) = \vec{0} \\ \Rightarrow (c_1 \lambda_1 \vec{v}_1 - c_1 \lambda_1 \vec{v}_1) &+ (c_2 \lambda_2 \vec{v}_1 - c_2 \lambda_1 \vec{v}_1) \dots + (c_k \lambda_k \vec{v}_1 - c_k \lambda_1 \vec{v}_1) = \vec{0} \\ \Rightarrow c_2 (\lambda_2 - \lambda_1) \vec{v}_2 &\dots + c_k (\lambda_k - \lambda_1) \vec{v}_k = \vec{0} \end{aligned}$$

By induction hypothesis  $\vec{v}_2, \dots, \vec{v}_k$  are linearly independent, so  $c_2 (\lambda_2 - \lambda_1) = \dots = c_k (\lambda_k - \lambda_1) = 0$ .

Then  $\lambda_2 - \lambda_1 \neq 0, \dots, \lambda_k - \lambda_1 \neq 0 \Rightarrow c_2 = \dots = c_k = 0$ . It follows that  $c_1 \vec{v}_1 = \vec{0}$ .

$c_1 \vec{v}_1 = \vec{0} \Rightarrow c_1 = 0$ . It contradicts to  $c_1 \neq 0$ , implying  $\vec{v}_1, \dots, \vec{v}_k$  are linearly independent.

□

It is also possible to have a repeated eigenvalue which does lead to more than one linearly independent eigenvector.

Example

Let  $J_n$ ,  $n = 2, 3, \dots$  be a square matrix with all its entries being 1, that is,

$$J_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

(a) Find all eigenvalues of  $J_n$ .

(b) Find  $n$  linearly independent eigenvectors.

Solution:

Observe that  $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \times 1 + \cdots + 1 \times 1 \\ \vdots \\ 1 \times 1 + \cdots + 1 \times 1 \end{pmatrix} = \begin{pmatrix} n \\ \vdots \\ n \end{pmatrix} = n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$  so  $n$  is an eigenvalue of  $J_n$

and one of its eigenvector is  $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$

In addition, see that  $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \times 1 - 1 \times 1 \\ \vdots \\ 1 \times 1 - 1 \times 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$

So 0 is also an eigenvalue of  $J_n$ ,  $n = 2, 3, \dots$  and one of its eigenvector is  $\begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$

For  $\lambda = 0$ ,  $J_n - 0I_n = J_n$ .

Solve  $\left( \begin{array}{cccc|c} 1 & 1 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 \end{array} \right) \sim \left( \begin{array}{cccc|c} 1 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{array} \right).$

So  $x_1 + x_2 + \cdots + x_n = 0$ . Then  $x_n = -x_1 - \cdots - x_{n-1}$ .

Let  $x_1 = a_1, x_2 = a_2, \dots, x_{n-1} = a_{n-1}$  where  $a_1, a_2, \dots, a_{n-1}$  are arbitrary real numbers.

Then

$$\begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ -x_1 - \cdots - x_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ -a_1 - \cdots - a_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 \\ 0 \\ \vdots \\ 0 \\ -a_1 \end{pmatrix} + \begin{pmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \\ -a_2 \end{pmatrix} + \cdots + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{n-1} \\ -a_{n-1} \end{pmatrix}$$

$$= a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}. \text{ Observe that}$$

$$a_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + \cdots + a_{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ -a_1 - a_2 - \cdots - a_{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a_1 = 0, a_2 = 0, \dots, a_{n-1} = 0$$

It follows that  $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$  are linearly independent.

Claim that  $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$  are linearly independent.

Consider  $x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + \cdots + x_{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} + x_n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \dots \dots (A).$

Both sides of (A) time  $J_n$ , then  $x_1 J_n \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + x_2 J_n \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + \cdots + x_{n-1} J_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} + x_n J_n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = J_n \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$

Since  $J_n \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} = \vec{0}, J_n \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} = \vec{0}, \dots, J_n \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} = \vec{0}$ , it follows that  $x_n n \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$  then  $x_n = 0$ .

$$\text{Thus, } x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} + \dots + x_{n-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

As  $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}$  are linearly independent,  $x_1 = x_2 = \dots = x_{n-1} = 0$

It follows that  $x_1 = x_2 = \dots = x_{n-1} = x_n = 0$  and  $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ -1 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$  are linearly independent.

□

### Theorem

If  $A$  is a real symmetric matrix, then

- (a) the eigenvalues are all real (but not necessarily distinct).
- (b) eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof:

(a)

Suppose  $A\vec{v} = \lambda\vec{v}$  where  $\vec{v} \neq \vec{0}$ .

Consider  $\overline{\vec{v}^t A\vec{v}} = \overline{\vec{v}^t \lambda\vec{v}} = \overline{\lambda\vec{v}^t \vec{v}} = \overline{\lambda(\overline{x_1 x_1} + \dots + \overline{x_n x_n})}$  where  $\overline{\vec{v}}^t = (\overline{x_1}, \dots, \overline{x_n})$  if  $\vec{v} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ .

Then  $\overline{\vec{v}^t A\vec{v}} = \overline{\vec{v}^t A\vec{v}} = \overline{\vec{v}^t A\vec{v}} = \overline{\vec{v}^t A\vec{v}}$ .

Observe that  $\vec{v}^t \overline{\vec{v}}$  is a  $1 \times 1$  matrix, so  $\vec{v}^t \overline{A\vec{v}} = \left( \vec{v}^t \overline{A\vec{v}} \right)^t = \overline{\vec{v}^t A\vec{v}} = \overline{\vec{v}^t A\vec{v}}$ .

As a result, we have  $\overline{\vec{v}^t A\vec{v}} = \vec{v}^t A\vec{v}$ . It follows that  $\vec{v}^t A\vec{v}$  is real.

We conclude that  $\lambda = \frac{\overrightarrow{v}^t A \overrightarrow{v}}{\overrightarrow{v}^t \overrightarrow{v}}$  is real (note that  $\overrightarrow{x_1}^t \overrightarrow{x_1} + \dots + \overrightarrow{x_n}^t \overrightarrow{x_n}$  is real).

(b)

Suppose  $A\overrightarrow{v_1} = \lambda_1 \overrightarrow{v_1}, A\overrightarrow{v_2} = \lambda_2 \overrightarrow{v_2}$  where  $\lambda_1 \neq \lambda_2, \overrightarrow{v_1}, \overrightarrow{v_2} \neq \vec{0}$ .

$$\lambda_1 \overrightarrow{v_1}^t \overrightarrow{v_2} = (\lambda_1 \overrightarrow{v_1})^t \overrightarrow{v_2} = (A\overrightarrow{v_1})^t \overrightarrow{v_2} = (\overrightarrow{v_1}^t A^t) \overrightarrow{v_2} = (\overrightarrow{v_1}^t A) \overrightarrow{v_2} = \overrightarrow{v_1}^t (A\overrightarrow{v_2}) = \overrightarrow{v_1}^t (\lambda_2 \overrightarrow{v_2}) = \lambda_2 \overrightarrow{v_1}^t \overrightarrow{v_2}$$

$$\Rightarrow (\lambda_1 - \lambda_2) \overrightarrow{v_1}^t \overrightarrow{v_2} = 0 \Rightarrow_{\lambda_1 - \lambda_2 \neq 0} \overrightarrow{v_1}^t \overrightarrow{v_2} = 0.$$

□

### Theorem

If  $A$  is an  $n \times n$  real symmetric matrix, then it is possible to find  $n$  linearly independent eigenvectors (even though the eigenvalues may not be distinct).

Proof: Omitted.

### Example

Consider  $J_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ , which is real symmetric.

$J_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$  has eigenvalue 3 of multiplicity 1 and eigenvalue 0 of multiplicity 2, all of them are real.

For eigenvalue 3, one of its corresponding eigenvectors is  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ .

For eigenvalue 0, its independent eigenvectors are  $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ .

Observe that  $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0$ .

$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$  are linearly independent.

□

An  $n \times n$  matrix is said to be diagonalizable if there is a non-singular  $n \times n$  matrix  $P$  such that  $D = P^{-1}AP$ , where  $D$  is a diagonal matrix.

We observe that:

$$\begin{aligned}\det(D - \lambda I) &= \det(P^{-1}AP - \lambda I) = \det(P^{-1}AP - \lambda P^{-1}IP) = \det[P^{-1}(A - \lambda I)P] \\ &= \det(P^{-1})\det(A - \lambda I)\det(P) = \det(A - \lambda I)\end{aligned}$$

Thus,  $A$  and  $P^{-1}AP = D$  have the same characteristic polynomial thus the same eigenvalues. We also notice that the eigenvalues of  $D$  are just the main diagonal entries.

### Theorem

If  $n \times n$  matrix  $A$  has  $n$  linearly independent eigenvectors then it is diagonalizable.

Remark:

Let  $A$  be a  $p \times m$  matrix and  $B$  an  $m \times n$  matrix. Recall that the product of  $AB$  can be evaluated in two different ways:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pm} \end{pmatrix} = A = (\vec{a_1} | \vec{a_2} | \cdots | \vec{a_m}), \text{ where } \vec{a_1} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{p1} \end{pmatrix}, \cdots, \vec{a_m} = \begin{pmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{pm} \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = B = (\vec{b_1} | \vec{b_2} | \cdots | \vec{b_n}), \text{ where } \vec{b_1} = \begin{pmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{pmatrix}, \cdots, \vec{b_n} = \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{mn} \end{pmatrix}$$

We observe that

$$\begin{aligned}AB &= \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pm} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix} = (\vec{Ab_1} | \vec{Ab_2} | \cdots | \vec{Ab_n}) \\ &= (\vec{b_{11}a_1 + b_{21}a_2 + \cdots + b_{m1}a_m} | \vec{b_{12}a_1 + b_{22}a_2 + \cdots + b_{m2}a_m} | \cdots | \vec{b_{1n}a_1 + b_{2n}a_2 + \cdots + b_{mn}a_m})\end{aligned}$$

Proof:

Suppose  $\vec{x_1}, \cdots, \vec{x_n}$  are linearly independent eigenvectors with corresponding eigenvalues  $\lambda_1, \cdots, \lambda_n$  such that  $A\vec{x_1} = \lambda_1\vec{x_1}, \cdots, A\vec{x_n} = \lambda_n\vec{x_n}$ .

Construct  $n \times n$  matrix  $P = (\vec{x_1} | \vec{x_2} | \cdots | \vec{x_n})$  with  $\vec{x_1}, \cdots, \vec{x_n}$  as its columns. Then eigenvectors of  $A$ . Then

$$\begin{aligned}
AP &= \left( A\vec{x}_1 \mid A\vec{x}_2 \mid \cdots \mid A\vec{x}_n \right) = \left( \lambda_1 \vec{x}_1 \mid \lambda_2 \vec{x}_2 \mid \cdots \mid \lambda_n \vec{x}_n \right) \\
&= \left( \lambda_1 \vec{x}_1 + 0\vec{x}_2 + \cdots + 0\vec{x}_n \mid 0\vec{x}_1 + \lambda_2 \vec{x}_2 + \cdots + 0\vec{x}_n \mid \cdots \mid 0\vec{x}_1 + \cdots + 0\vec{x}_{n-1} + \lambda_n \vec{x}_n \right), \\
&= \left( \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n \right) \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} = PD \\
, \text{ where } D &= \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} = \text{diag}(\lambda_1, \dots, \lambda_n).
\end{aligned}$$

As  $\vec{x}_1, \dots, \vec{x}_n$  are linearly independent,  $P^{-1}$  exist. Thus we have  $P^{-1}AP = P^{-1}PD = D$

### Example

Show that  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$  is diagonalizable.

Proof:

For  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ , we have eigenvalues  $\lambda_1 = 5$ ,  $\lambda_2 = -1$  (they are distinct).

For  $\lambda_1 = 5$ , one of the corresponding eigenvector is  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

For  $\lambda_2 = -1$ , one of the corresponding eigenvector is  $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ .

Let  $P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$ , as the eigenvectors of  $A$  are linearly independent,  $P^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}^{-1}$  exists and

$$P^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}. \text{ Also } P^{-1}AP = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}.$$

It concludes that  $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$  is diagonalizable.

□

### Example

Is  $A = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}$  diagonalizable?

Solution:

$$|A - \lambda I| = \begin{vmatrix} 5 - \lambda & 3 \\ -3 & -1 - \lambda \end{vmatrix} = (\lambda - 2)^2 \text{ and } \lambda = 2 \text{ is the only eigenvalue.}$$

$$\begin{pmatrix} 3 & 3 \\ -3 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k \begin{pmatrix} -1 \\ 1 \end{pmatrix}, k \neq 0$$

As there is only one linearly independent eigenvector,  $A = \begin{pmatrix} 5 & 3 \\ -3 & -1 \end{pmatrix}$  is not diagonalizable.

□

### Example

Diagonalize  $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ , if possible.

Proof:

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 3 & 3 \\ -3 & -5 - \lambda & -3 \\ 3 & 3 & 1 - \lambda \end{vmatrix} = -\lambda^3 - 3\lambda^2 + 4 = -(\lambda - 1)(\lambda + 2)^2$$

When  $\lambda = 1$

$$\begin{pmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 3 & 3 & 0 & | & 0 \\ -3 & -6 & -3 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & -3 & -3 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & | & 0 \\ 0 & -1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix}$$

$$\Rightarrow x_2 = -x_3, x_1 = -x_2 = x_3 \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, k \neq 0$$

When  $\lambda = -2$

$$\begin{pmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 3 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{pmatrix} \Rightarrow x_3 = t, x_2 = s, x_1 = -s - t$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s - t \\ s \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, t^2 + s^2 \neq 0$$

$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$  has three linearly independent eigenvectors, therefore  $A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$  is

diagonalizable and  $\begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

□

### Example

Diagonalisation is useful if it is required to raise a matrix  $A$  to a high power, that is,  $A^m$ .

If  $P^{-1}AP = D$ , then  $A = PDP^{-1}$  and

$$A^m = (PDP^{-1}) \cdots (PDP^{-1}) = PDP^{-1}PDP^{-1} \cdots PDP^{-1}PDP^{-1} = PDIDID \cdots IDIDP^{-1} = PD^mP^{-1} \text{ and}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n \end{pmatrix} = \text{diag}(\lambda_1, \dots, \lambda_n) \Rightarrow D^m = \begin{pmatrix} \lambda_1^m & 0 & \cdots & 0 \\ 0 & \lambda_2^m & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_n^m \end{pmatrix} = \text{diag}(\lambda_1^m, \dots, \lambda_n^m)$$

□

An important application of diagonalisation occurs in the solution of systems of differential equations.

### Example

Consider a system of  $n$  first order linear differential equations for the dependent variables  $x_1, x_2, \dots, x_n$  as follows.

$$\begin{cases} \frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \dots\dots (E), \text{ where the } a\text{'s are given constants. The system (E) can be written} \\ \vdots \\ \frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n \end{cases}$$

as the following matrix form  $\frac{d\vec{u}}{dt} = A\vec{u} \dots (F)$ , where  $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$  and  $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$ .

Suppose  $A$  is diagonalizable, then there exists nonsingular matrix  $P$  such that  $D = P^{-1}AP$ , where  $D$  is a diagonal matrix.

Let  $\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = P^{-1} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$ , then

$$\frac{d}{dt} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = \frac{d}{dt} P^{-1} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = P^{-1} \frac{d}{dt} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = P^{-1} A \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = DP^{-1} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = D \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}.$$

Observe that  $D$  is diagonal so each row of the system is of the form  $\frac{d}{dt} y_i(t) = d_j y_i(t)$ .

The solution of  $\frac{d}{dt} \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix} = D \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$  is easy to obtain. Once  $\begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}$  is known, then

$$\begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} = P \begin{pmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{pmatrix}.$$

□

### Example

(a) Use Gauss-Jordan elimination to compute the inverse of  $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ .

(b) Let  $A$  be a  $3 \times 3$  matrix which has eigenvalues  $-1, 1, 0$  with corresponding eigenvectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \text{ With the help of (a), find } A^{10}.$$

**Solution:**

(a)

$$\left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_2 - r_1} \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{r_1 - r_3} \left( \begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\xrightarrow{r_1 - r_2} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)$$

$$\text{So } \left( \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & -1 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right)^{-1} = \left( \begin{array}{ccc|ccc} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right).$$

(b)

$$\text{We have } A \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\begin{aligned}
A &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
\Rightarrow A^{10} &= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}^{10} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} (-1)^{10} & 0 & 0 \\ 0 & 1^{10} & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

□

### Example

(a) Let  $B$  be a  $3 \times 3$  matrix and  $\lambda_1, \lambda_2, \lambda_3$  are the eigenvalues of  $B$ .

Write down the characteristic equation of  $B$  and show that  $\lambda_1 \lambda_2 \lambda_3 = \det B$ .

(b) Let  $A_{3 \times 3}$  be a real symmetric matrix. Suppose  $\det A = 1$ ,  $A \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$ ,  $A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$ ,

- (i) use part (a), or otherwise, to find the eigenvalues of  $A$ ;
- (ii) find a matrix  $P$  having its columns to be the unit eigenvectors of  $A$  and show that  $P^T P = I$ ;
- (iii) show that  $A$  is diagonalizable and find the matrix  $A$ .

Solution:

(a)

$$\begin{aligned}
\det(\lambda I - B) &= \det \begin{pmatrix} \lambda - b_{11} & -b_{12} & -b_{13} \\ -b_{21} & \lambda - b_{22} & -b_{23} \\ -b_{31} & -b_{32} & \lambda - b_{33} \end{pmatrix} = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3) \\
&= \lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\lambda^2 + (\lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3)\lambda - \lambda_1 \lambda_2 \lambda_3
\end{aligned}$$

Let  $\lambda = 0$ , then  $\det(-B) = -\lambda_1 \lambda_2 \lambda_3 \Rightarrow (-1)^3 \det B = -\lambda_1 \lambda_2 \lambda_3 \Rightarrow \det B = \lambda_1 \lambda_2 \lambda_3$ .

(b)

(i)

$$A \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}, A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} \Rightarrow A \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix} = (-2) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

So we have, say  $\lambda_1 = -1, \lambda_2 = -2$ .

$$\det A = 1 = \lambda_1 \lambda_2 \lambda_3, \lambda_1 = -1, \lambda_2 = -2 \Rightarrow 1 = 2\lambda_3 \Rightarrow \lambda_3 = \frac{1}{2}. \text{ As a result, } \lambda_1 = -1, \lambda_2 = -2, \lambda_3 = \frac{1}{2}.$$

(Observe that the real symmetric  $A$  has distinct simple (of multiplicity 1) eigenvalues, therefore,  $A$  has the orthonormal (orthogonal unit) eigenvectors which are linearly independent.)

(ii)

For  $\lambda_1 = -1$ ,  $A \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = (-1) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \Rightarrow A \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix} = (-1) \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{pmatrix}.$

For  $\lambda_1 = -2$ ,  $A \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = (-2) \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow A \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} = (-2) \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}.$

Suppose  $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ . Since  $A$  is real symmetric, we have

$$\begin{cases} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 0 \\ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 0 \end{cases} \Leftrightarrow \begin{cases} x + 2y + z = 0 \\ x - z = 0 \end{cases} \Leftrightarrow \begin{cases} y = -z \\ x = z \end{cases}. \text{ Let } z = 1, \text{ then } \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \text{ is an eigenvector of } A \text{ for } \lambda_3 = \frac{1}{2}$$

and the corresponding unit eigenvector is  $\begin{pmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}.$

Let  $P = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ . Observe that  $P^T = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.$

And  $P^T P = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$

(Observe matrix  $P$  which is formed by the orthonormal eigenvectors of  $A$  has its inverse just its transpose)

(iii)

$$\begin{aligned}
A \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} &= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \\
\Rightarrow A &= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix}^{-1} \\
&= \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{2}} & \frac{1}{2\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{2}} & \frac{1}{2\sqrt{3}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \\
&= \begin{pmatrix} -\frac{1}{6} - 1 + \frac{1}{6} & -\frac{2}{6} - \frac{1}{6} & -\frac{1}{6} + 1 + \frac{1}{6} \\ -\frac{2}{6} - \frac{1}{6} & -\frac{4}{6} + \frac{1}{6} & -\frac{2}{6} - \frac{1}{6} \\ -\frac{1}{6} + 1 + \frac{1}{6} & -\frac{2}{6} - \frac{1}{6} & -\frac{1}{6} - 1 + \frac{1}{6} \end{pmatrix} = \begin{pmatrix} -1 & -\frac{1}{2} & 1 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -1 \end{pmatrix}
\end{aligned}$$

□

-End-