

Due date: Monday, 30 October 2023 (Week 9) before 11:59 p.m.

- Answer **all** questions. Show **all** steps.
- Please submit the softcopy of your work (as a **single pdf file**) to Canvas on or before the due date.

Questions

1. Let $z = f(x, y)$, where $x = uv^2$, $y = u^3v$ and f is a function whose first and second order partial derivatives exist. Given that

$$\left. \frac{\partial f}{\partial x} \right|_{(x,y)=(4,2)} = 5, \quad \left. \frac{\partial f}{\partial y} \right|_{(x,y)=(4,2)} = 1,$$

$$\left. \frac{\partial^2 f}{\partial x^2} \right|_{(x,y)=(4,2)} = -4, \quad \left. \frac{\partial^2 f}{\partial y^2} \right|_{(x,y)=(4,2)} = -2, \quad \left. \frac{\partial^2 f}{\partial x \partial y} \right|_{(x,y)=(4,2)} = \left. \frac{\partial^2 f}{\partial y \partial x} \right|_{(x,y)=(4,2)} = 1.$$

Find the values of $\left. \frac{\partial z}{\partial u} \right|_{(u,v)=(1,2)}$ and $\left. \frac{\partial^2 z}{\partial u^2} \right|_{(u,v)=(1,2)}$.

2. Suppose $2z^3 - 3xz + xy^2 = 3$ determines a function $z = z(x, y)$ of x and y locally at $(x, y, z) = (1, 2, 1)$.
- Find the tangent plane approximation of $z(x, y)$ at $(x, y, z) = (1, 2, 1)$.
 - Find the quadratic surface approximation of $z(x, y)$ at $(x, y, z) = (1, 2, 1)$.
3. Consider the function $f(x, y) = e^{3x} \cos(2y)$.
- Use Taylor's formula to find a linear approximation of $f(x, y)$ at the origin.
 - Estimate the error in the linear approximation if $|x| \leq 0.1$ and $|y| \leq 0.1$.
(You may leave your answer in terms of e^c , where c is a known number.)
4. Find all stationary points of the function $f(x, y) = xye^{-(x^2+y^2)}$ and determine their nature.
5. Let $f(x, y) = x^2 - 2xy + y^2 - y$. Find the unit vectors $\vec{u}$ and the values of $D_{\vec{u}}f(2, 1)$ for which
- $D_{\vec{u}}f(2, 1)$ is the largest;
 - $D_{\vec{u}}f(2, 1)$ is the smallest;
 - $D_{\vec{u}}f(2, 1) = 0$;
 - $D_{\vec{u}}f(2, 1) = -2$.

6. **Discovery Question.** (Read Lecture Notes of Chapter 2 (Part B) Example 38 on page 40).

Consider a point $P(x_0, y_0)$ and a parabola $y = ax^2 + bx + c$. The value

$$[y_0 - (ax_0^2 + bx_0 + c)]^2$$

is called the square of the vertical displacement of the data point $P(x_0, y_0)$ from the parabola $y = ax^2 + bx + c$. Our goal is to determine the parabola $y = ax^2 + bx + c$ such that the sum S of the squares of the vertical displacements of the data points

$$P_1(x_1, y_1) = P_1(-2, 6),$$

$$P_2(x_2, y_2) = P_2(-1, 2),$$

$$P_3(x_3, y_3) = P_3(1, 1),$$

$$P_4(x_4, y_4) = P_4(2, 4.5)$$

from the parabola is the smallest. In other words, determine the values of a , b and c such that S is the smallest, where

$$S = \sum_{i=1}^4 (y_i - ax_i^2 - bx_i - c)^2.$$

Solve this question by the following steps:

- Find the only stationary point (a_0, b_0, c_0) of S .
- Using the Taylor series for triple variables, you can expand S at (a_0, b_0, c_0) as

$$\begin{aligned} S(a, b, c) &= S(a_0, b_0, c_0) \\ &+ [S_a(a_0, b_0, c_0)(a - a_0) + S_b(a_0, b_0, c_0)(b - b_0) + S_c(a_0, b_0, c_0)(c - c_0)] \\ &+ \frac{1}{2} [S_{aa}(a_0, b_0, c_0)(a - a_0)^2 + S_{bb}(a_0, b_0, c_0)(b - b_0)^2 + S_{cc}(a_0, b_0, c_0)(c - c_0)^2 \\ &+ 2S_{ab}(a_0, b_0, c_0)(a - a_0)(b - b_0) + 2S_{ac}(a_0, b_0, c_0)(a - a_0)(c - c_0) \\ &+ 2S_{bc}(a_0, b_0, c_0)(b - b_0)(c - c_0)] \end{aligned}$$

Rewrite the above $S(a, b, c) - S(a_0, b_0, c_0)$ in terms of the **quadratic form**:

$$S(a, b, c) - S(a_0, b_0, c_0) = \frac{1}{2} \begin{bmatrix} a - a_0 & b - b_0 & c - c_0 \end{bmatrix} \begin{bmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ab} & S_{bb} & S_{bc} \\ S_{ac} & S_{bc} & S_{cc} \end{bmatrix} \begin{bmatrix} a - a_0 \\ b - b_0 \\ c - c_0 \end{bmatrix}$$

- Show that the matrix

$$A = \begin{bmatrix} S_{aa} & S_{ab} & S_{ac} \\ S_{ab} & S_{bb} & S_{bc} \\ S_{ac} & S_{bc} & S_{cc} \end{bmatrix}$$

in the above quadratic form is **positive definite**. (You may use any method that you have learnt from Chapter 1 to show that the matrix A is positive definite.)

Then $S(a, b, c) - S(a_0, b_0, c_0) \geq 0$ for all values of a, b, c . Therefore, S has a global minimum at (a_0, b_0, c_0) .

- End -