

EE 3210 Signals and Systems

Test 2

Name: _____ (Student No.: _____)

Answer **ALL FIVE** questions:

Question 1 (14 marks)

Calculate the Fourier transforms of the following signals.

(a) $\delta(t+1) + \delta(t-1)$ (7 marks)

(b) $e^{-2(t-1)}u(t-1)$ (7 marks)

Solution:

(a) Method 1: Use the definition of Fourier transform:

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} [\delta(t+1) + \delta(t-1)] e^{-j\Omega t} dt \\ &= \int_{-\infty}^{\infty} \delta(t+1) e^{-j\Omega t} dt + \int_{-\infty}^{\infty} \delta(t-1) e^{-j\Omega t} dt \\ &= e^{j\Omega} + e^{-j\Omega} \\ &= 2 \cos \Omega \end{aligned}$$

Method 2: Use the s transform and its properties.

Use Table 9.1 in Lecture notes, we have

$$\delta(t+1) \longleftrightarrow e^s, \text{ ROC: all } s$$

$$\delta(t-1) \longleftrightarrow e^{-s}, \text{ ROC: all } s$$

$$\delta(t+1) + \delta(t-1) \longleftrightarrow e^s + e^{-s} \text{ ROC: all } s$$

When $s = j\Omega$, we have the Fourier transform $e^{j\Omega} + e^{-j\Omega} = 2 \cos \Omega$

(b) Method 1: Use the definition of Fourier transform:

$$\begin{aligned} X(j\Omega) &= \int_{-\infty}^{\infty} e^{-2(t-1)} u(t-1) e^{-j\Omega t} dt \\ &= \int_1^{\infty} e^{-2(t-1)} e^{-j\Omega t} dt \\ &= e^2 \int_1^{\infty} e^{(-2-j\Omega)t} dt \\ &= e^2 \left. \frac{e^{(-2-j\Omega)t}}{-2-j\Omega} \right|_1^{\infty} \\ &= \frac{e^{-j\Omega}}{j\Omega + 2} \end{aligned}$$

Method 2: Use the s transform and its properties.

Use Table 9.1 in Lecture notes, we have

$$e^{-2t} u(t) \longleftrightarrow \frac{1}{s+2}, \text{ ROC: } \sigma > -2$$

Use the time-shifting properties, we have

$$e^{-2(t-1)} u(t-1) \longleftrightarrow \frac{e^{-s}}{s+2}, \text{ ROC: } \sigma > -2$$

When $s = j\Omega$, we have the Fourier transform $\frac{e^{-j\Omega}}{j\Omega + 2}$

Question 2 (16 marks)

Determine the z-transform for the following discrete-time signals. Sketch the pole-zero plot and indicate the region of convergence (ROC). Indicate whether or not the Discrete time Fourier Transform of the signal exists.

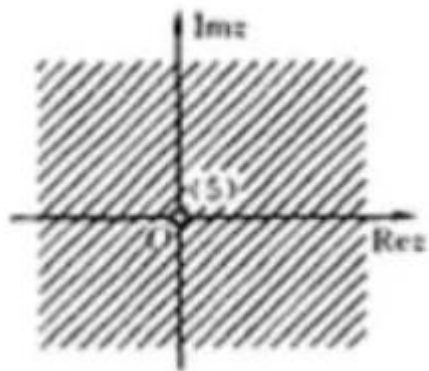
(a) $x[n] = \delta[n+5]$ (8 marks)

(b) $x[n] = \left(-\frac{1}{3}\right)^n u[-n-2]$ (8 marks)

Solution:

(a) Base on Table 8.1 in Lecture Notes, we have

$$\delta[n+5] \longleftrightarrow z^5, \text{ ROC: } |z| < \infty$$



Since the ROC includes the unit circle, the DTFT of the signal exists.

(b) Method 1: Use the definition of z transform

$$X(z) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{3}\right)^n u[-n-2] \cdot z^{-n} = \sum_{n=2}^{\infty} \left(-\frac{1}{3}\right)^{-n} z^n = \sum_{n=2}^{\infty} (-3z)^n$$

The $X(z)$ converges if $|-3z| < 1$, i.e., $|z| < \frac{1}{3}$. Therefore,

$$X(z) = \frac{(-3z)^2}{1+3z} = \frac{9z^2}{1+3z}, \text{ ROC: } |z| < \frac{1}{3}$$

Method 2:

Base on Table 8.1 in Lecture Notes, we have

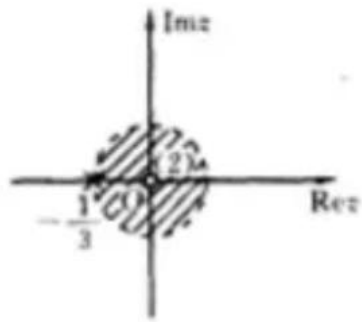
$$-\left(-\frac{1}{3}\right)^n u[-n-1] \longleftrightarrow \frac{1}{1+\frac{1}{3}z^{-1}}, \text{ ROC: } |z| < \frac{1}{3}$$

Use the time-shifting property of the z transform, we have

$$-\left(-\frac{1}{3}\right)^{n+1} u[-(n+1)-1] \longleftrightarrow \frac{z}{1+\frac{1}{3}z^{-1}}, \text{ ROC: } |z| < \frac{1}{3}$$

$$\left(-\frac{1}{3}\right)^n u[-n-2] \longleftrightarrow \frac{3z}{1+\frac{1}{3}z^{-1}} = \frac{9z^2}{1+3z}, \text{ ROC: } |z| < \frac{1}{3}$$

Since the ROC does not include the unit circle, the DTFT of the signal does not exist.



Question 3 (23 marks)

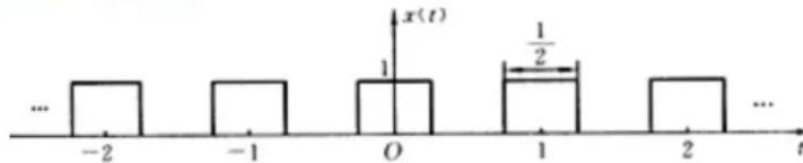
Consider a continuous-time linear time-invariant (LTI) system with the impulse response $h(t) = e^{-4|t|}$. Suppose that the input signal $x(t)$ is a periodic signal with fundamental period $T = 1$. Within the interval $(-0.5, 0.5)$, $x(t)$ is:

$$x(t) = \begin{cases} 1, & -\frac{1}{4} \leq t \leq \frac{1}{4} \\ 0, & \text{else} \end{cases}$$

- (a) Plot the signal $x(t)$. (5 marks)
- (b) Compute the Fourier series representation of the input signal $x(t)$. (6 marks)
- (c) Determine the system frequency response $H(j\Omega)$. (6 marks)
- (d) Find the Fourier series representation of the system output signal $y(t)$. (6 marks)

Solution:

(a)



(b)

$$\begin{aligned} a_k &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt \\ &= \int_{-\frac{1}{4}}^{\frac{1}{4}} e^{-jk2\pi t} dt = \frac{\sin(k\pi/2)}{k\pi} \end{aligned}$$

(c)

$$\begin{aligned} H(j\Omega) &= \int_{-\infty}^{\infty} h(t) e^{-j\Omega t} dt \\ &= \int_{-\infty}^{\infty} e^{-4|t|} e^{-j\Omega t} dt \\ &= \int_{-\infty}^0 e^{4t} e^{-j\Omega t} dt + \int_0^{\infty} e^{-4t} e^{-j\Omega t} dt \\ &= \frac{1}{4 - j\Omega} + \frac{1}{4 + j\Omega} \\ &= \frac{8}{16 + \Omega^2} \end{aligned}$$

(d)

$$\begin{aligned}
y(t) &= \sum_{k=-\infty}^{\infty} a_k H(jk\Omega_0) e^{jk\Omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi/2)}{k\pi} \cdot \frac{8}{16 + (k\Omega_0)^2} \cdot e^{jk\Omega_0 t} \\
&= \sum_{k=-\infty}^{\infty} \frac{\sin(k\pi/2)}{k\pi} \cdot \frac{8}{16 + 4\pi^2 k^2} \cdot e^{jk2\pi t}
\end{aligned}$$

Question 4 (23 marks)

Given a continuous-time periodic signal $x(t)$:

$$x(t) = \sin(100t + 1) + 2 \cos(200t + 2)$$

- (a) Find the fundamental frequency of $x(t)$. **(5 marks)**
- (b) Determine the minimum sampling frequency which avoids aliasing. **(6 marks)**
- (c) The $x(t)$ is sampled with a sampling period $T = 0.05\text{s}$ to produce the discrete-time signal $x[n]$. Determine $x[n]$. **(7 marks)**
- (d) Is $x[n]$ periodic? **(5 marks)**

Solution:

- (a) There are only two frequency components in $x(t)$: 100 and 200. Apparently, the fundamental frequency is 100.
- (b) According to sampling theorem, the sampling frequency must exceed 400 (or $200 / \pi$ in Hz).
- (c) $x[n] = x(t) |_{t=nT} = \sin(5n + 1) + 2 \cos(10n + 2)$
- (d) $x[n]$ is not periodic as there is no integer N which satisfies $x[n] = x[n + N]$.

Question 5 (24 marks)

Consider a discrete-time LTI system whose input and output signals are

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4} \left(\frac{1}{2}\right)^{n-1} u[n-1] \text{ and } y[n] = \left(\frac{1}{3}\right)^n u[n], \text{ respectively.}$$

- (a) Determine the z transform of the input signal $X(z)$. Specific its ROC. **(6 marks)**
- (b) Determine the z transform of the output signal $Y(z)$. Specific its ROC. **(6 marks)**
- (c) Based on (a) and (b), compute the frequency response of this system $H(z)$. Specific its zeros and poles. **(6 marks)**
- (d) Based on (c), find the difference equation relating the input and output signals. **(6 marks)**

Solution:

- (a) Based on Table 8.1 in Lecture Notes, we have

$$\left(\frac{1}{2}\right)^n u[n] \longleftrightarrow \frac{1}{1 - \frac{1}{2} z^{-1}}, \text{ ROC: } |z| > \frac{1}{2}$$

Use the time-shifting property, we have

$$\left(\frac{1}{2}\right)^{n-1} u[n-1] \longleftrightarrow \frac{z^{-1}}{1 - \frac{1}{2} z^{-1}}, \text{ ROC: } |z| > \frac{1}{2}$$

$$\text{Therefore, } X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}} - \frac{1}{4} \frac{z^{-1}}{1 - \frac{1}{2} z^{-1}} = \frac{z - 1/4}{z - 1/2}, \text{ ROC: } |z| > \frac{1}{2}$$

- (b) Based on Table 8.1 in Lecture Notes, we have

$$y[n] = \left(\frac{1}{3}\right)^n u[n] \longleftrightarrow Y(z) = \frac{1}{1 - \frac{1}{3} z^{-1}} = \frac{z}{z - 1/3}, \text{ ROC: } |z| > \frac{1}{3}$$

$$(c) H(z) = \frac{Y(z)}{X(z)} = \frac{z(z - 1/2)}{(z - 1/3)(z - 1/4)}. \text{ Zeros: } 0, \frac{1}{2}. \text{ Poles: } \frac{1}{3}, \frac{1}{4}$$

- (d) Performing cross-multiplication of $H(z) = \frac{Y(z)}{X(z)} = \frac{z(z - 1/2)}{(z - 1/3)(z - 1/4)}$, we have

$$(1 - 7z^{-1}/12 + z^{-2}/12)Y(z) = (1 - z^{-1}/2)X(z).$$

Take the inverse z transform, we obtain the difference equation:

$$y[n] - \frac{7}{12} y[n-1] + \frac{1}{12} y[n-2] = x[n] - \frac{1}{2} x[n-1]$$