

Test 2 Solution

13 Nov 2023: 6:00pm~6:45pm (YEUNG LT-15)

- As a closed-book test, you are not allowed to use any written materials online/offline.
- Write your answers on the question papers directly.
- Non-graphical calculators are allowed. Laptops/phones must be switched off.
- Unless otherwise specified, SI units are assumed in the questions and 3 significant figures should be used in numerical answers.
- The total score is 100. There are a total of **6 pages**.

The following formulas may or may not be useful.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \mathbf{E}(x, y, z) = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{\mathbf{a}}_R \quad d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{Id\mathbf{L} \times \hat{\mathbf{a}}_R}{R^2}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \oint_S \mathbf{D} \cdot d\mathbf{S} = Q \quad \oint_L \mathbf{H} \cdot d\mathbf{L} = I$$

$$\nabla \cdot \mathbf{D} = \rho_v \quad \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \epsilon_r \mathbf{E} \quad \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) = \mu_0 \mu_r \mathbf{H}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ (F/m)} \quad \mu_0 = 4\pi \times 10^{-7} \text{ (H/m)}$$

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \hat{\mathbf{a}}_\rho \quad \mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3} (2\cos\theta \hat{\mathbf{a}}_r + \sin\theta \hat{\mathbf{a}}_\theta) \quad \mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\mathbf{a}}_\phi \quad \mathbf{B} = \mu_0 n I \hat{\mathbf{a}}_z$$

$$V_{AB} = -\int_B^A \mathbf{E} \cdot d\mathbf{L} \quad \nabla^2 V = -\frac{\rho_v}{\epsilon} \quad V = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$\nabla^2 = \nabla \cdot \nabla$$

$$\nabla = \frac{\partial}{\partial x} \hat{\mathbf{a}}_x + \frac{\partial}{\partial y} \hat{\mathbf{a}}_y + \frac{\partial}{\partial z} \hat{\mathbf{a}}_z$$

$$\nabla f = \frac{\partial f}{\partial \rho} \hat{\mathbf{a}}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi + \frac{\partial f}{\partial z} \hat{\mathbf{a}}_z$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{a}}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\mathbf{a}}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\mathbf{a}}_\phi$$

$$\nabla \times \mathbf{F} = \frac{1}{\rho} \begin{vmatrix} \hat{\mathbf{a}}_\rho & \rho \hat{\mathbf{a}}_\phi & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_\rho & \rho F_\phi & F_z \end{vmatrix}$$

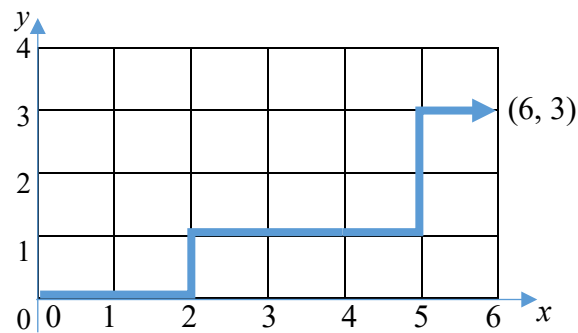
$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin \theta} \begin{vmatrix} \hat{\mathbf{a}}_r & r \hat{\mathbf{a}}_\theta & r \sin \theta \hat{\mathbf{a}}_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ F_r & r F_\theta & r \sin \theta F_\phi \end{vmatrix}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} (\rho F_\rho) + \frac{\partial}{\partial \phi} F_\phi + \frac{\partial}{\partial z} (\rho F_z) \right]$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 \sin \theta F_r) + \frac{\partial}{\partial \theta} (r \sin \theta F_\theta) + \frac{\partial}{\partial \phi} (r F_\phi) \right]$$

Question 1 (6 points)

On the xy -plane, consider the path P (below) that goes from the $(0, 0)$ to $(6, 3)$

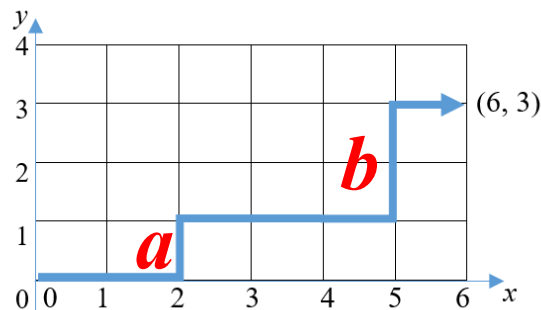


Find the path integral $\int_P (3x\hat{\mathbf{y}}) \cdot d\mathbf{L}$

Solution:

For version (a),

$$\int_P (3x\hat{\mathbf{y}}) \cdot d\mathbf{L} = \int_a (3x) \cdot dy + \int_b (3x) \cdot dy = 3 \times 2 \times 1 + 3 \times 5 \times 2 = 36$$



For version (b), we instead have:

$$\int_P (7y\hat{\mathbf{x}}) \cdot d\mathbf{L} = 7 \times (5 - 2) + 21 \times (6 - 5) = 42$$

Question 2 (14 points)

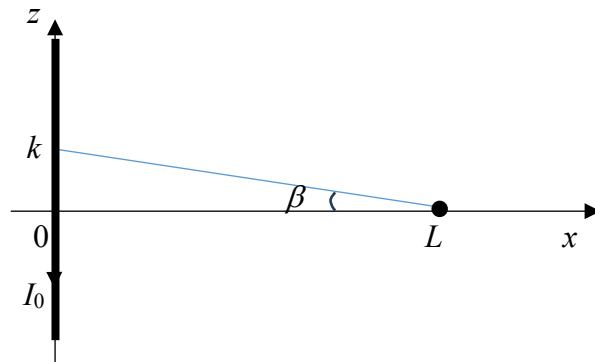
- (a) Consider a field $\mathbf{F} = 2104 \hat{\mathbf{y}}$, what is its divergence at $(x, y) = (20, 23)$?
- (b) Consider a charge of +1 nC and a charge of +12 pC. Is the force between them attractive or repulsive?
- (c) Consider two long wires each carrying 3 A of current towards the +x-direction. Is the force between them attractive or repulsive?
- (d) Consider a capacitor with a capacitance of 21 μF . How much charge is stored when 3 V is applied across the capacitor?
- (e) Consider a metallic circular rod of radius $a = 5 \text{ mm}$. It is infinitely long, but is uniformly charged with +5 nC/m. What is the magnitude of the E-field inside the rod?

Solution:

- (a) The divergence of a field $\mathbf{F} = 2104 \hat{\mathbf{y}}$ at $(x, y) = (20, 23)$ is 0.
- (b) The force between a charge of +1 nC and a charge of +12 pC is repulsive. (The reverse is true for version (b) with -12pC.)
- (c) The force between two wires is attractive.
- (d) $Q = CV = 21 \times 10^{-6} \times 3 = 6.3 \times 10^{-5} \text{ (C)}$. (The values are changed for the different version(s)).
- (e) The magnitude of the E-field inside the metallic rod is 0. This is true for ALL versions because the electrostatic field inside a perfect conductor is ALWAYS zero.

Question 3 (5 points)

On the xy -plane, the figure shows an infinitely long current I_0 that flows towards $-\hat{z}$. The magnetic field $\mathbf{B}$ is measured at a $(x, z) = (L, 0)$.



Consider a current element from $z = k$ to $k+dk$, find $d\mathbf{B}$. Express the answer in terms of the angle β only.

Solution:

For the hypotenuse $\mathbf{R}$ with the associated unit vector $\hat{\mathbf{a}}$

$$\begin{aligned}
 d\mathbf{B} &= \frac{\mu_0}{4\pi} \frac{Id\mathbf{L} \times \hat{\mathbf{a}}}{R^2} \\
 &= \frac{-\mu_0 I_0 dk \cos\beta}{4\pi \frac{L^2 + k^2}{}} \\
 &= \frac{-\mu_0 I_0 d(L \tan\beta) \cos\beta}{4\pi \frac{L^2 + (L \tan\beta)^2}{}} \hat{\mathbf{y}} \\
 &= \frac{-\mu_0 I_0 L (\sec\beta)^2 \cos\beta d\beta}{4\pi \frac{(L \sec\beta)^2}{}} \hat{\mathbf{y}} \\
 &= \frac{-\mu_0 I_0 \cos\beta d\beta}{4\pi L} \hat{\mathbf{y}}
 \end{aligned}$$

Question 4 (15 points)

Consider a static field:

$$\mathbf{F}(x, y) = 3xy^{-4}\hat{\mathbf{x}} + y^{-3}\hat{\mathbf{y}},$$

The field goes to zero as both x and y go to infinity.

- (a) Find the divergence of $\mathbf{F}$.
- (b) Find the curl of $\mathbf{F}$.
- (c) State whether $\mathbf{F}$ represents an electrostatic field $\mathbf{E}$ or a magnetostatic field $\mathbf{B}$. Please state the reasoning.

Solution:

Students should note that in the questions is expressed using the Cartesian coordinates (i.e. rectangular coordinates). There is NO need to invoke the more complicated forms of the operators (in cylindrical/spherical coordinates).

Also, when answering without the use of a computer, the bolded fonts below are for vectors. So the “arrow” signs are needed for every vector. This has been emphasized through the semester.

- (a) The divergence of $\mathbf{F}$ is

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(3xy^{-4}) + \frac{\partial}{\partial y}(y^{-3}) = 3y^{-4} - 3y^{-4} = 0$$

- (b) The curl of $\mathbf{F}$ is

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy^{-4} & y^{-3} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 3xy^{-4} & y^{-3} \end{vmatrix} \hat{\mathbf{z}} = 0 - \frac{\partial(3xy^{-4})}{\partial y} \hat{\mathbf{z}} = 12xy^{-5}\hat{\mathbf{z}}$$

- (c) $\mathbf{F}$ represents a magnetostatic field $\mathbf{B}$

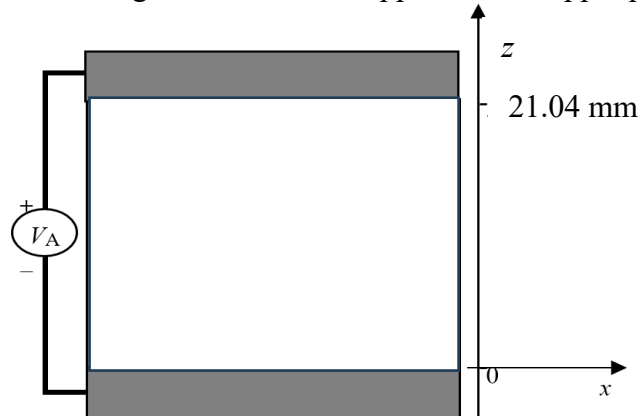
For an electrostatic field $\mathbf{E}$, $\nabla \times \mathbf{E}$ should be zero.

For a magnetostatic field $\mathbf{B}$, $\nabla \cdot \mathbf{B}$ should be zero.

Since $\nabla \cdot \mathbf{F} = 0$ and $\nabla \times \mathbf{F} \neq 0$, we can conclude that $\mathbf{F}$ represents a magnetostatic field $\mathbf{B}$.

Question 5 (30 points)

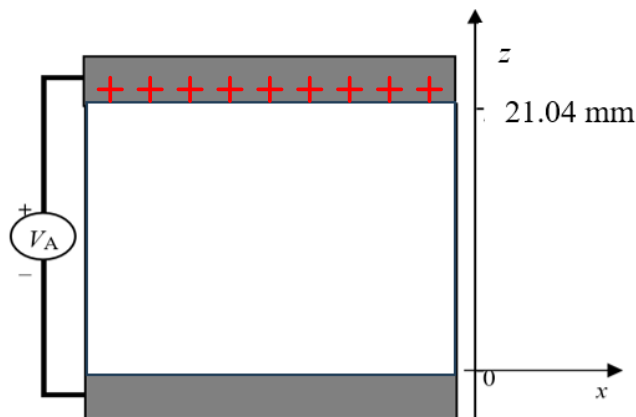
The diagram below shows a parallel plate capacitor. From $z = 0$ to 21.04 mm, there is a piece of dielectric with a relative permittivity $\epsilon_r = 8$. It is sandwiched between two parallel metallic plates. The cross-section area of each plate is 2.3 m^2 . The lower plate is grounded. A positive voltage of $+V_A = 3\text{V}$ is applied to the upper plate.



- (a) Where are the positive charges? In addition to your answer, please clearly indicate these charges on the diagram above.
- (b) State two boundary conditions for the potential $V(z)$.
- (c) State the Laplace's equation. Then, deduce the potential $V(z)$ in the dielectric.
- (d) Calculate for the charges stored in the upper plate. A numerical answer is needed. [Hint: A formulas covered in the course can be used.].

Solution:

- (a) The positive charges are at the lower surface of the upper metallic plate.



- (b) The boundary condition is
$$\begin{cases} V(21.04) = 3 \\ V(0) = 0 \end{cases}$$

(c) The Laplace's equation is

$$\nabla^2 V = 0$$

This is due to the absence of free charges in the dielectric. In other words, ρ_v in the formula sheet becomes zero. Some students have “ ρ_v ” for the charge density on the right-hand side of the equation. However, it should be zero in the absence of free-charge inside the dielectric. Please study Chapter 6 of the lecture notes for a better understanding.

Here the problem depends only on z . So the Laplace's equation reduces to:

$$\frac{\partial^2}{\partial z^2} V(z) = 0$$

$$\frac{\partial}{\partial z} V(z) = A$$

$$V(z) = Az + B$$

where A and B are integration constants. The problem specified the dependence on z (only). The solution has to be in z accordingly.

From the boundary conditions:

$$\begin{cases} V(21.04 \times 10^{-3}) = 3 \\ V(0) = 0 \end{cases}$$

$$\begin{cases} 21.04 \times 10^{-3} A = 3 \\ B = 0 \end{cases}$$

$$\begin{cases} A = 142.6 \\ B = 0 \end{cases}$$

Therefore

$$V(z) = 142.6z$$

where z is in the unit of meters and V is in the unit of Volts.

(d) Since electric field is the negative gradient of potential,

$$\mathbf{E} = -\nabla V = -142.6\hat{\mathbf{z}}$$

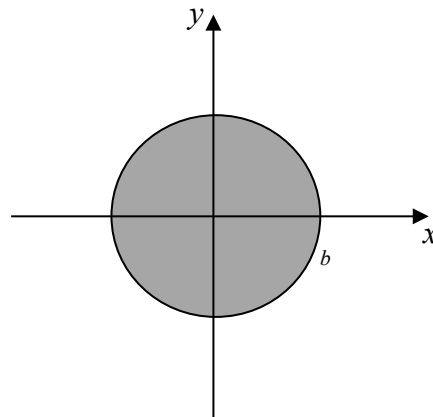
$$D = \epsilon_0 \epsilon_r E = (8.854 \times 10^{-12})(8)(142.6) = 1.01 \times 10^{-8} \text{ (C/m}^2\text{)} = 10.1 \text{ (nC/m}^2\text{)}$$

$$Q = DS = 10.1 \times 2.3 = 23.23 \text{ (nC)}$$

Again, the vectors should have the “arrows” and the del-operator should not be associated with dots/crosses when applied for “grad”.

Question 6 (30 points)

In the cylindrical coordinate system (ρ, ϕ, z) , consider a **solid cylinder** along the z -axis with radius b . The figure below shows the cross-section of the structure. The charge density of the structure for $\rho < b$ is $A \exp(\rho^2/b^2)$ for some constant A in C/m^3 .



Here, we consider a Gaussian surface S as a cylinder centered at the z -axis from $z = 0$ to L with a radius ρ .

- In terms of the cylindrical coordinates (ρ, ϕ, z) and the associated unit vectors, express the differential area $d\mathbf{S}$ on the curved part of the Gaussian surface.
- In observing the symmetry, simplify the expression of the field $\mathbf{D}(\rho, \phi, z)$. Be clear about its direction. Be clear about whether it depends on all the three coordinates.
- Apply Gauss's law, find $|\mathbf{D}|$ for any $\rho > 0$.

Solution:

(a) $d\mathbf{S} = \rho d\phi dz \hat{\mathbf{a}}_\rho$

(b) $\mathbf{D}(\rho, \phi, z) = D(\rho) \hat{\mathbf{a}}_\rho$

This equation has to be precise. The left-hand side should have an arrow for the vector field. The right-hand side should NOT have “phi” or “z”.

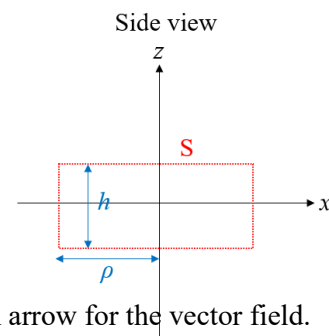
(c) From Gauss's law, we have $\oint_S \mathbf{D}(\rho, \phi, z) \cdot d\mathbf{S} = \oint_S D(\rho) \cdot d\mathbf{S} = Q_{enc}$,

where Q_{enc} is the total charge enclosed by the Gaussian surface.

We can construct a Gaussian surface with a height of h and a radius of ρ .

$$\oint_S \mathbf{D}(\rho) \cdot d\mathbf{S} = \int_{top} \mathbf{D}(\rho) \cdot d\mathbf{S} + \int_{bottom} \mathbf{D}(\rho) \cdot d\mathbf{S} + \int_{side} \mathbf{D}(\rho) \cdot d\mathbf{S}$$

Because $\mathbf{D}$ is along $\hat{\mathbf{a}}_\rho$, we have $\int_{top} \mathbf{D}(\rho) \cdot d\mathbf{S} = \int_{bottom} \mathbf{D}(\rho) \cdot d\mathbf{S} = 0$



As for the side surface, $\mathbf{D}$ is always parallel to $d\mathbf{S}$ and $|\mathbf{D}(\rho)| = D(\rho)$ is constant because ρ is constant. So

$$\oint_S \mathbf{D}(\rho) \cdot d\mathbf{S} = D(\rho) \int_{side} dS = 2\pi\rho h D(\rho) = Q_{enc}$$

$$\therefore D(\rho) = \frac{Q_{enc}}{2\pi\rho h}$$

For $0 \leq \rho < b$,

$$\begin{aligned} Q_{enc} &= \int_0^\rho \int_0^{2\pi} \int_0^h A \exp\left(\frac{\rho^2}{b^2}\right) \rho \, d\rho d\phi dz \\ &= A \int_0^\rho \exp(\rho^2/b^2) \rho d\rho \int_0^{2\pi} d\phi \int_0^h dz \\ &= A(2\pi)(h) \int_0^\rho \exp(\rho^2/b^2) \rho d\rho \\ &= A(2\pi)(h) \left(\frac{b^2}{2}\right) \left(\exp\left(\frac{\rho^2}{b^2}\right) - 1\right) \end{aligned}$$

Note that the “rho” inside the integration is regarded as a “dummy” variable.

By substitution,

$$D(\rho) = \frac{Q_{enc}}{2\pi\rho h} = \frac{Ab^2 \left(\exp\left(\frac{\rho^2}{b^2}\right) - 1\right)}{2\rho}$$

For $b \leq \rho$,

$$\begin{aligned} Q_{enc} &= A(2\pi)(h) \left(\frac{b^2}{2}\right) (\exp(1) - 1) = A(2\pi)(h) \left(\frac{b^2}{2}\right) (1.72) \\ \therefore D(\rho) &= \frac{Q_{enc}}{2\pi\rho h} = \frac{0.86Ab^2}{\rho} \end{aligned}$$

~END~