

EE1001

Foundations of Digital Techniques and Data Analysis

Essence of Optimization – Simplex Method



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A Simple Maximization Problem

- Two plants: wheat and barley;
- 8 acres of land, and 10 gallons;
 - Wheat
 - 1. 2 gallons/acre
 - 2. \$5,000/acre
 - Barley
 - 1. 1 gallons/acre
 - 2. \$3,000/acre
- What is the maximum profit he can make?

Formulation

- x : the number of acres of wheat with $x \geq 0$
- y : the number of acres of barley with $y \geq 0$

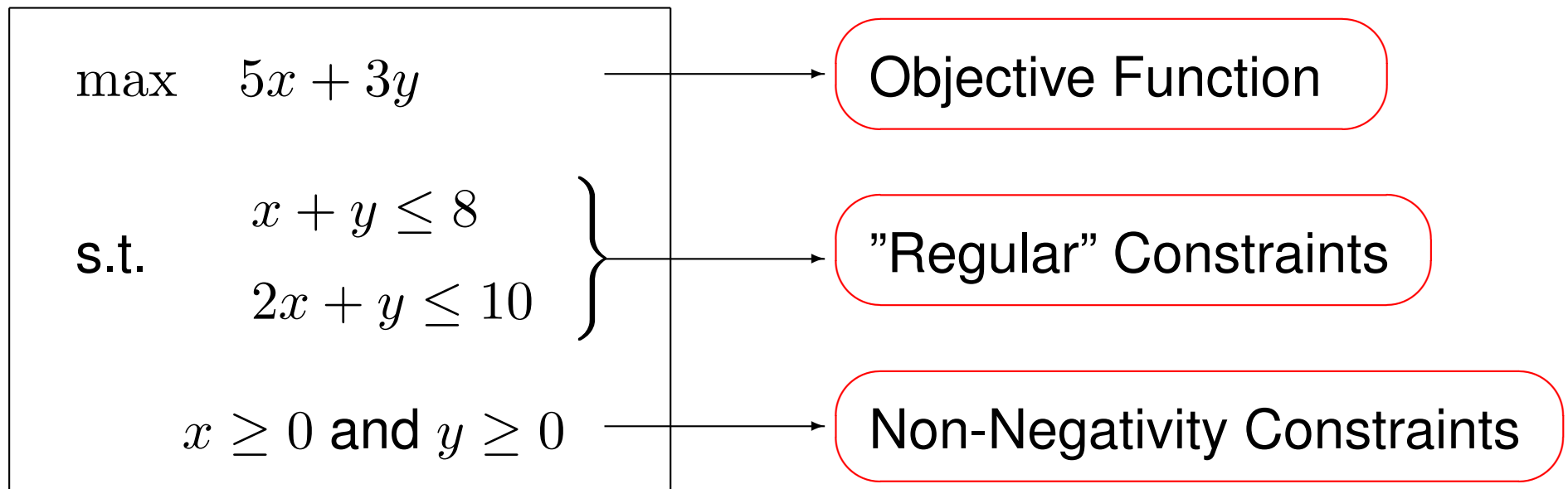
$$8 \text{ acres of land} \Rightarrow x + y \leq 8$$

$$10 \text{ gallons} \Rightarrow 2x + y \leq 10$$

- Objective function:

$$\max \text{ total profit} = 5x + 3y \text{ (thousand dollars)}$$

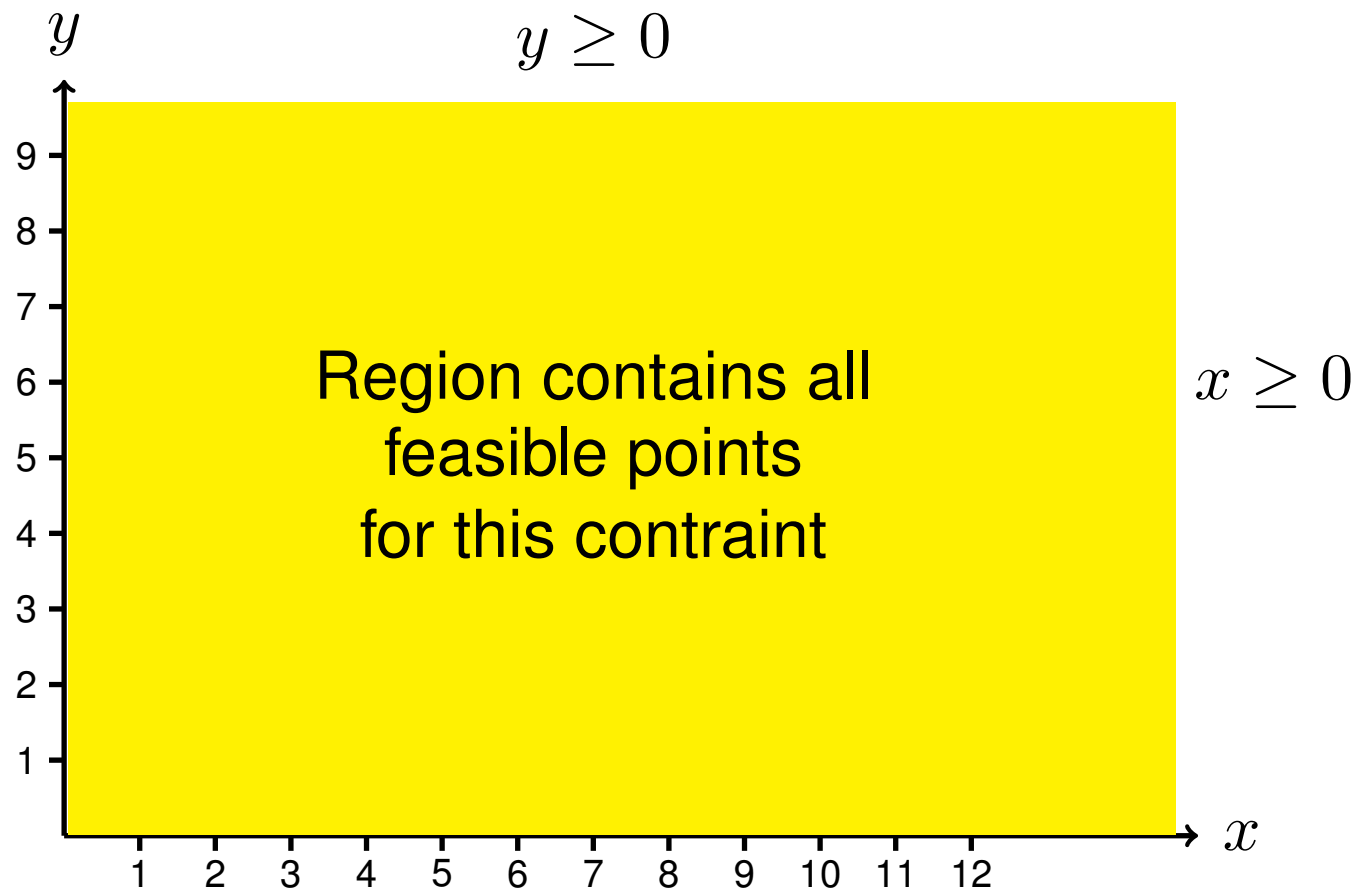
Formulation (Cont.)



Graphical Solution

■ Non-negativity constraints graphed

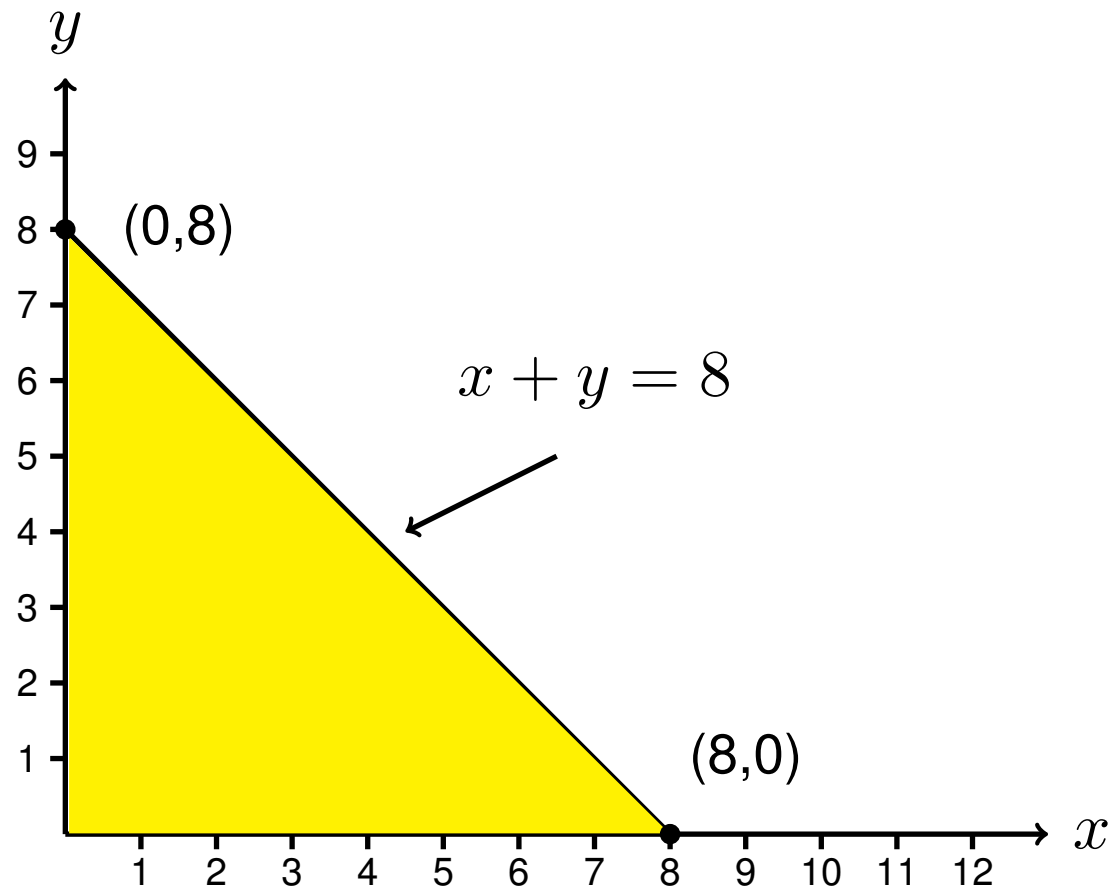
■ $x \geq 0, y \geq 0$.



Graphical Solution (Cont.)

■ First regular constraint Graphed

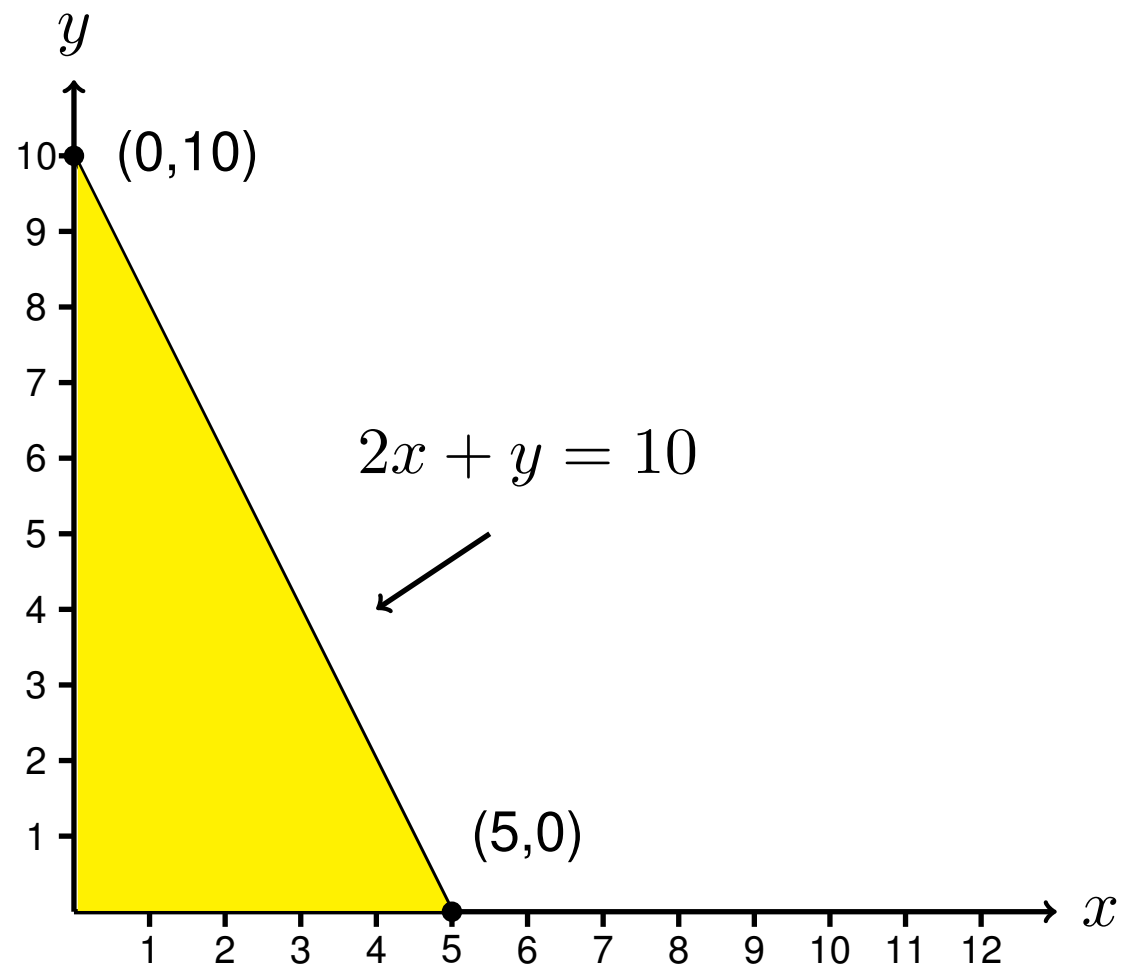
■ $x + y \leq 8$.



Graphical Solution (Cont.)

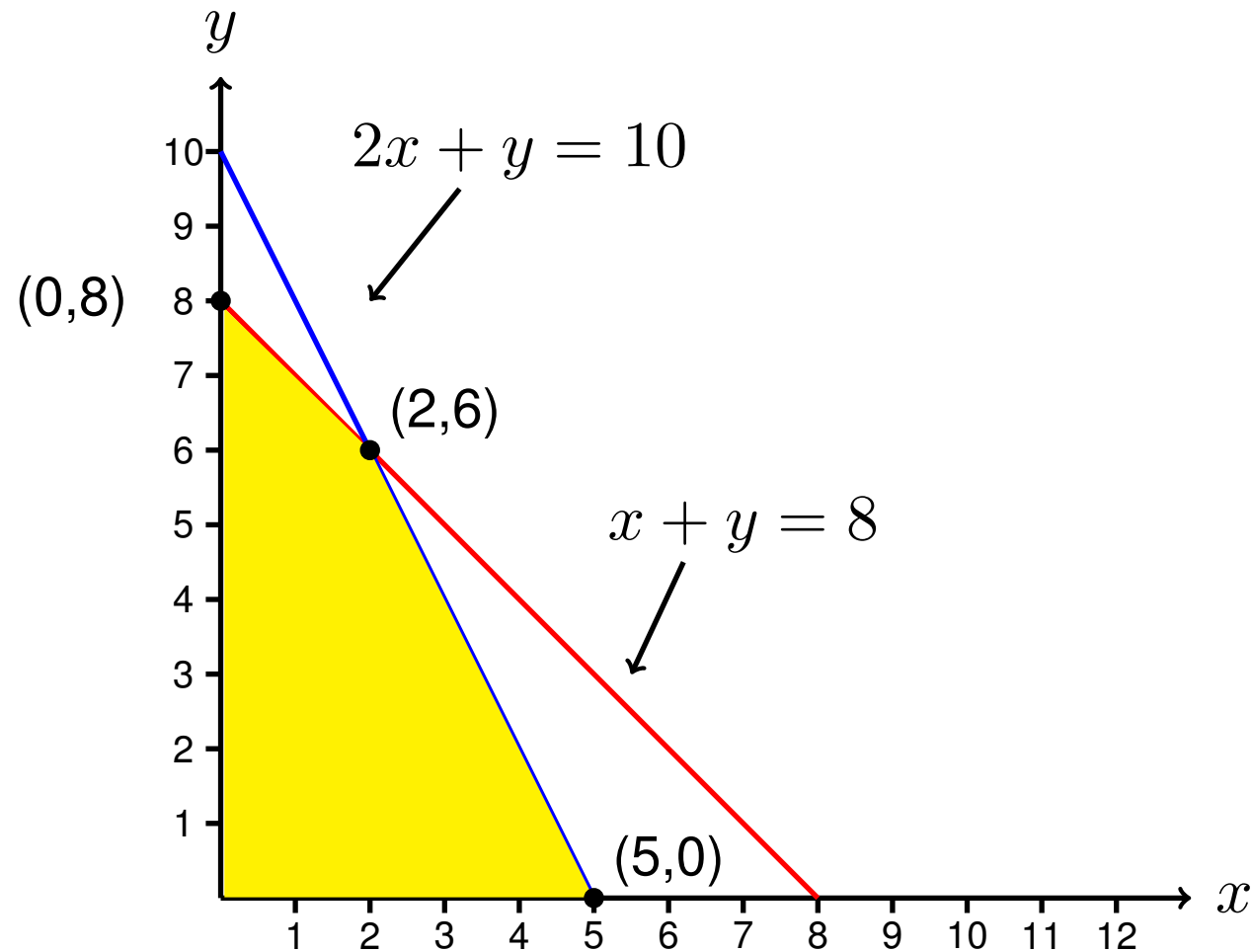
■ Second regular constraint Graphed

■ $2x + y \leq 10$.



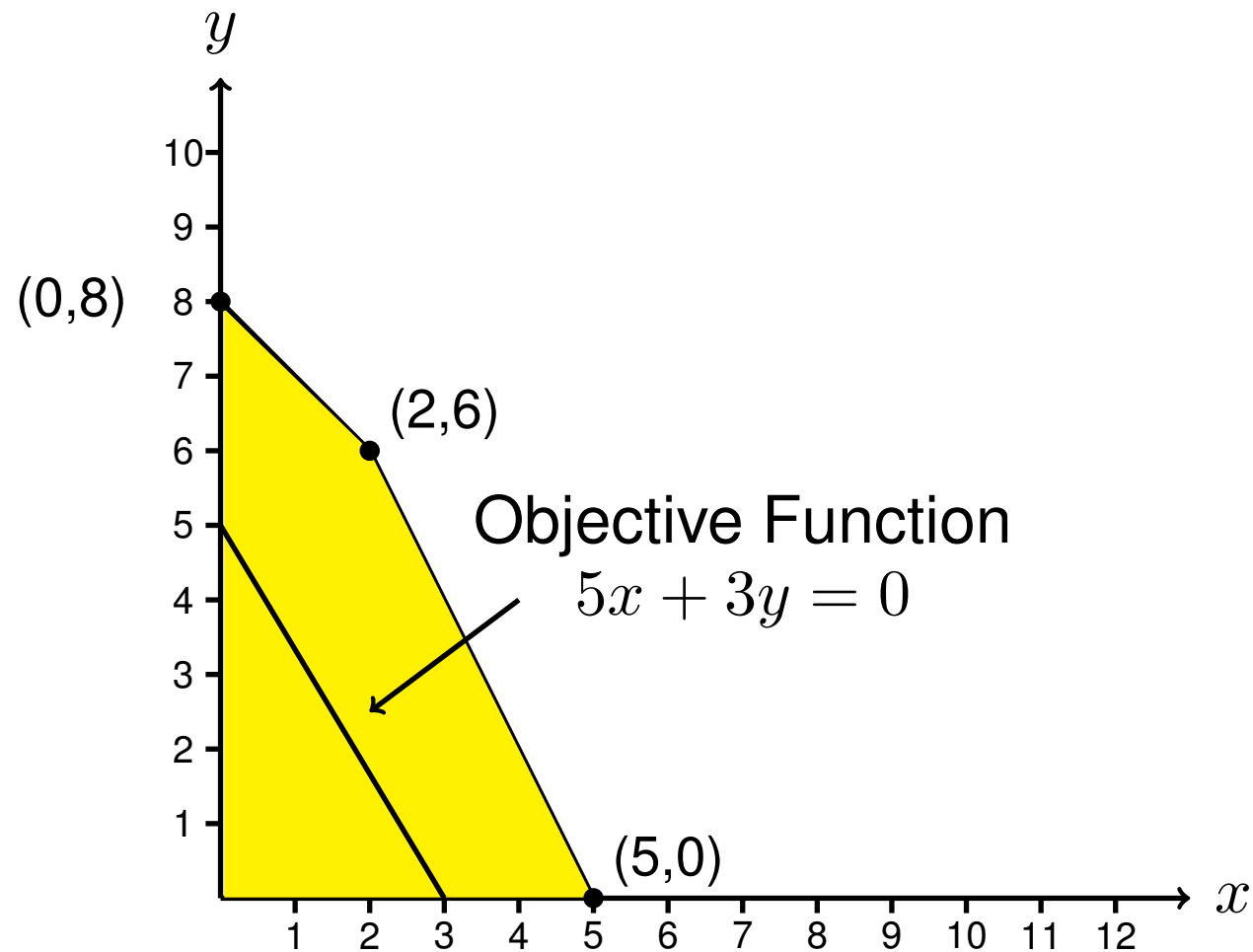
Graphical Solution (Cont.)

■ Combined-Constraint Graph Showing Feasible Region



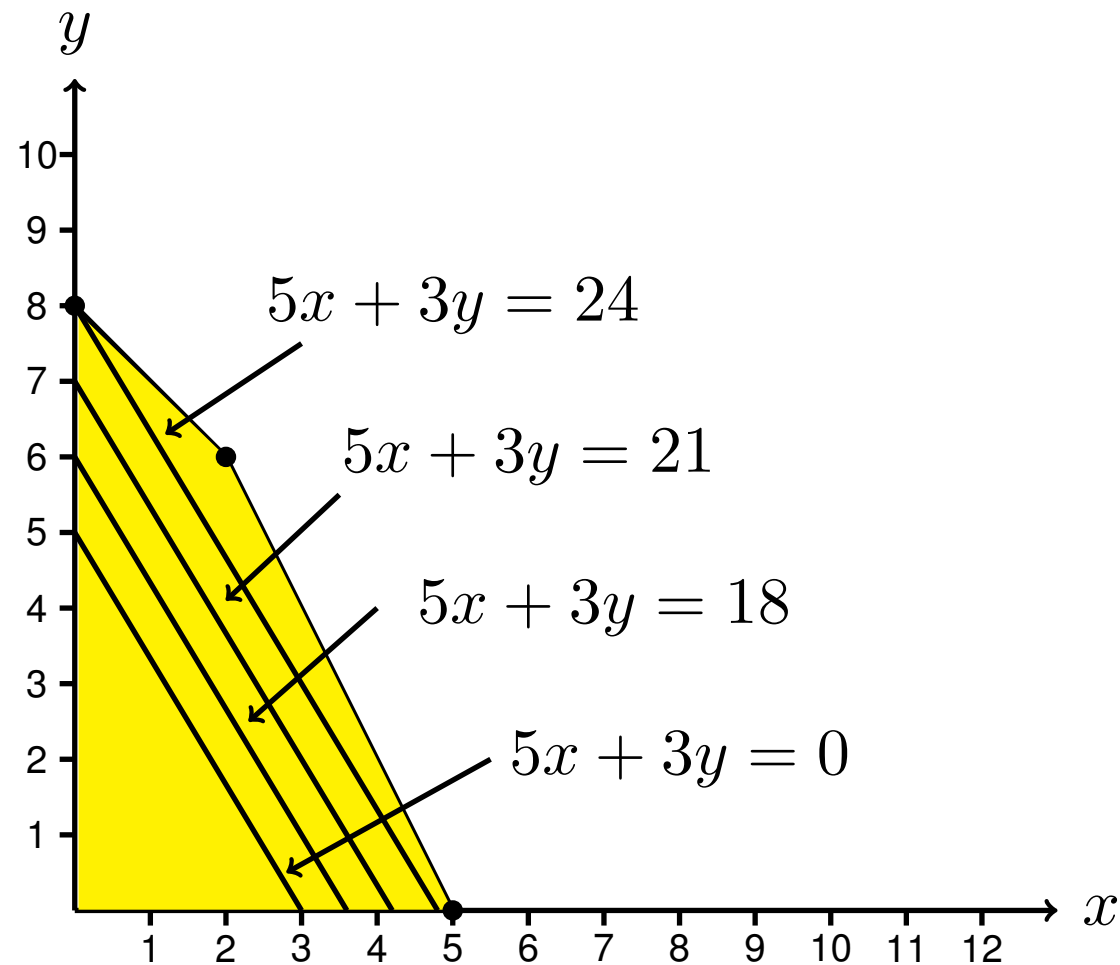
Graphical Solution (Cont.)

■ Objective Function Line



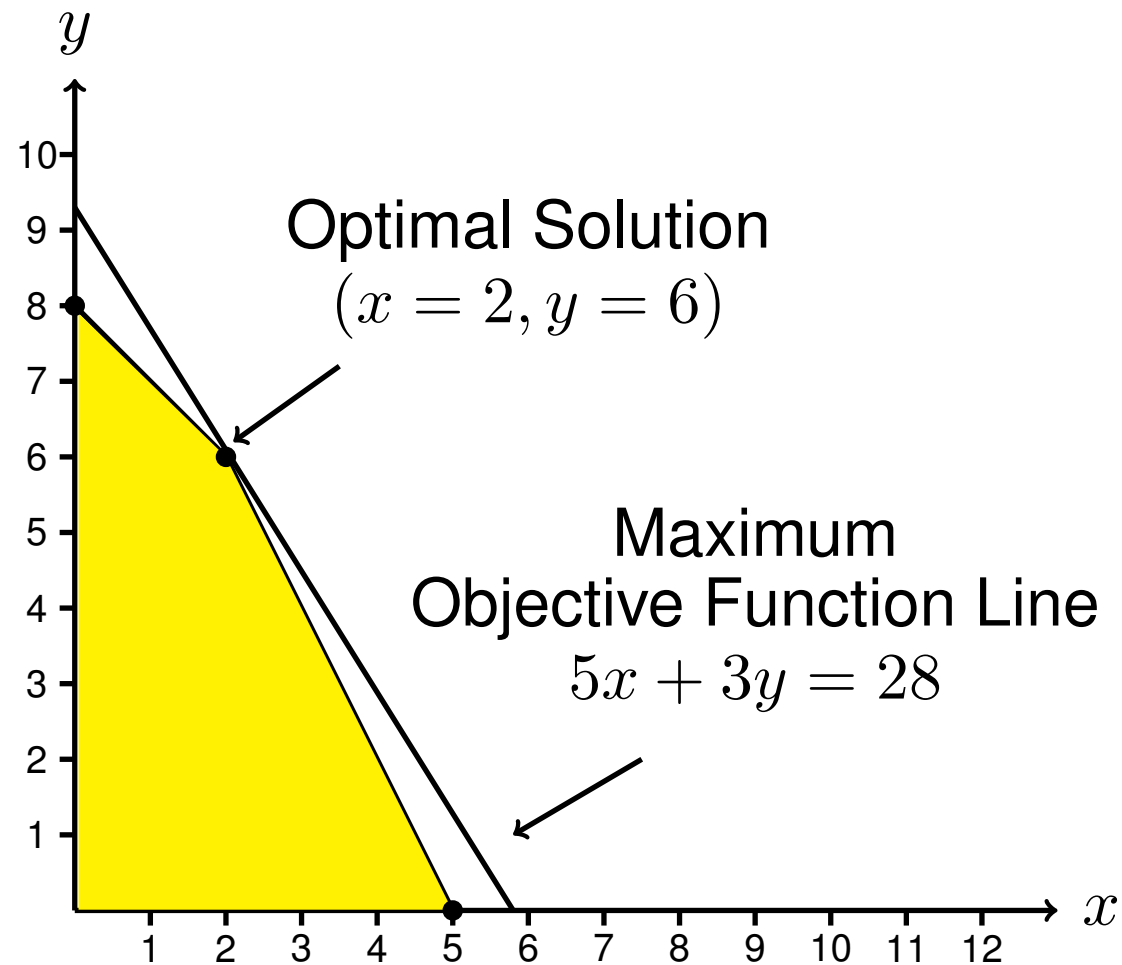
Graphical Solution (Cont.)

■ Selected Objective Function Line



Graphical Solution (Cont.)

■ Optimal Solution



A Simple Minimization Problem

- Special fertilizer mixing two fertilizers, A and B.
- The mixture is to contain:
 1. Phosphate: at least 45 units
 2. Nitrate: at least 36 units
 3. Ammonium: at least 40 units
- Fertilizer A:
 - 5 units of phosphate
 - 2 units of nitrate
 - 2 units of ammonium
 - Cost: 1/pound
- Fertilizer B:
 - 3 units of phosphate
 - 3 units of nitrate
 - 5 units of ammonium
 - Cost: 2/pound
- How many pounds of each fertilizer should the shop use to minimize their cost?

Formulation

■ x : the number of pounds of fertilizer A with $x \geq 0$

■ y : the number of pounds of fertilizer B with $y \geq 0$

■ Constraints

The units of phosphate has to be at least 45 $\Rightarrow 5x + 3y \geq 45$

The units of nitrate must be at least 36 $\Rightarrow 2x + 3y \geq 36$

The units of ammonium must be at least 40 $\Rightarrow 2x + 5y \geq 40$

■ Objective function:

$$\min \text{ total cost} = x + 2y$$

Formulation (Cont.)

$$\min \quad x + 2y$$

$$\text{s.t.} \quad \left. \begin{array}{l} 5x + 3y \geq 45 \\ 2x + 3y \geq 36 \\ 2x + 5y \geq 40 \end{array} \right\}$$

$$x \geq 0 \text{ and } y \geq 0$$

Objective Function

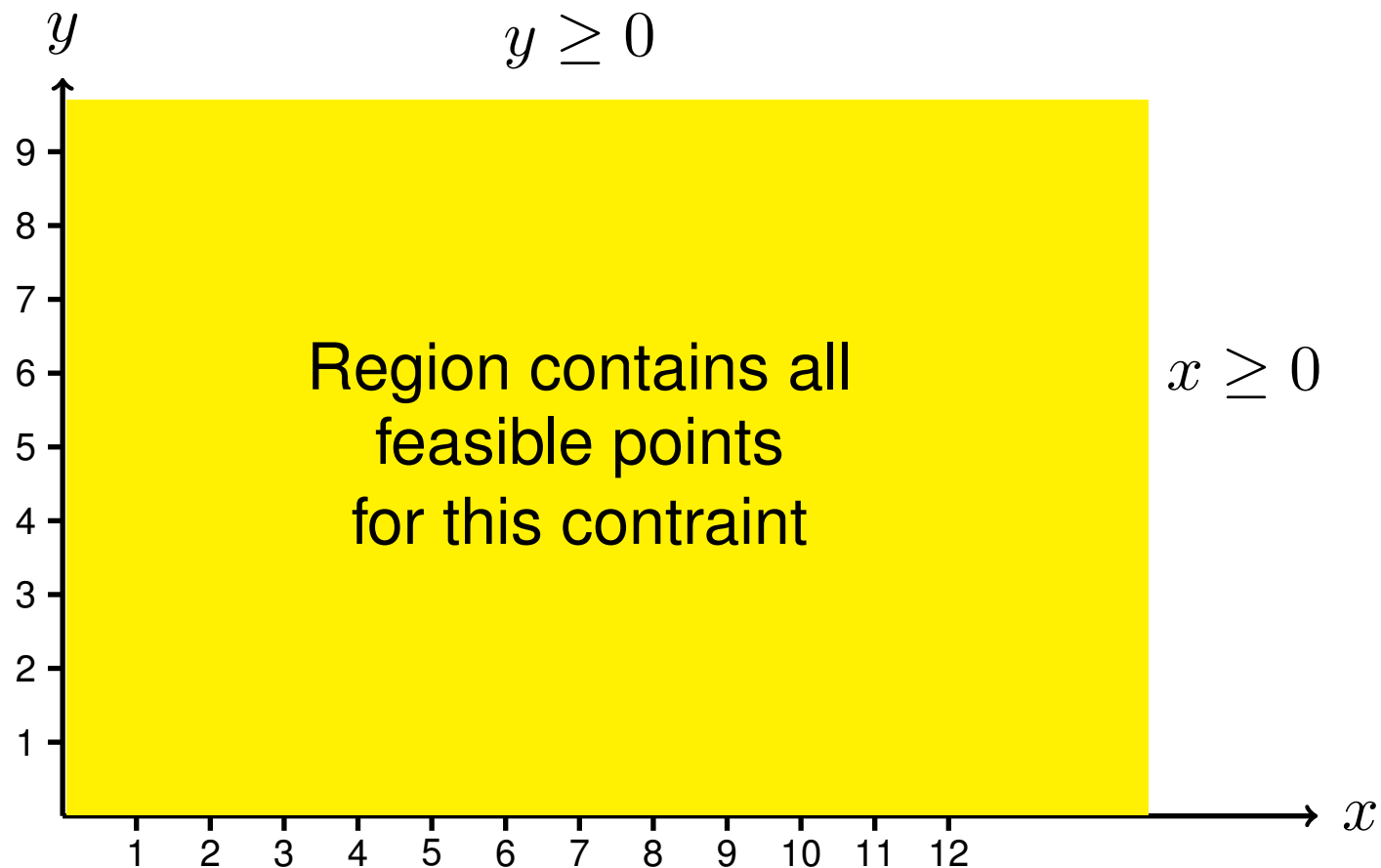
"Regular" Constraints

Non-Negativity Constraints

Graphical Solution

■ Non-negativity constraints graphed

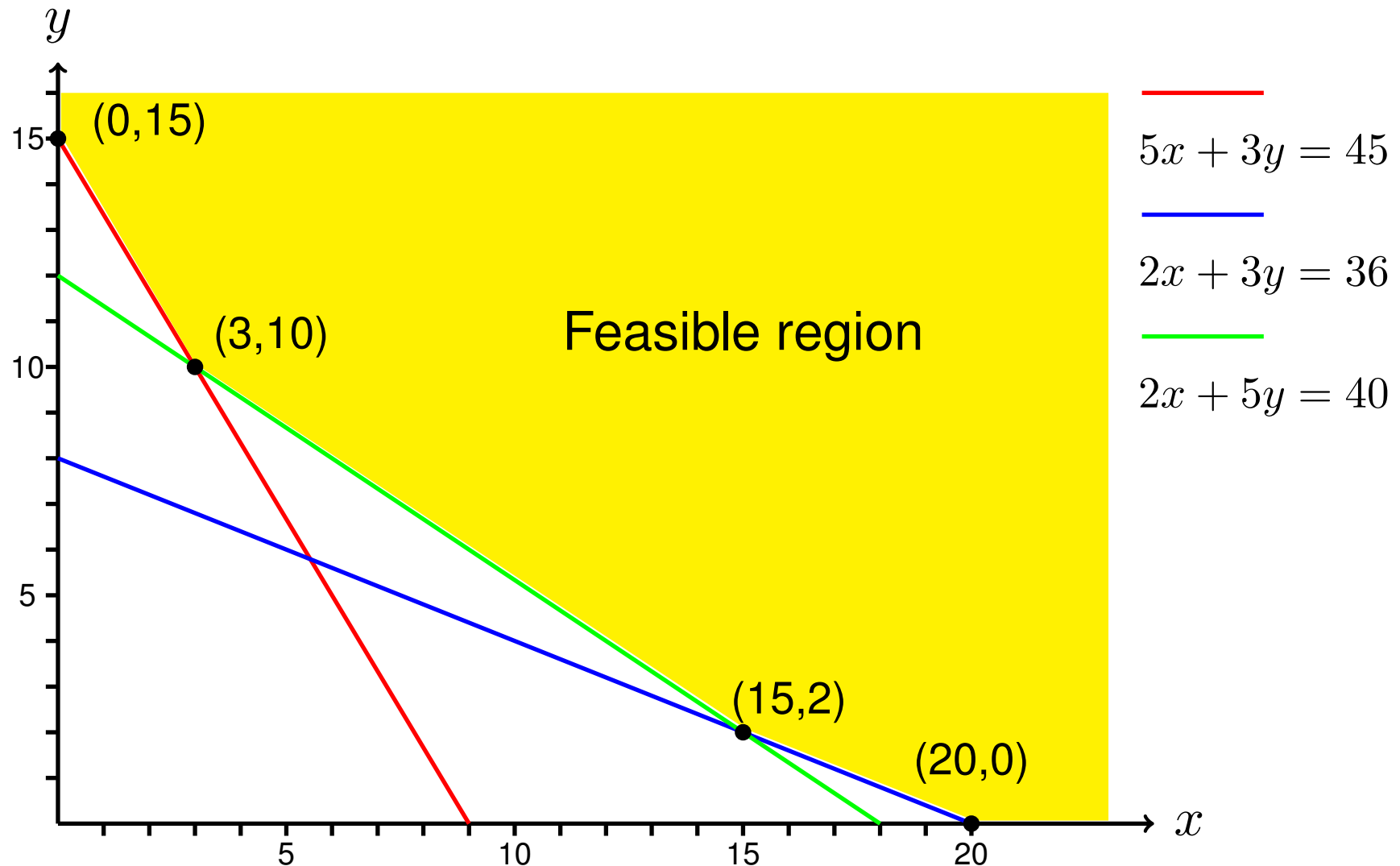
■ $x \geq 0, y \geq 0$.



Graphical Solution (Cont.)

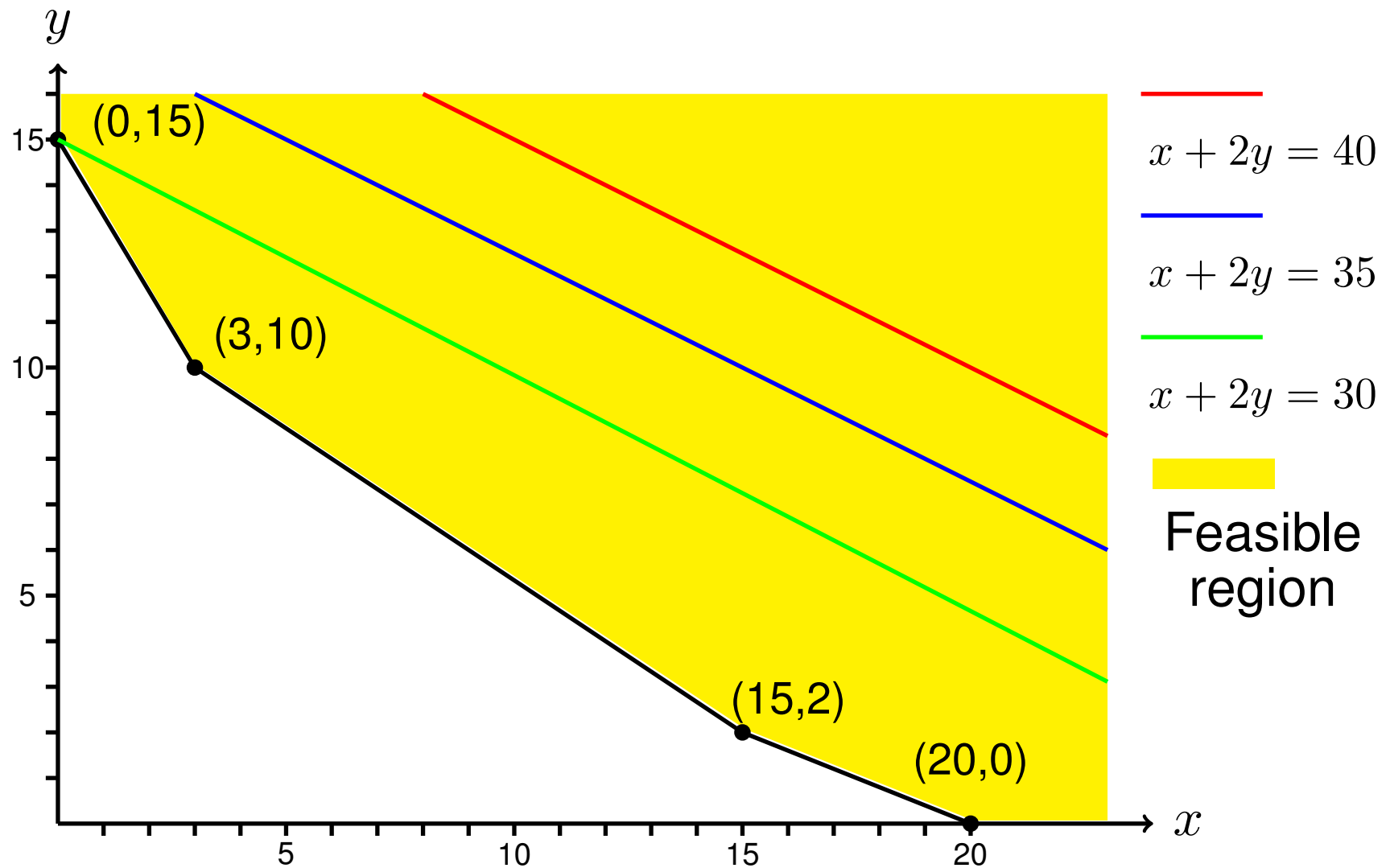
■ Regular constraints Graphed

■ $5x + 3y \geq 45$, $2x + 3y \geq 36$, $2x + 5y \geq 40$.



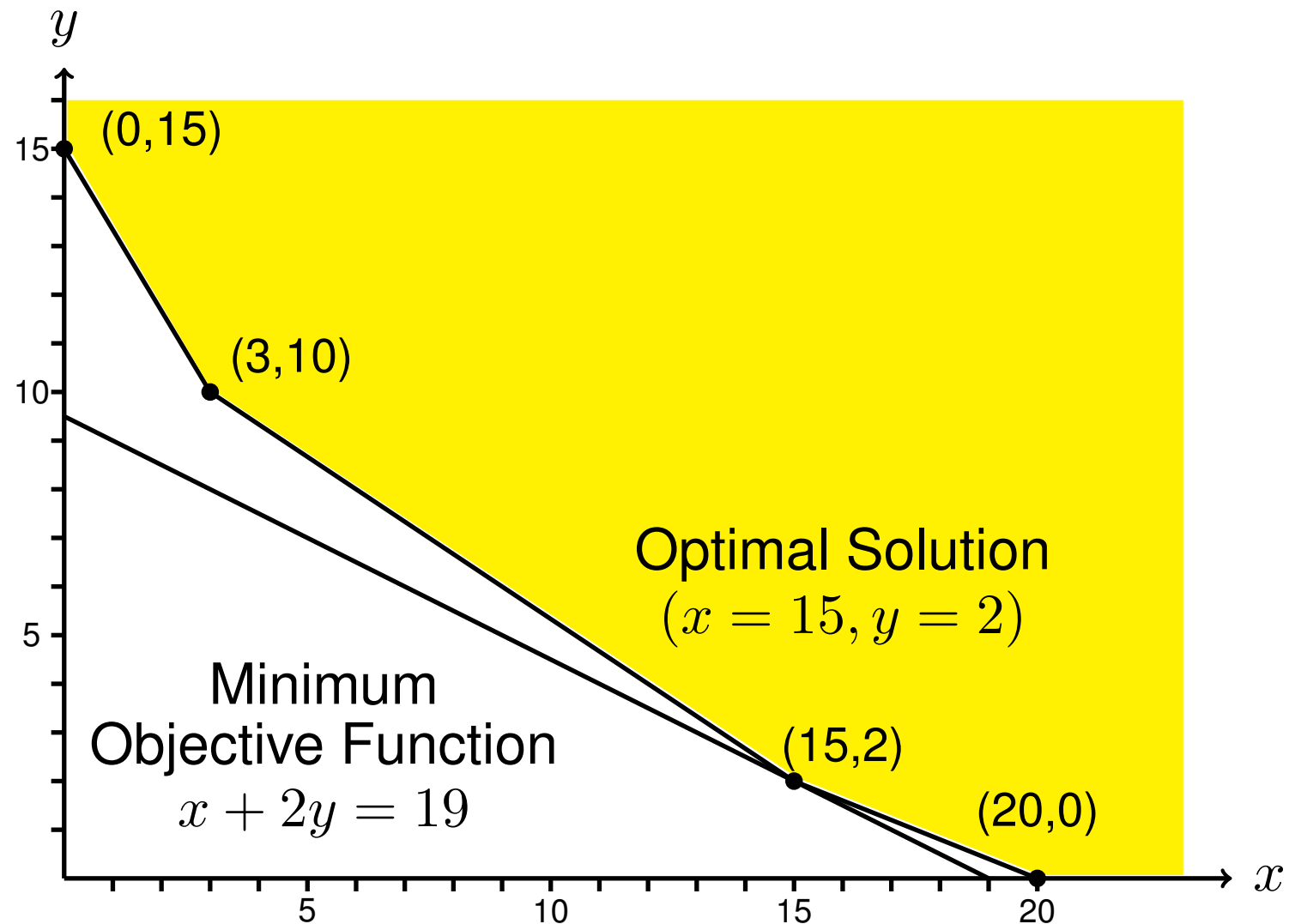
Graphical Solution (Cont.)

■ Selected objective function line



Graphical Solution (Cont.)

■ Optimal Solution



What is A Linear Programming Problem

Example 1

$$\begin{array}{ll}\max & 5x + 3y \\ \text{s.t.} & \left. \begin{array}{l} x + y \leq 8 \\ 2x + y \leq 10 \end{array} \right\} \\ & x \geq 0 \text{ and } y \geq 0\end{array}$$

Example 2

$$\begin{array}{ll}\min & x + 2y \\ \text{s.t.} & \left. \begin{array}{l} 5x + 3y \geq 45 \\ 2x + 3y \geq 36 \\ 2x + 5y \geq 40 \end{array} \right\} \\ & x \geq 0 \text{ and } y \geq 0\end{array}$$

■ Objective (or cost) function

- maximize $c_1x_1 + \cdots + c_nx_n$, or
- minimize $c_1x_1 + \cdots + c_nx_n$, or
- find any feasible solution

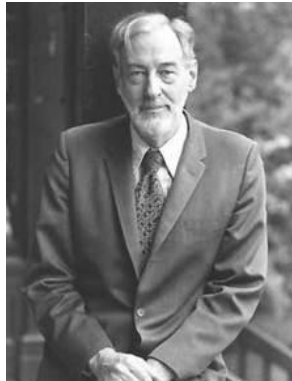
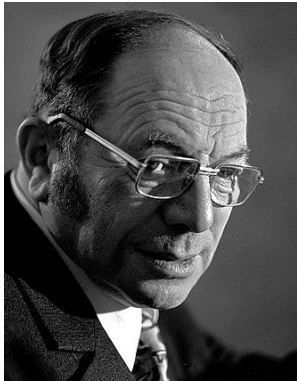
■ "Regular" constraints

- $a_1x_1 + \cdots + a_nx_n \leq b$, or
- $a_1x_1 + \cdots + a_nx_n \geq b$, or
- $a_1x_1 + \cdots + a_nx_n = b$, or
- any combination

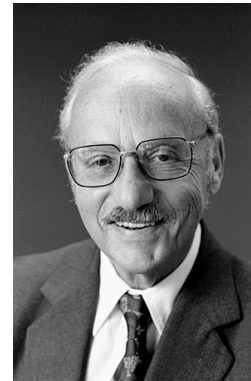
■ Nonnegative constraints $x_1 \geq 0, \cdots, x_n \geq 0$

Background on LP Problem

L. Kantorovich and T. C. Koopmans solved the LP problem based on different areas, and shared 1975 Nobel prize in economics.



World War II



1947

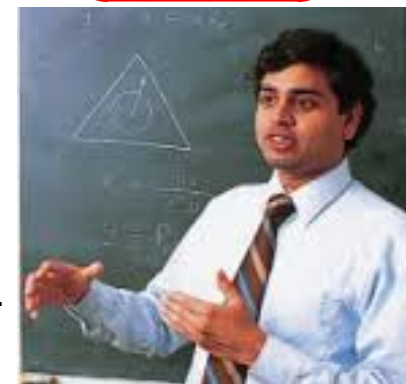
George Bernard Dantzig develops simplex method, and takes National Medal of Science Awards from Gerald R. Ford in 1976.



Leonid Khachiyan first solves LP problem in polynomial time. The New York Times published an article "A Soviet Discovery Rocks of Mathematics" in 1979.

1979

N. Karmarkar proposed a projective method. The algorithm was much faster in practical LP than the simplex method.



1984

Background on LP Problem (Cont.)

- Further developed over following decades.
- Applications of LP problem
 - Economics,
 - Business management,
 - Transportation,
 - Technology,
 - Planning,
 - Production,
 - . . .



Guidelines for Model Formulation

- Understand the problem thoroughly.
- Describe the objective.
- Describe each constraint.
- Define the decision variable.
- Write the objective in terms of the decision variables.
- Write the constraints in terms of the decision variables.

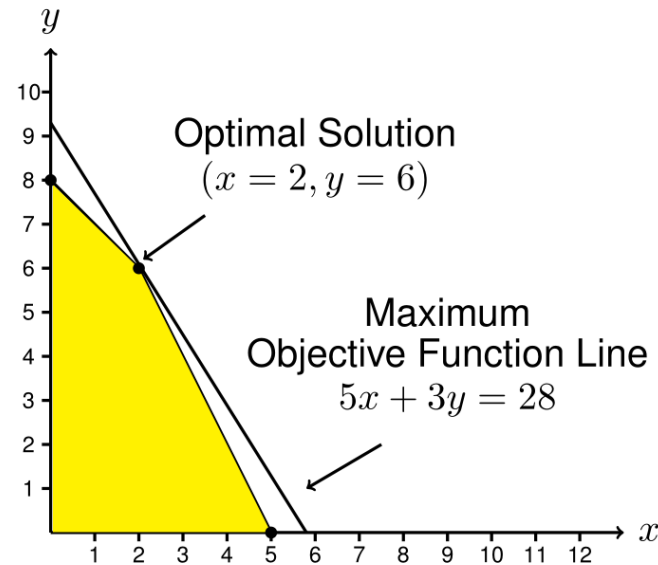
Geometric View of Simplex Method

- Simplex (Dantzig 1947)
- A **polytope** in n-dimensional space
 - Each inequality corresponds to a half-space.
 - The feasible set is the intersection of the half-spaces.
 - This corresponds to a polytope.
 - The optimal solution is at a **vertex** (i.e., a corner).

Geometric View of Simplex Method (Cont.)

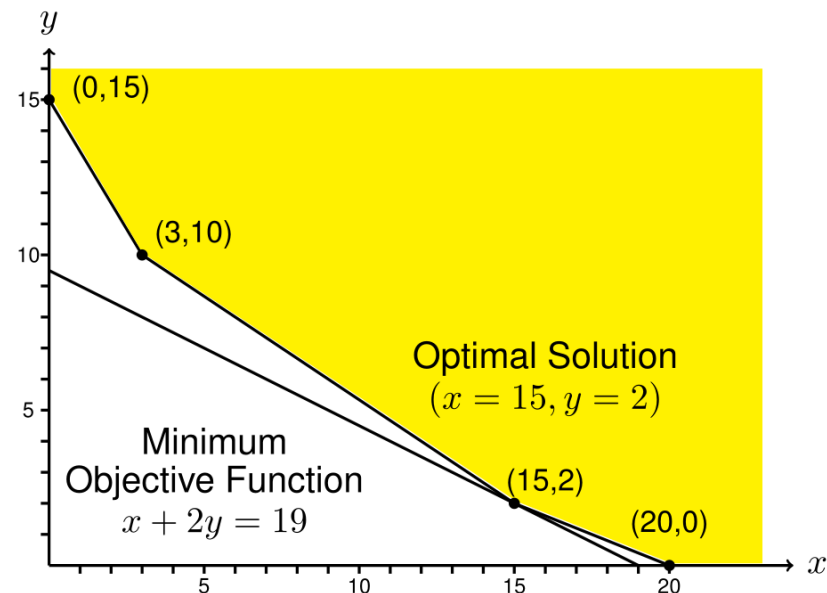
■ Example 1

Vertex	$5x + 3y$	$\max 5x + 3y$
(0,8)	24	
(2,6)	28	✓
(5,0)	25	



■ Example 2

Vertex	$x + 2y$	$\min x + 2y$
(0,15)	30	
(3,10)	23	
(15,2)	19	✓
(20,0)	20	



Feasible Solution for LP Problem

- A Halloween necklace with orange and black beads
- Constraints:
 - At least 12 inches long, but no more than 24 inches long
 - At least 5 inches of black beads
 - The length of black beads is at least twice the length of orange beads
- Objective: find any feasible solution

Formulation

■ x : the number of inches of black beads with $x \geq 0$

■ y : the number of inches of orange beads with $y \geq 0$

■ Constraints

■ At least 12 inches long, but no more than 24 inches long

$$12 \leq x + y \leq 24$$

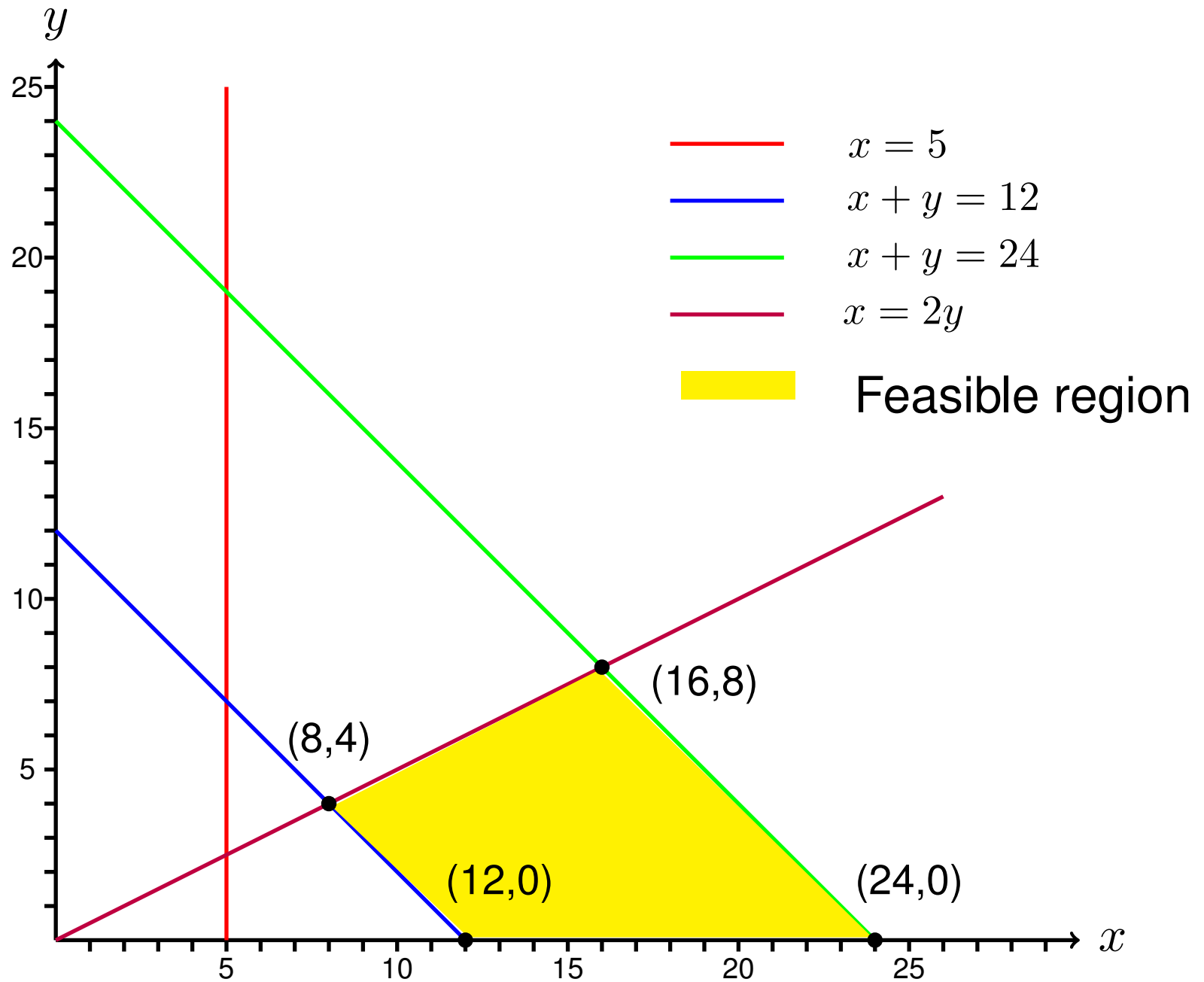
■ At least 5 inches of black beads

$$x \geq 5$$

■ The length of black beads is at least twice the length of orange beads

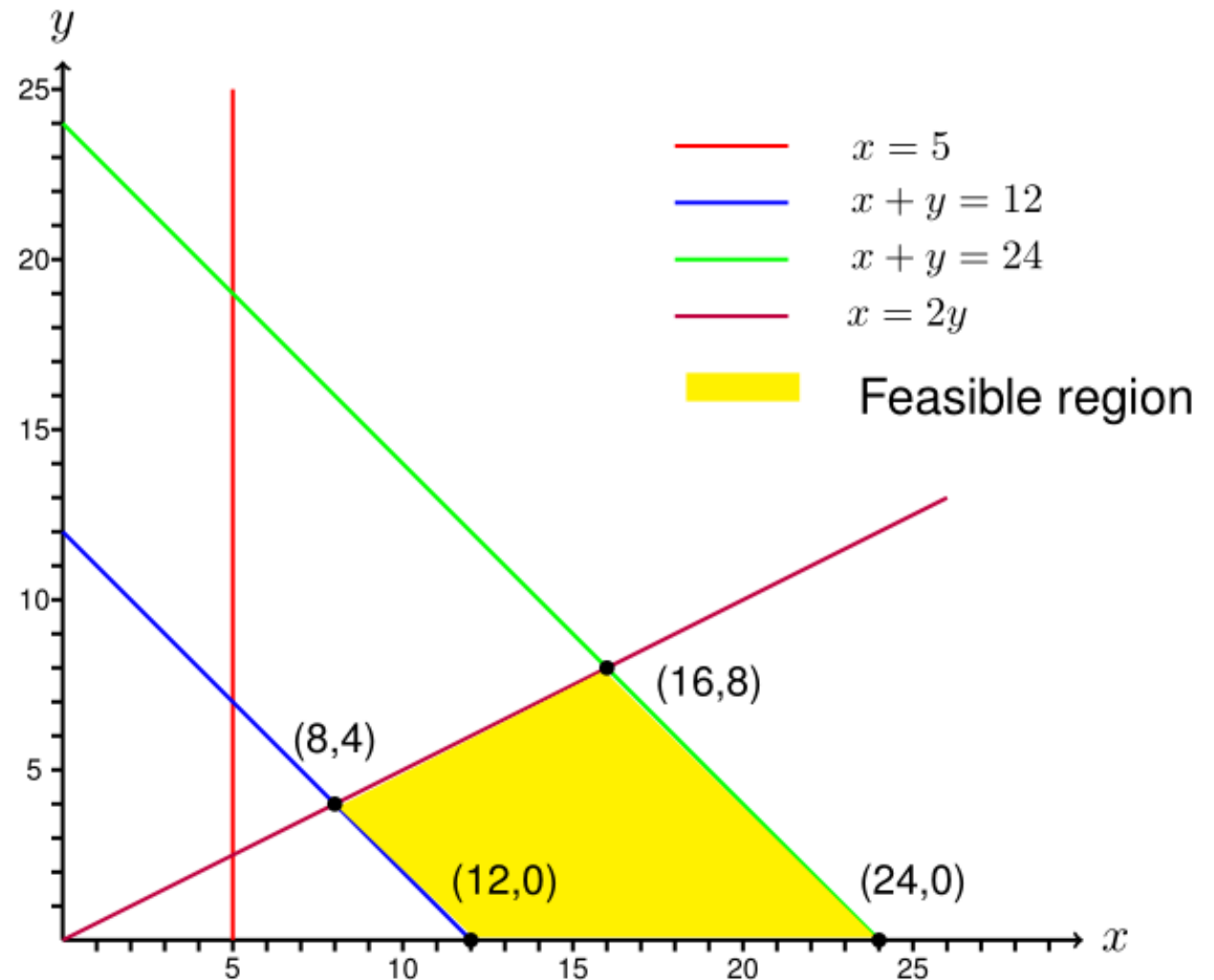
$$x \geq 2y$$

Graphical Solution



Any Feasible Solution

Vertex	$x + y$
(8,4)	12
(12,0)	12
(16,8)	24
(24,0)	24

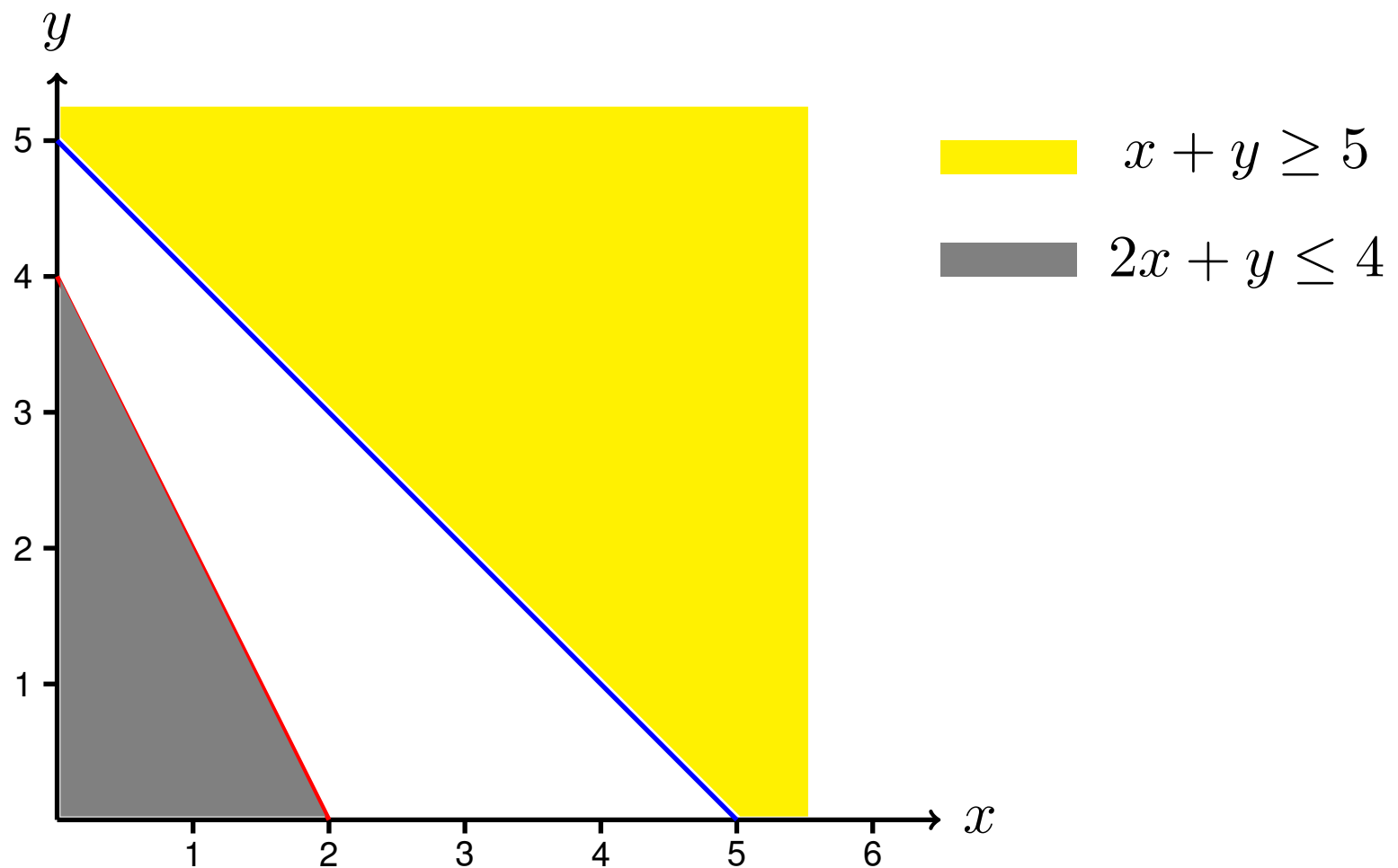


Infeasible LP Problem

- Consider the following LP problem

$$\begin{array}{ll}\max & x + 2y \\ \text{s.t.} & \left. \begin{array}{l} 2x + y \leq 4 \\ x + y \geq 5 \end{array} \right\} \\ & x \geq 0 \text{ and } y \geq 0\end{array}$$

Graphical Solution



There is NO feasible region

Infeasibility

- No solution to the LP problem satisfies all the constraints, including the non-negativity conditions.
- Graphically, this means a feasible region does not exist.
- Cause include:
 - A formulation error has been made.
 - Too many restrictions have been placed on the problem.

Unbounded LP Problem

- Unbounded

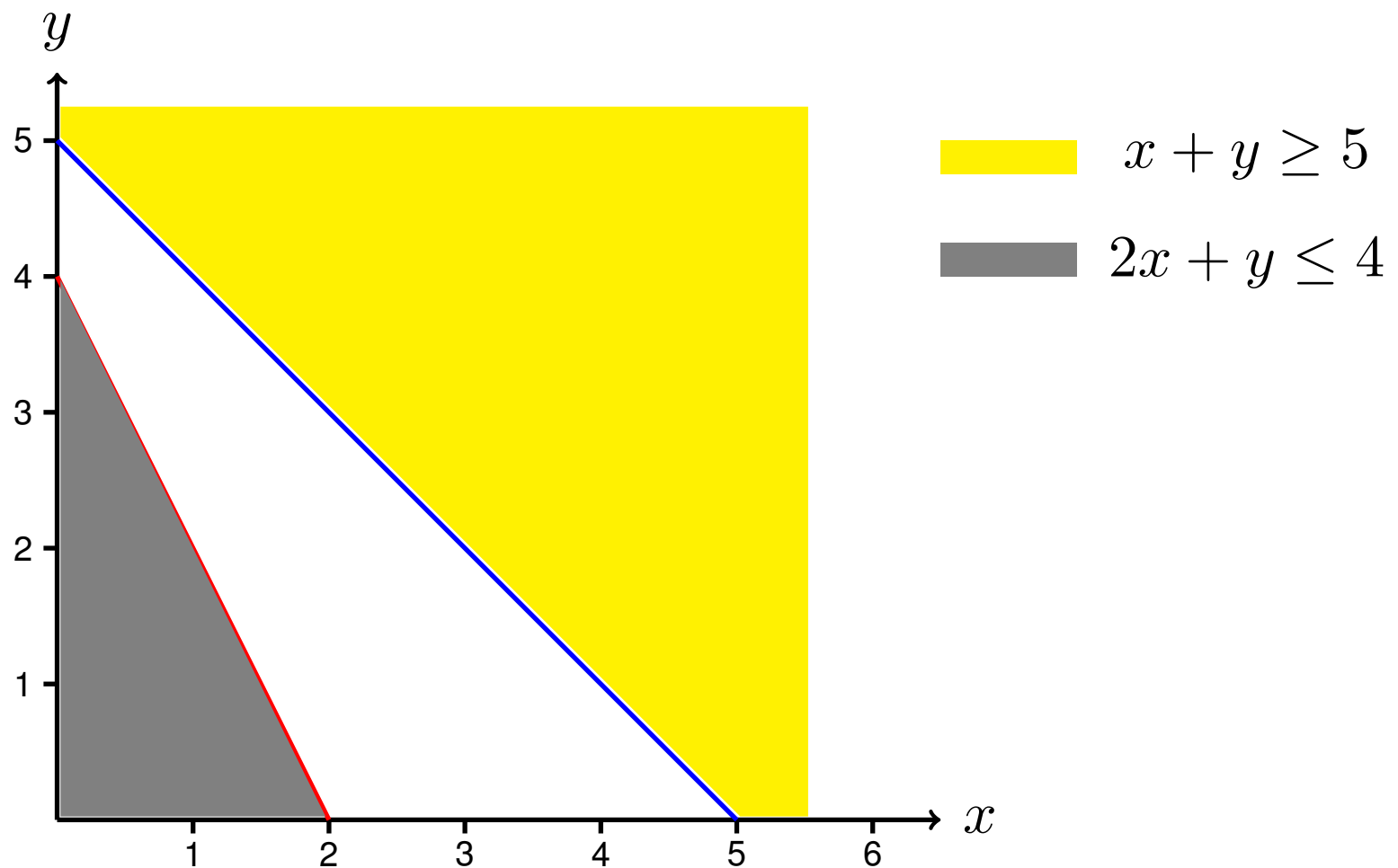
- The solution to a maximization LP problem is **unbounded** if the value of the solution may be made indefinitely large without violating any of the constraints.
- For real problems, this is the results of improper formulation.

Unbounded LP Problem

- Consider the following LP problem

$$\begin{array}{ll}\max & x + 2y \\ \text{s.t.} & \left. \begin{array}{l} x + y \geq 2 \\ 3x + 4y \geq 12 \end{array} \right\} \\ & x \geq 0 \text{ and } y \geq 0\end{array}$$

Graphical Solution



There is NO feasible region

Summary of LP Problem

- An LP problem consists of a **linear objective function** and a set of **linear constraints**—All equations involved are linear equations.
- LP problems are among the easiest optimization problems which can be solved using a variety of methods, such as **simplex** and **interior point** methods.
- LP algorithms have been used to solve large-scale problems in science, technology, military, commercial, and civil sectors.
- The defining feature of practical LP problems is the problem size: They are **BIG** problems, involving thousands of variables.